CS 70 Discrete Mathematics and Probability Theory
Fall 2021 HW 8

Due: Saturday 10/23, 4:00 PM
Grace period until Saturday 10/23, 5:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Probability Warm-Up

(a) Suppose that we have a bucket of 30 red balls and 70 blue balls. If we pick 20 balls uniformly out of the bucket, what is the probability of getting exactly k red balls (assuming $0 \leq k \leq 20$) if the sampling is done **with** replacement, i.e. after we take a ball out the bucket we return the ball back to the bucket for the next round?

(b) Same as part (a), but the sampling is **without** replacement, i.e. after we take a ball out the bucket we **do not** return the ball back to the bucket.

(c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

2 Five Up

Say you toss a coin five times, and record the outcomes. For the three questions below, you can assume that order matters in the outcome, and that the probability of heads is some p in $0 < p < 1$, but *not* that the coin is fair ($p = 0.5$).

(a) What is the size of the sample space, $|\Omega|$?

(b) How many elements of Ω have exactly three heads?

(c) How many elements of Ω have three or more heads?
(*Hint:* Argue by symmetry.)

For the next three questions, you can assume that the coin is fair (i.e. heads comes up with $p = 0.5$, and tails otherwise).

(d) What is the probability that you will observe the sequence HHHTT? What about HHHHT?

(e) What is the chance of observing at least one head?

(f) What about the chance of observing three or more heads?

For the final three questions, you can instead assume the coin is biased so that it comes up heads with probability $p = \frac{2}{3}$.

(g) What is the chance of observing the outcome HHHTT? What about HHHHT?

(h) What about the chance of at least one head?

(i) What about the chance of ≥ 3 heads?

3 Easter Eggs

You made the trek to Soda for a Spring Break-themed homework party, and every attendee gets to leave with a party favor. You're given a bag with 20 chocolate eggs and 40 (empty) plastic eggs. You pick 5 eggs (uniformly) without replacement.

(a) What is the probability that the first egg you drew was a chocolate egg?

(b) What is the probability that the second egg you drew was a chocolate egg?

(c) Given that the first egg you drew was an empty plastic one, what is the probability that the fifth egg you drew was also an empty plastic egg?

4 Past Probabilified

In this question we review some of the past CS70 topics, and look at them probabilistically.
For the following experiments,

i. Define an appropriate sample space Ω.

ii. Give the probability function $\mathbb{P}(\omega)$.

iii. Compute $\mathbb{P}(E_1)$ given event E_1.

iv. Compute $\mathbb{P}(E_2)$ given event E_2.

(a) Fix a prime $p > 2$, and uniformly sample twice with replacement from $\{0, \ldots, p-1\}$ (assume we have two $\{0, \ldots, p-1\}$-sided fair dice and we roll them). Then multiply these two numbers with each other in $(\bmod p)$ space.
$E_1 =$ The resulting product is 0.
$E_2 =$ The product is $(p-1)/2$.

(b) Make a graph on n vertices by sampling uniformly at random from all possible edges, (assume for each edge we flip a coin and if it is head we include the edge in the graph and otherwise we exclude that edge from the graph).
$E_1 =$ The graph is complete.
$E_2 =$ vertex v_1 has degree d.

(c) Create a random stable matching instance by having each person's preference list be a random permutation of the opposite entity's list (make the preference list for each individual job and each individual candidate a random permutation of the opposite entity's list). Finally, create a uniformly random pairing by matching jobs and candidates up uniformly at random (note that in this pairing, (1) a candidate cannot be matched with two different jobs, and a job cannot be matched with two different candidates (2) the pairing does not have to be stable). $E_1 =$ All jobs have distinct favorite candidates.
$E_2 =$ The resulting pairing is the candidate-optimal stable pairing.

5 Ball-and-Bin Counting Problems

Say you have 5 bins, and randomly throw 7 balls into them.

1. What is the probability that the first bin has precisely 3 balls in it?
2. What is the probability that the third bin has at least 3 balls in it?
3. What is the probability that at least one of the bins has precisely 3 balls in it?