CS 70 Discrete Mathematics and Probability Theory
Spring 2025 Rao

HW 07

1 Counting on Graphs + Symmetry

Note 10

(a) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

(b) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

(d) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

Solution:

(a) Without considering symmetries there are 6! ways to color the faces of the cube. The number of equivalent colorings, for any given coloring, is $24 = 6 \times 4$: 6 comes from the fact that every given face can be rotated to face any of the six directions. 4 comes from the fact that after we decide the direction of a certain face, we can rotate the cube around this axis in 4 different ways (including no further rotations). Hence there are $6!/24 = 30$ distinct colorings.

(b) Without considering symmetries there are $n!$ ways to color the beads on the bracelet. Due to rotations, there are n equivalent colorings for any given coloring. Hence taking into account symmetries, there are $(n-1)!$ distinct colorings. Note: if in addition to rotations, we also consider flips/mirror images, then the answer would be $(n-1)!/2$.

(c) There are $\binom{n}{2} = n(n-1)/2$ possible edges, and each edge is either present or not. So the answer is $2^{n(n-1)/2}$. (Recall that $2^m = \sum_{k=0}^{m} \binom{m}{k}$, where $m = n(n-1)/2$ in this case.)

(d) The number k of vertices in a cycle is at least 3 and at most n. Without accounting for duplicates, the number of cycles of length k can be counted by choosing any ordered sequence of k vertices from the graph. Hence, there are $n!/(n-k)!$ k-length cycles. We count cycles inverted (abc = cba) and rotated (abc = bca = cab) to be non-distinct cycles. Since every k-length cycle can be inverted in one way and rotated in k-1 ways, we divide $n!/(n-k)!$ by 2 to account for inversions, and by k to account for rotations. Hence the total number of distinct cycles is

$$\sum_{k=3}^{n} \frac{n!}{(n-k)! \cdot 2k}.$$

2 Proofs of the Combinatorial Variety

Note 10 Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c=m$.)

Solution:

(a) Suppose we have n people and want to pick some of them to form a special committee. Moreover, suppose we want to pick a leader from among the committee members - how many ways can we do this?

We can do so by first picking the committee members, and then choosing the leader from among the chosen members. We can pick a committee of size k in $\binom{n}{k}$ ways, and once we have picked the committee, we have k choices for which member becomes the leader. In order to account for all possible committee sizes, we need to sum over all valid values of k, hence we get the expression

$$\sum_{k=0}^{n} k \cdot \binom{n}{k},$$

which is exactly the left hand side of the identity we want to prove.

Now, we can also count this set by first picking the leader for the committee, then choosing the rest of committee. We have n choices for the leader, and then among the remaining $n-1$ people, we can pick any subset to form the rest of the committee. Picking a subset of size k can be done in $\binom{n-1}{k}$ ways, hence summing over k, we get the expression

$$n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k},$$

which is exactly the right hand side of the identity we want to prove.

(b) Suppose we have n distinguishable red pencils, n distinguishable blue pencils, and n distinguishable green pencils ($3n$ pencils total), and want to choose m of these pencils to bring to class. How many ways can be do this?

We can do so by just picking the m pencils without considering color, as they are all distinguishable. There are $\binom{3n}{m}$ ways of doing this, which is exactly the right hand side of the identity we want to prove.

We can also count this set by picking some red pencils, the picking some blue pencils, and then finally picking some green pencils. We can pick a red pencils, b blue pencils, and c green pencils (with the tacit assumption that $a+b+c=m$) in $\binom{n}{a}\cdot\binom{n}{b}\cdot\binom{n}{c}$ ways. Finally, in order to account for all possible distributions of pencils, we need to sum over all valid triples (a,b,c), which gives us the expression

$$\sum_{a+b+c=m}\binom{n}{a}\cdot\binom{n}{b}\cdot\binom{n}{c},$$

which is exactly the left hand side of the identity we want to prove.

3 Strings

Note 10 Show your work/justification for all parts of this problem.

(a) How many different strings of length 5 can be constructed using the characters A, B, C?

(b) How many different strings of length 5 can be constructed using the characters A, B, C that contain at least one of each character?

Solution:

(a) The number of different strings of length 5 is 3^5 since each position have 3 different choices.

(b) Let E_A be the set of strings that the character A is not used in the string. We define E_B, E_C similarly. Then the total number of "bad" strings is $|E_A \cup E_B \cup E_C|$.

By the Principle of Inclusion and Exclusion,

$$|E_A \cup E_B \cup E_C| = |E_A| + |E_B| + |E_C| - |E_A \cap E_B| - |E_A \cap E_C| - |E_B \cap E_C| + |E_A \cap E_B \cap E_C| = 3\cdot 2^5 - 3\cdot 1 = 93$$

where $|E_A \cap E_B| = |E_B \cap E_C| = |E_C \cap E_A| = 1$, and $|E_A \cap E_B \cap E_C| = 0$. Thus, the total number of valid string is $3^5 - 93 = 150$

4 Unions and Intersections

Note 11 Given:

- X is a countable, non-empty set. For all $i \in X$, A_i is an uncountable set.
- Y is an uncountable set. For all $i \in Y$, B_i is a countable set.

For each of the following, decide if the expression is "Always countable", "Always uncountable", "Sometimes countable, Sometimes uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

(a) $X \cap Y$

(b) $X \cup Y$

(c) $\bigcup_{i \in X} A_i$

(d) $\bigcap_{i \in X} A_i$

(e) $\bigcup_{i \in Y} B_i$

(f) $\bigcap_{i \in Y} B_i$

Solution:

(a) Always countable. $X \cap Y$ is a subset of X, which is countable.

(b) Always uncountable. $X \cup Y$ is a superset of Y, which is uncountable.

(c) Always uncountable. Let x be any element of X. A_x is uncountable. Thus, $\bigcup_{i \in X} A_i$, a superset of A_x, is uncountable.

(d) Sometimes countable, sometimes uncountable.

Countable: When the A_i are disjoint, the intersection is empty, and thus countable. For example, let $X = \mathbb{N}$, let $A_i = \{i\} \times \mathbb{R} = \{(i,x) \mid x \in \mathbb{R}\}$. Then, $\bigcap_{i \in X} A_i = \varnothing$.

Uncountable: When the A_i are identical, the intersection is uncountable. Let $X = \mathbb{N}$, let $A_i = \mathbb{R}$ for all i. $\bigcap_{i \in X} A_i = \mathbb{R}$ is uncountable.

(e) Sometimes countable, sometimes uncountable.

Countable: Make all the B_i identical. For example, let $Y = \mathbb{R}$, and $B_i = \mathbb{N}$. Then, $\bigcup_{i \in Y} B_i = \mathbb{N}$ is countable.

Uncountable: Let $Y = \mathbb{R}$. Let $B_i = \{i\}$. Then, $\bigcup_{i \in Y} B_i = \mathbb{R}$ is uncountable.

(f) Always countable. Let y be any element of Y. B_y is countable. Thus, $\bigcap_{i \in Y} B_i$, a subset of B_y, is also countable.

5 Count It!

Note 11

For each of the following collections, determine and briefly explain whether it is finite, countably infinite (like the natural numbers), or uncountably infinite (like the reals):

(a) The integers which divide 8.

(b) The integers which 8 divides.

(c) The functions from $\mathbb{N}$ to $\mathbb{N}$.

(d) The set of strings over the English alphabet. (Note that the strings may be arbitrarily long, but each string has finite length. Also the strings need not be real English words.)

(e) The set of finite-length strings drawn from a countably infinite alphabet, $\mathscr{C}$.

(f) The set of infinite-length strings over the English alphabet.

Solution:

(a) Finite. They are $\{-8,-4,-2,-1,1,2,4,8\}$.

(b) Countably infinite. We know that there exists a bijective function $f:\mathbb{N}\to\mathbb{Z}$. Then the function $g(n)=8f(n)$ is a bijective mapping from $\mathbb{N}$ to integers which 8 divides.

(c) Uncountably infinite. We use Cantor's Diagonalization Proof:

Let $\mathscr{F}$ be the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We can represent a function $f\in\mathscr{F}$ as an infinite sequence $(f(0),f(1),\cdots)$, where the i-th element is $f(i)$. Suppose towards a contradiction that there is a bijection from $\mathbb{N}$ to $\mathscr{F}$:

$$\begin{aligned}
0 &\longleftrightarrow (f_0(0), f_0(1), f_0(2), f_0(3), \dots)\\
1 &\longleftrightarrow (f_1(0), f_1(1), f_1(2), f_1(3), \dots)\\
2 &\longleftrightarrow (f_2(0), f_2(1), f_2(2), f_2(3), \dots)\\
3 &\longleftrightarrow (f_3(0), f_3(1), f_3(2), f_3(3), \dots)\\
&\vdots
\end{aligned}$$

Consider the function $g:\mathbb{N}\to\mathbb{N}$ where $g(i)=f_i(i)+1$ for all $i\in\mathbb{N}$. We claim that the function g is not in our finite list of functions. Suppose for contradiction that it were, and that it was the n-th function $f_n(\cdot)$ in the list, i.e., $g(\cdot)=f_n(\cdot)$. However, $f_n(\cdot)$ and $g(\cdot)$ differ in the n-th argument, i.e. $f_n(n)\neq g(n)$, because by our construction $g(n)=f_n(n)+1$. Contradiction!

(d) Countably infinite. The English language has a finite alphabet (52 characters if you count only lower-case and upper-case letters, or more if you count special symbols – either way, the alphabet is finite).

We will now enumerate the strings in such a way that each string appears exactly once in the list. We will use the same trick as used in Lecture note 10 to enumerate the elements of $\{0,1\}^*$ We get our bijection by setting $f(n)$ to be the n-th string in the list. List all strings of length 1 in lexicographic order, and then all strings of length 2 in lexicographic order, and then strings of length 3 in lexicographic order, and so forth. Since at each step, there are only finitely many strings of a particular length ℓ, any string of finite length appears in the list. It is also clear that each string appears exactly once in this list.

(e) Countably infinite. Let $\mathscr{C} = \{a_1, a_2, \dots\}$ denote the alphabet. (We are making use of the fact that the alphabet is countably infinite when we assume there is such an enumeration.) We will provide two solutions:

Alternative 1: We will enumerate all the strings similar to that in part (b), although the enumeration requires a little more finesse. Notice that if we tried to list all strings of length 1, we would be stuck forever, since the alphabet is infinite! On the other hand, if we try to restrict our alphabet and only print out strings containing the first character $a \in \mathscr{C}$, we would also have a similar problem: the list

$$a, aa, aaa, \dots$$

also does not end.

The idea is to restrict *both* the length of the string and the characters we are allowed to use:

1. List all strings containing only a_1 which are of length at most 1.
2. List all strings containing only characters in $\{a_1, a_2\}$ which are of length at most 2 and have not been listed before.
3. List all strings containing only characters in $\{a_1, a_2, a_3\}$ which are of length at most 3 and have not been listed before.
4. Proceed onwards.

At each step, we have restricted ourselves to a finite alphabet with a finite length, so each step is guaranteed to terminate. To show that the enumeration is complete, consider any string s of length ℓ; since the length is finite, it can contain at most ℓ distinct a_i from the alphabet. Let k denote the largest index of any a_i which appears in s. Then, s will be listed in step $\max(k, \ell)$, so it appears in the enumeration. Further, since we are listing only those strings that have not appeared before, each string appears exactly once in the listing.

Alternative 2: We will encode the strings into ternary strings. Recall that we used a similar trick in Lecture note 10 to show that the set of all polynomials with natural coefficients is countable. Suppose, for example, we have a string: $S = a_5 a_2 a_7 a_4 a_6$. Corresponding to each of the characters in this string, we can write its index as a binary string: $(101, 10, 111, 100, 110)$. Now, we can construct a ternary string where "2" is inserted as a separator between each binary string. Thus we map the string S to a ternary string: 101210211121002110. It is clear that this mapping is injective, since the original string S can be uniquely recovered from this ternary string. Thus we have an injective map to $\{0,1,2\}^*$. From note 11, we know that the set $\{0,1,2\}^*$ is countable, and hence the set of all strings with finite length over $\mathscr{C}$ is countable.

(f) Uncountably infinite. We can use a diagonalization argument. First, for a string s, define $s[i]$ as the i-th character in the string (where the first character is position 0), where $i \in \mathbb{N}$ because the strings are infinite. Now suppose for contradiction that we have an enumeration of strings s_i for all $i \in \mathbb{N}$: then define the string s' as $s'[i] = (\text{the next character in the alphabet after } s_i[i])$, where the character after z loops around back to a. Then s' differs at position i from s_i for all

$i \in \mathbb{N}$, so it is not accounted for in the enumeration, which is a contradiction. Thus, the set is uncountable.

Alternative 1: The set of all infinite strings containing only as and bs is a subset of the set we're counting. We can show a bijection from this subset to the real interval $\mathbb{R}[0,1]$, which proves the uncountability of the subset and therefore entire set as well: given a string in $\{a,b\}^*$, replace the as with 0s and bs with 1s and prepend $'0.'$ to the string, which produces a unique binary number in $\mathbb{R}[0,1]$ corresponding to the string.