Due: Saturday, 3/28, 4:00 PM
Grace period until Saturday, 3/28, 6:00 PM
Remember to show your work for all problems!

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Post-Midterm Check-In Form

Please fill out this required form to let us know how the midterm went for you and what feedback you have for us: https://tinyurl.com/cs70sp26midtermfeedback!

2 Counting Warmup

Note 12 In discussion, we used the combinatorial identity: $\binom{a+b}{a} = \binom{a+b}{b}$. You will prove this important identity in two ways.

(a) Prove this identity algebraically (i.e. using the definition of $\binom{n}{k}$).

(b) Now, prove the identity using a combinatorial proof.

3 Counting, Counting, and More Counting

Note 12 The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

(a) How many ways are there to arrange n 1s and k 0s into a sequence?

(b) How many 19-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?

(c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.

(i) How many different 13-card bridge hands are there?

(ii) How many different 13-card bridge hands are there that contain no aces?

(iii) How many different 13-card bridge hands are there that contain all four aces?

(iv) How many different 13-card bridge hands are there that contain exactly 4 spades?

(d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?

(e) How many 99-bit strings are there that contain more ones than zeros? (Do not leave your answer as a summation. It will simplify nicely, think of how to use symmetry to help you.)

(f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.

(i) How many different anagrams of ALABAMA are there?

(ii) How many different anagrams of MONTANA are there?

(g) How many different anagrams of ABCDEF are there if:

(i) C is the left neighbor of E

(ii) C is on the left of E (and not necessarily E's neighbor)

(h) We have 8 balls, numbered 1 through 8, and 25 bins. How many different ways are there to distribute these 8 balls among the 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(i) How many different ways are there to throw 8 identical balls into 25 bins? Assume the bins are distinguishable (e.g., numbered 1 through 25).

(j) We throw 8 identical balls into 6 bins. How many different ways are there to distribute these 8 balls among the 6 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 6).

(k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**

4 Proofs of the Combinatorial Variety

Note 12

Prove each of the following identities using a combinatorial proof.

(a) For every positive integer $n > 1$,

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

(b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a,b,c) such that $a+b+c=m$.)

5 Five Up

Note 13

Say you toss a coin five times, and record the outcomes. For the three questions below, you can assume that order matters in the outcome, and that the probability of heads is some p in $0 < p < 1$, but *not* that the coin is fair ($p = 0.5$).

(a) What is the size of the sample space, $|\Omega|$?

(b) How many elements of Ω have exactly three heads?

(c) How many elements of Ω have three or more heads?

For the next three questions, you can assume that the coin is fair (i.e. heads comes up with $p = 0.5$, and tails otherwise).

(d) What is the probability that you will observe the sequence HHHTT? What about HHHHT?

(e) What is the probability of observing at least one head?

(f) What is the probability you will observe more heads than tails?

6 Aces

Note 13

Consider a standard 52-card deck of cards, which has 4 suits (hearts, diamonds, clubs, and spades) with 13 cards in each suit. Each suit has one ace. Hearts and diamonds are red, while clubs and spades are black.

(a) Find the probability of getting an ace or a red card, when drawing a single card.

(b) Find the probability of getting an ace or a spade, but not both, when drawing a single card.

(c) Find the probability of getting the ace of diamonds when drawing a 5 card hand.

(d) Find the probability of getting exactly 2 aces when drawing a 5 card hand.

(e) Find the probability of getting at least 1 ace when drawing a 5 card hand.

(f) Find the probability of getting at least 1 ace or at least 1 heart when drawing a 5 card hand.

7 Past Probabilified

Note 13

In this question we review some of the past CS70 topics, and look at them probabilistically.
For the following experiments, define an appropriate sample space Ω, and give the probability

function $\mathbb{P}[\omega]$ for each $\omega \in \Omega$. Then compute the probabilities of the events E_1 and E_2.

(a) Fix a prime $p > 2$, and uniformly sample twice with replacement from $\{0, \ldots, p-1\}$ (assume we have two $\{0, \ldots, p-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other in $(\bmod p)$ space.

$$E_1 = \text{The resulting product is } 0.$$
$$E_2 = \text{The product is } (p-1)/2.$$

(b) Make a graph on n vertices by sampling uniformly at random from all possible edges, (assume for each edge we flip a coin and if it is head we include the edge in the graph and otherwise we exclude that edge from the graph).

$$E_1 = \text{The graph is complete.}$$
$$E_2 = \text{vertex } v_1 \text{ has degree } d.$$