CS 70 Discrete Mathematics and Probability Theory
Fall 2021

HW 9

Due: Saturday 10/30, 4:00 PM
Grace period until Saturday 10/30, 5:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Let's Talk Probability

(a) When is $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$ true? What is the general expression for $\mathbb{P}[A \cup B]$ that is always true?

(b) When is $\mathbb{P}[A \cap B] = \mathbb{P}[A] \cdot \mathbb{P}[B]$ true? What is the general expression for $\mathbb{P}[A \cap B]$ that is always true?

(c) If A and B are disjoint, does that imply they're independent?

2 Independent Complements

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

(a) Prove or disprove: $\overline{A}$ and $\overline{B}$ must be independent.

(b) Prove or disprove: A and $\overline{B}$ must be independent.

(c) Prove or disprove: A and $\overline{A}$ must be independent.

(d) Prove or disprove: It is possible that $A = B$.

3 Conditional Practice

(a) Suppose you have 3 bags. Two of them contain a \$10 bill and a \$5 bill, and the third contains two \$5 bills. Suppose you pick one of these bags uniformly at random, you draw a bill from the bag without looking uniformly at random. Suppose it turns out to be a \$5 bill. If a you draw the remaining bill from the bag, what is the probability that it, too, is a \$5 bill? Show your calculations.

(b) Now suppose that you have a large number of bags, and that each of them contain either a gold or a silver coin (every bag contains exactly one coin). Moreover, these bags are either colored red, blue, or yellow (every bag is exactly one of these colors). Half of the bags are red and a third of the bags are blue. Moreover, two thirds of the red bags and one fourth of the blue bags contain gold coins. Lastly, a randomly chosen bag has a $\frac{1}{2}$ probability of containing a silver coin. Suppose that you pick a bag at random and find that it contains a silver coin. What is the probability that the bag you picked was yellow?

4 Monty Hall's Revenge

Due to a quirk of the television studio's recruitment process, Monty Hall has ended up drawing all the contestants for his game show from among the ranks of former CS70 students. Unfortunately for Monty, the former students' amazing probability skills have made his cars-and-goats gimmick unprofitable for the studio. Monty decides to up the stakes by asking his contestants to generalise to three new situations with a variable number of doors, goats, and cars:

(a) There are n doors for some $n > 2$. One has a car behind it, and the remaining $n-1$ have goats. As in the ordinary Monty Hall problem, Monty will reveal one door with a goat behind it after you make your first selection. How would switching affect the odds that you select the car?

(Hint: Think about the size of the sample space for the experiment where you *always* switch. How many of those outcomes are favorable?)

(b) Again there are $n > 2$ doors, one with a car and $n-1$ with goats, but this time Monty will reveal $n-2$ doors with goats behind them instead of just one. How does switching affect the odds of winning in this modified scenario?

(c) Finally, imagine there are $k < n-1$ cars and $n-k$ goats behind the $n > 2$ doors. After you make your first pick, Monty will reveal $j < n-k$ doors with goats. What values of j,k maximize the relative improvement in your odds of winning if you choose to switch? (i.e. what j,k maximizes the ratio between your odds of winning when you switch, and your odds of winning when you do not switch?)

5 Random Polynomials

Consider the following scenarios, where we apply probability to polynomials. We generate a new real polynomial Q, by picking 6 numbers in the set $\{0,1,2,3,4,5,6\}$ independently and uniformly at random (with replacement), for a polynomial of the form

$$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

The first number we pick is a_5, the second number is a_4, etc.

(a) What is probability we have a polynomial of degree less than 4?

(b) What is the probability that the polynomial has degree at least 4?

(c) Now, consider only degree 5 polynomials that were randomly generated using the scheme described above. What is the probability that the sum of its coefficients is equal to 6?

6 (Un)conditional (In)equalities

Let us consider a sample space $\Omega = \{\omega_1, \ldots, \omega_N\}$ of size $N > 2$ and two probability functions $\mathbb{P}_1$ and $\mathbb{P}_2$ on it. That is, we have two probability spaces: $(\Omega, \mathbb{P}_1)$ and $(\Omega, \mathbb{P}_2)$.

(a) Suppose that for every subset $A \subseteq \Omega$ of size $|A| = 2$ and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1[\omega \mid A] = \mathbb{P}_2[\omega \mid A]$. Is it necessarily true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$? That is, if $\mathbb{P}_1$ and $\mathbb{P}_2$ are equal conditional on events of size 2, are they equal unconditionally? (*Hint*: Remember that probabilities must add up to 1.)

(b) Suppose that for every subset $A \subseteq \Omega$ of size $|A| = k$, where k is some fixed element in $\{2, \ldots, N\}$, and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1[\omega \mid A] = \mathbb{P}_2[\omega \mid A]$. Is it necessarily true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$?

For the following two parts, assume that $\Omega = \left\{(a_1, \ldots, a_k) \mid \sum_{j=1}^{k} a_j = n\right\}$ is the set of configurations of n balls into k labeled bins, and let $\mathbb{P}_1$ be the probabilities assigned to these configurations by throwing the balls independently one after another and they will land into any of the k bins uniformly at random, and let $\mathbb{P}_2$ be the probabilities assigned to these configurations by uniformly sampling one of these configurations.

(c) Let A be the event that all n balls are in exactly one bin.

   (i) What are $\mathbb{P}_1[\omega \mid A]$ and $\mathbb{P}_2[\omega \mid A]$ for any $\omega \in A$?

   (ii) Repeat part (i) for $\omega \in \Omega \setminus A$.

   (iii) Is it true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$?

(d) For the special case of $n = 9$ and $k = 3$, provide two outcomes B and C, so that $\mathbb{P}_1[B] < \mathbb{P}_2[B]$ and $\mathbb{P}_1[C] > \mathbb{P}_2[C]$. Provide justification.