CS 70 Discrete Mathematics and Probability Theory
Fall 2021 **HW 10**

1 Cliques in Random Graphs

Consider the graph $G = (V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. Let's call the event that S forms a clique E_S. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1, \ldots, v_\ell\}$ and $V_2 = \{w_1, \ldots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \leq n^k$. (You might find this useful in part (e))

(e) Prove that the probability that the graph contains a k-clique, for $k \geq 4\log_2 n + 1$, is at most $1/n$.

Solution:

(a) Between every pair of verticles, there is either an edge or not. Since there are two choices for each of the $\binom{n}{2}$ pairs of vertices, the size of the sample space is $2^{\binom{n}{2}}$.

(b) For a fixed set of k vertices to be a k-clique, all of the $\binom{k}{2}$ pairs of those vertices have to be connected by an edge. The probability of this event is $1/2^{\binom{k}{2}}$.

(c) E_{V_1} and E_{V_2} are independent if and only if V_1 and V_2 share at most one vertex: If V_1 and V_2 share at most one vertex, then since edges are added independently of each other, we have

$$\begin{aligned}\mathbb{P}(E_{V_1} \cap E_{V_2}) &= \mathbb{P}(\text{all edges in } V_1 \text{ and all edges in } V_2 \text{ are present}) \\ &= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2}} \\ &= \mathbb{P}(E_{V_1}) \cdot \mathbb{P}(E_{V_2}).\end{aligned}$$

Conversely, if V_1 and V_2 share at least two vertices, then their intersection $V_3 = V_1 \cap V_2$ has at least 2 elements, and whence

$$\mathbb{P}(E_{V_1} \cap E_{V_2}) = \left(\frac{1}{2}\right)^{\binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} - \binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2} - \binom{|V_3|}{2}}$$
$$= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} + \binom{|V_2|}{2} - \binom{|V_3|}{2}} \neq \mathbb{P}(E_{V_1}) \cdot \mathbb{P}(E_{V_2}).$$

(d) The algebraic solution is an application of the definition of $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!}$$
$$\leq n \cdot (n-1) \cdots (n-k+1)$$
$$\leq n^k$$

(e) Let A_S denote the event that S is a k-clique, where $S \subseteq V$ is of size k. Then, the event that the graph contains a k-clique can be described as the union of A_S's over all $S \subseteq V$ of size k. Using the union bound,

$$\mathbb{P}\left[\bigcup_{S \subseteq V, |S|=k} A_S\right] \leq \sum_{S \subseteq V, |S|=k} \mathbb{P}[A_S] = \sum_{S \subseteq V, |S|=k} \frac{1}{2^{\binom{k}{2}}}.$$

Now, since there are $\binom{n}{k}$ ways of choosing a subset $S \subseteq V$ of size k, the right-hand side of the above equality is

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{\binom{n}{k}}{2^{k(k-1)/2}} \leq \frac{n^k}{\left(2^{(k-1)/2}\right)^k} \leq \frac{n^k}{\left(2^{(4\log n+1-1)/2}\right)^k} = \frac{n^k}{\left(2^{2\log n}\right)^k} = \frac{n^k}{n^{2k}} = \frac{1}{n^k} \leq \frac{1}{n}.$$

2 Random Variables Warm-Up

Let X and Y be random variables, each taking values in the set $\{0, 1, 2\}$, with joint distribution

$$\begin{array}{lll} \mathbb{P}[X=0, Y=0] = 1/3 & \mathbb{P}[X=0, Y=1] = 0 & \mathbb{P}[X=0, Y=2] = 1/3 \\ \mathbb{P}[X=1, Y=0] = 0 & \mathbb{P}[X=1, Y=1] = 1/9 & \mathbb{P}[X=1, Y=2] = 0 \\ \mathbb{P}[X=2, Y=0] = 1/9 & \mathbb{P}[X=2, Y=1] = 1/9 & \mathbb{P}[X=2, Y=2] = 0. \end{array}$$

(a) What are the marginal distributions of X and Y?

(b) What are $\mathbb{E}[X]$ and $\mathbb{E}[Y]$?

(c) Let I be the indicator that $X = 1$, and J be the indicator that $Y = 1$. What are $\mathbb{E}[I]$, $\mathbb{E}[J]$ and $\mathbb{E}[IJ]$?

(d) In general, let I_A and I_B be the indicators for events A and B in a probability space $(\Omega, \mathbb{P})$. What is $\mathbb{E}[I_A I_B]$, in terms of the probability of some event?

Solution:

(a) By the law of total probabilty

$$\mathbb{P}[X=0] = \mathbb{P}[X=0, Y=0] + \mathbb{P}[X=0, Y=1] + \mathbb{P}[X=0, Y=2] = 1/3 + 0 + 1/3 = 2/3$$

and similarly

$$\mathbb{P}[X=1] = 0 + 1/9 + 0 = 1/9$$
$$\mathbb{P}[X=2] = 1/9 + 1/9 + 0 = 2/9.$$

As a sanity check, these three numbers are all positive and they add up to $2/3 + 1/9 + 2/9 = 1$ as they should. The same kind of calculation gives

$$\mathbb{P}[Y=0] = 1/3 + 0 + 1/9 = 4/9$$
$$\mathbb{P}[Y=1] = 0 + 1/9 + 1/9 = 2/9$$
$$\mathbb{P}[Y=2] = 1/3.$$

(b) From the above marginal distributions, we can compute

$$\mathbb{E}[X] = 0\mathbb{P}[X=0] + 1\mathbb{P}[X=1] + 2\mathbb{P}[X=2] = 5/9$$
$$\mathbb{E}[Y] = 0\mathbb{P}[Y=0] + 1\mathbb{P}[Y=1] + 2\mathbb{P}[Y=2] = 8/9$$

(c) We know that taking the expectation of an indicator for some event gives the probability of that event, so

$$\mathbb{E}[I] = \mathbb{P}[X=1] = 1/9$$
$$\mathbb{E}[J] = \mathbb{P}[Y=1] = 2/9.$$

The random variable IJ is equal to one if $I=1$ and $J=1$, and is zero otherwise. In other words, it is the indicator for the event that $I=1$ *and* $J=1$:

$$\mathbb{E}[IJ] = \mathbb{P}[I=1, J=1] = 1/9.$$

(d) By what we said in the previous part of the solution, $I_A I_B$ is the indicator for the event $A \cap B$, so

$$\mathbb{E}[I_A I_B] = \mathbb{P}[A \cap B].$$

3 Maybe Lossy Maybe Not

Let us say that Alice would like to send a message to Bob, over some channel. Alice has a message of length 4.

(a) Packets are dropped with probability p. If Alice sends 5 packets, what is probability that Bob can successfully reconstruct Alice's message using polynomial interpolation?

(b) Again, packets can be dropped with probability p. The channel may additionally corrupt 1 packet after deleting packets. Alice realizes this and sends 8 packets for a message of length 4. What is the probability that Bob receives enough packets to successfully reconstruct Alice's message using Berlekamp-Welch?

(c) Again, packets can be dropped with probability p. This time, packets may be corrupted with probability q. A packet being dropped is independent of whether or not is corrupted (i.e. a packet may be both corrupted and dropped). Consider the original scenario where Alice sends 5 packets for a message of length 4. What is probability that Bob can correctly reconstruct Alice's message using polynomial interpolation on all of the points he receives?

Solution:

(a) Alice's message requires a polynomial of degree 3, which can be uniquely identified by 4 points. Thus, at least 4 points need to make it across the channel. The probability that Bob can recover the message is thus the probability that at most one packet is lost. Since the packets are lost with probability with probability p, we have the probability of losing 1 packet is

$$\binom{5}{1}(1-p)^4 p.$$

The probability of losing 0 packets is $(1-p)^5$. Thus, the probability of losing 0 or 1 packets is

$$\binom{5}{1}(1-p)^4 p + (1-p)^5.$$

This is the probability that Bob receives 4 packets, meaning he can successfully reconstruct the 3-degree polynomial.

(b) Bob needs $n+2k=6$ packets to guarantee successful reconstruction of Alice's message. There are a total of 8 packets sent, so this guarantee occurs only if 0 packets, 1 packet or 2 packets are lost. The probability of 0 packets lost is

$$(1-p)^8.$$

The probability of one packet lost is

$$\binom{8}{1}p(1-p)^7.$$

The probability of two packets lost is

$$\binom{8}{2} p^2 (1-p)^6.$$

Thus, the probability of success is

$$(1-p)^8 + \binom{8}{1} p(1-p)^7 + \binom{8}{2} p^2 (1-p)^6.$$

(c) Again, Bob can reconstruct the message if none of the packets are corrupted. We use the same idea as in Part (a). The probability that none of the packets are corrupted is $(1-q)^5$. We know that *on top of* being uncorrupted, we can only at lose at most 1 packet. Thus, we can either lose one packet, which has probability

$$\binom{5}{1} p(1-p)^4.$$

Or, we can lose no packets, which has probability $(1-p)^5$. Yet another possibility is if exactly one packet is corrupted, but that packet is also dropped; in this case, we can recover the message, so long as no other packets are corrupted or dropped. This occurs with probability

$$\binom{5}{1} pq(1-p)^4(1-q)^4.$$

As a result, we have the following.

$$(1-q)^5 \left(5p(1-p)^4 + (1-p)^5\right) + 5pq(1-p)^4(1-q)^4.$$

4 Class Enrollment

Lydia has just started her CalCentral enrollment appointment. She needs to register for a marine science class and CS 70. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling successfully in the marine science class on each attempt is μ and the probability of enrolling successfully in CS 70 on each attempt is λ. Also, these events are independent.

(a) Suppose Lydia begins by attempting to enroll in the marine science class everyday and gets enrolled in it on day M. What is the distribution of M?

(b) Suppose she is not enrolled in the marine science class after attempting each day for the first 5 days. What is the conditional distribution of M given $M > 5$?

(c) Once she is enrolled in the marine science class, she starts attempting to enroll in CS 70 from day $M+1$ and gets enrolled in it on day C. Find the expected number of days it takes Lydia to enroll in both the classes, i.e. $\mathbb{E}[C]$.

Suppose instead of attempting one by one, Lydia decides to attempt enrolling in both the classes from day 1. Let M be the number of days it takes to enroll in the marine science class, and C be the number of days it takes to enroll in CS 70.

(d) What is the distribution of M and C now? Are they independent?

(e) Let X denote the day she gets enrolled in her first class and let Y denote the day she gets enrolled in both the classes. What is the distribution of X?

(f) What is the expected number of days it takes Lydia to enroll in both classes now, i.e. $\mathbb{E}[Y]$.

(g) What is the expected number of classes she will be enrolled in by the end of 14 days?

Solution:

(a) $M \sim \text{Geometric}(\mu)$.

(b) Given that $M > 5$, the random variable M takes values in $\{6,7,\dots\}$. For $i = 6,7,\dots,$

$$\mathbb{P}[M = i | M > 5] = \frac{\mathbb{P}[M = i \wedge M > 5]}{\mathbb{P}[M > 5]} = \frac{\mathbb{P}[M = i]}{\mathbb{P}[M > 5]} = \frac{\mu(1-\mu)^{i-1}}{(1-\mu)^5} = \mu(1-\mu)^{i-6}.$$

If K denotes the additional number of days it takes to get enrolled in the marine science class after day 5, i.e. $K = M - 5$, then conditioned on $M > 5$, the random variable K has the geometric distribution with parameter μ. Note that this is the same as the distribution of M. This is known as the memoryless property of geometric distribution.

(c) We have $C - M \sim \text{Geometric}(\lambda)$. Thus $\mathbb{E}[M] = 1/\mu$ and $\mathbb{E}[C - M] = 1/\lambda$. And hence $\mathbb{E}[C] = \mathbb{E}[M] + \mathbb{E}[C - M] = 1/\mu + 1/\lambda$.

(d) $M \sim \text{Geometric}(\mu)$, $C \sim \text{Geometric}(\lambda)$. Yes they are independent.

(e) We have $X = \min\{M,C\}$ and $Y = \max\{M,C\}$. We also use the following definition of the minimum:

$$\min(m,c) = \begin{cases} m & \text{if } m \leq c; \\ c & \text{if } m > c. \end{cases}$$

Now, for all $k \in \{1,2,\dots\}$, $\min(M,C) = k$ is equivalent to $(M = k) \cap (C \geq k)$ or $(C = k) \cap (M > k)$. Hence,

$$\begin{aligned} \mathbb{P}[X = k] &= \mathbb{P}[\min(M,C) = k] \\ &= \mathbb{P}[(M = k) \cap (C \geq k)] + \mathbb{P}[(C = k) \cap (M > k)] \\ &= \mathbb{P}[M = k] \cdot \mathbb{P}[C \geq k] + \mathbb{P}[C = k] \cdot \mathbb{P}[M > k] \end{aligned}$$

(since M and C are independent)

$$= [(1-\mu)^{k-1}\mu](1-\lambda)^{k-1} + [(1-\lambda)^{k-1}\lambda](1-\mu)^k$$

(since M and C are geometric)

$$= ((1-\mu)(1-\lambda))^{k-1}(\mu+\lambda(1-\mu))$$
$$= (1-\mu-\lambda+\lambda\mu)^{k-1}(\mu+\lambda-\mu\lambda).$$

But this final expression is precisely the probability that a geometric r.v. with parameter $\mu+\lambda-\mu\lambda$ takes the value k. Hence $X \sim \text{Geom}(\mu+\lambda-\mu\lambda)$.

An alternative, slightly cleaner approach is to work with the *tail probabilities* of the geometric distribution, rather than with the usual point probabilities as above. In other words, we can work with $\mathbb{P}[X \geq k]$ rather than with $\mathbb{P}[X = k]$; clearly the values $\mathbb{P}[X \geq k]$ specify the values $\mathbb{P}[X = k]$ since $\mathbb{P}[X = k] = \mathbb{P}[X \geq k] - \mathbb{P}[X \geq (k+1)]$, so it suffices to calculate them instead. We then get the following argument:

$$\begin{aligned}
\mathbb{P}[X \geq k] &= \mathbb{P}[\min(M,C) \geq k] \\
&= \mathbb{P}[(M \geq k) \cap (C \geq k)] \\
&= \mathbb{P}[M \geq k] \cdot \mathbb{P}[C \geq k] && \text{since } M, C \text{ are independent} \\
&= (1-\mu)^{k-1}(1-\lambda)^{k-1} && \text{since } M, C \text{ are geometric} \\
&= ((1-\mu)(1-\lambda))^{k-1} \\
&= (1-\mu-\lambda+\mu\lambda)^{k-1}.
\end{aligned}$$

This is the tail probability of a geometric distribution with parameter $\mu+\lambda-\mu\lambda$, so we are done.

(f) From part (e) we get $\mathbb{E}[X] = 1/(\mu+\lambda-\mu\lambda)$. From part (d) we have $\mathbb{E}[M] = 1/\mu$ and $\mathbb{E}[C] = 1/\lambda$. We now observe that $\min\{m,c\} + \max\{m,c\} = m + c$. Using linearity of expectation we get $\mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[M] + \mathbb{E}[C]$. Thus $\mathbb{E}[Y] = 1/\mu + 1/\lambda - 1/(\mu+\lambda-\mu\lambda)$.

(g) Let I_M and I_C be the indicator random variables of the events "$M \leq 14$" and "$C \leq 14$" respectively. Then $I_M + I_C$ is the number of classes she will be enrolled in within 14 days. Hence the answer is $\mathbb{E}[I_M] + \mathbb{E}[I_C] = \mathbb{P}[M \leq 14] + \mathbb{P}[C \leq 14] = 1 - (1-\mu)^{14} + 1 - (1-\lambda)^{14}$

5 Boutique Store

Consider a boutique store in a busy shopping mall. Every hour, a large number of people visit the mall, and each independently enters the boutique store with some small probability. The store owner decides to model X, the number of customers that enter her store during a particular hour, as a Poisson random variable with mean λ.

Suppose that whenever a customer enters the boutique store, they leave the shop without buying anything with probability p. Assume that customers act independently, i.e. you can assume that they each flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as Y and the number of them that do not buy anything as Z (so $X = Y + Z$).

(a) What is the probability that $Y = k$ for a given k? How about $\mathbb{P}[Z = k]$? *Hint*: You can use the identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

(b) State the name and parameters of the distribution of Y and Z.

(c) Prove that Y and Z are independent. In particular, prove that for every pair of values y, z, we have $\mathbb{P}[Y = y, Z = z] = \mathbb{P}[Y = y]\mathbb{P}[Z = z]$.

Solution:

(a) We consider all possible ways that the event $Y = k$ might happen: namely, $k + j$ people enter the store ($X = k + j$) and then exactly k of them choose to buy something. That is,

$$\begin{aligned}
\mathbb{P}[Y = k] &= \sum_{j=0}^{\infty} \mathbb{P}[X = k + j] \cdot \mathbb{P}[Y = k \mid X = k + j] \\
&= \sum_{j=0}^{\infty} \left(\frac{\lambda^{k+j}}{(k+j)!} e^{-\lambda} \right) \cdot \left(\binom{k+j}{k} p^j (1-p)^k \right) \\
&= \sum_{j=0}^{\infty} \frac{\lambda^{k+j}}{(k+j)!} e^{-\lambda} \cdot \frac{(k+j)!}{k!j!} p^j (1-p)^k \\
&= \frac{(\lambda(1-p))^k e^{-\lambda}}{k!} \cdot \sum_{j=0}^{\infty} \frac{(\lambda p)^j}{j!} \\
&= \frac{(\lambda(1-p))^k e^{-\lambda}}{k!} \cdot e^{\lambda p} \\
&= \frac{(\lambda(1-p))^k e^{-\lambda(1-p)}}{k!}.
\end{aligned}$$

The case for Z is completely analogous:

$$\mathbb{P}[Z = k] = \frac{(\lambda p)^k e^{-\lambda p}}{k!}$$

(b) Y follows the Poisson distribution with parameter $\lambda(1-p)$ and Z follows the Poisson distribution with parameter λp.

(c) The joint distribution of Y and Z is given by

$$\begin{aligned}
\mathbb{P}(Y=y,Z=z) &= \sum_{x=0}^{\infty} \mathbb{P}(X=x,Y=y,Z=z) \\
&= \sum_{x=0}^{\infty} \mathbb{P}(Y=y,Z=z \mid X=x)\mathbb{P}(X=x) \\
&= \mathbb{P}(Y=y,Z=z \mid X=y+z)\mathbb{P}(X=y+z) \\
&= \frac{\cancel{(y+z)!}}{y!z!} p^z (1-p)^y \frac{\mathrm{e}^{-\lambda}\lambda^{y+z}}{\cancel{(y+z)!}} \\
&= \frac{\mathrm{e}^{-\lambda(1-p)}(\lambda(1-p))^y}{y!} \cdot \frac{\mathrm{e}^{-\lambda p}(\lambda p)^z}{z!} \\
&= \mathbb{P}(Y=y)\cdot\mathbb{P}(Z=z).
\end{aligned}$$

Since $\mathbb{P}(Y=y,Z=z)=\mathbb{P}(Y=y)\cdot\mathbb{P}(Z=z)$ for all $y,z\in\mathbb{N}$, we get that Y and Z are independent.

6 Swaps and Cycles

We'll say that a permutation $\pi=(\pi(1),\ldots,\pi(n))$ contains a *swap* if there exist $i,j\in\{1,\ldots,n\}$ so that $\pi(i)=j$ and $\pi(j)=i$, where $i\neq j$.

(a) What is the expected number of swaps in a random permutation?

(b) In the same spirit as above, we'll say that π contains a *s-cycle* if there exist $i_1,\ldots,i_s\in\{1,\ldots,n\}$ with $\pi(i_1)=i_2,\pi(i_2)=i_3,\ldots,\pi(i_s)=i_1$. Compute the expectation of the number of s-cycles.

Solution:

(a) As a warm-up, let's compute the probability that 1 and 2 are swapped. There are $n!$ possible permutations, and $(n-2)!$ of them have $\pi(1)=2$ and $\pi(2)=1$. This means

$$\mathbb{P}[(1,2)\text{ are a swap}] = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$

There was nothing special about 1 and 2 in this calculation, so for any $\{i,j\}\subset\{1,\ldots,n\}$, the probability that i and j are swapped is the same as above. Let's write $I_{i,j}$ for the indicator that i and j are swapped, and N for the total number of swaps, so that

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{\{i,j\}\subset\{1,\ldots,n\}} I_{i,j}\right] = \sum_{\{i,j\}\subset\{1,\ldots n\}} \mathbb{P}[(i,j)\text{ are swapped}] = \frac{1}{n(n-1)}\binom{n}{2} = \frac{1}{2}.$$

(b) The idea here is quite similar to the above, so we'll be a little less verbose in the exposition. However, as a first aside we need the notion of a *cyclic ordering* of s elements from a set $\{1,\ldots n\}$. We mean by this a labelling of the s beads of a necklace with elements of the set, where we say that labelings of the beads are the same if we can move them along the string to turn one into the other. For example, $(1,2,3,4)$ and $(1,2,4,3)$ are different cyclic orderings, but $(1,2,3,4)$ and $(2,3,4,1)$ are the same. There are

$$\binom{n}{s}\frac{s!}{s} = \frac{n!}{(n-s)!}\frac{1}{s}$$

possible cyclic orderings of length s from a set with n elements, since if we first count all subsets of size s, and then all permutations of each of those subsets, we have overcounted by a factor of s.

Now, let N be a random variable counting the number of s-cycles, and for each cyclic ordering $(i_1,\ldots,i_s)$ of s elements of $\{1,\ldots,n\}$, let $I_{(i_1,\ldots,i_s)}$ be the indicator that $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_s) = i_1$. There are $(n-s)!$ permutations in which $(i_1,\ldots,i_s)$ form an s-cycle (since we are free to do whatever we want to the remaining $(n-s)$ elements of $\{1,\ldots,n\}$), so the probability that $(i_1,\ldots,i_s)$ are such a cycle is $\frac{(n-s)!}{n!}$, and

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{(i_1,\ldots,i_s)\text{ cyclic ordering}} I_{(i_1,\ldots,i_s)}\right] = \frac{n!}{(n-s)!}\frac{1}{s}\frac{(n-s)!}{n!} = \frac{1}{s}.$$