CS 70 Discrete Mathematics and Probability Theory
Fall 2021 HW 10

Due: Saturday 11/6, 4:00 PM
Grace period until Saturday 11/6, 5:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Cliques in Random Graphs

Consider the graph $G = (V,E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v, we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

(a) What is the size of the sample space?

(b) A k-clique in graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. Let's call the event that S forms a clique E_S. What is the probability of E_S for a particular set S of k vertices?

(c) Suppose that $V_1 = \{v_1, \dots, v_\ell\}$ and $V_2 = \{w_1, \dots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.

(d) Prove that $\binom{n}{k} \le n^k$. (You might find this useful in part (e))

(e) Prove that the probability that the graph contains a k-clique, for $k \ge 4\log_2 n + 1$, is at most $1/n$.

2 Random Variables Warm-Up

Let X and Y be random variables, each taking values in the set $\{0,1,2\}$, with joint distribution

$$\begin{array}{lll}
\mathbb{P}[X=0,Y=0]=1/3 & \mathbb{P}[X=0,Y=1]=0 & \mathbb{P}[X=0,Y=2]=1/3 \\
\mathbb{P}[X=1,Y=0]=0 & \mathbb{P}[X=1,Y=1]=1/9 & \mathbb{P}[X=1,Y=2]=0 \\
\mathbb{P}[X=2,Y=0]=1/9 & \mathbb{P}[X=2,Y=1]=1/9 & \mathbb{P}[X=2,Y=2]=0.
\end{array}$$

(a) What are the marginal distributions of X and Y?

(b) What are $\mathbb{E}[X]$ and $\mathbb{E}[Y]$?

(c) Let I be the indicator that $X = 1$, and J be the indicator that $Y = 1$. What are $\mathbb{E}[I]$, $\mathbb{E}[J]$ and $\mathbb{E}[IJ]$?

(d) In general, let I_A and I_B be the indicators for events A and B in a probability space $(\Omega, \mathbb{P})$. What is $\mathbb{E}[I_A I_B]$, in terms of the probability of some event?

3 Maybe Lossy Maybe Not

Let us say that Alice would like to send a message to Bob, over some channel. Alice has a message of length 4.

(a) Packets are dropped with probability p. If Alice sends 5 packets, what is probability that Bob can successfully reconstruct Alice's message using polynomial interpolation?

(b) Again, packets can be dropped with probability p. The channel may additionally corrupt 1 packet after deleting packets. Alice realizes this and sends 8 packets for a message of length 4. What is the probability that Bob receives enough packets to successfully reconstruct Alice's message using Berlekamp-Welch?

(c) Again, packets can be dropped with probability p. This time, packets may be corrupted with probability q. A packet being dropped is independent of whether or not is corrupted (i.e. a packet may be both corrupted and dropped). Consider the original scenario where Alice sends 5 packets for a message of length 4. What is probability that Bob can correctly reconstruct Alice's message using polynomial interpolation on all of the points he receives?

4 Class Enrollment

Lydia has just started her CalCentral enrollment appointment. She needs to register for a marine science class and CS 70. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling successfully in the marine science class on each attempt is μ and the probability of enrolling successfully in CS 70 on each attempt is λ. Also, these events are independent.

(a) Suppose Lydia begins by attempting to enroll in the marine science class everyday and gets enrolled in it on day M. What is the distribution of M?

(b) Suppose she is not enrolled in the marine science class after attempting each day for the first 5 days. What is the conditional distribution of M given $M > 5$?

(c) Once she is enrolled in the marine science class, she starts attempting to enroll in CS 70 from day $M + 1$ and gets enrolled in it on day C. Find the expected number of days it takes Lydia to enroll in both the classes, i.e. $\mathbb{E}[C]$.

(d) Suppose instead of attempting one by one, Lydia decides to attempt enrolling in both the classes from day 1. Let M be the number of days it takes to enroll in the marine science class, and C be the number of days it takes to enroll in CS 70. What is the distribution of M and C now? Are they independent?

(e) Let X denote the day she gets enrolled in her first class and let Y denote the day she gets enrolled in both the classes. What is the distribution of X?

(f) What is the expected number of days it takes Lydia to enroll in both classes now, i.e. $\mathbb{E}[Y]$.

(g) What is the expected number of classes she will be enrolled in by the end of 14 days?

5 Boutique Store

Consider a boutique store in a busy shopping mall. Every hour, a large number of people visit the mall, and each independently enters the boutique store with some small probability. The store owner decides to model X, the number of customers that enter her store during a particular hour, as a Poisson random variable with mean λ.

Suppose that whenever a customer enters the boutique store, they leave the shop without buying anything with probability p. Assume that customers act independently, i.e. you can assume that they each flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as Y and the number of them that do not buy anything as Z (so $X = Y + Z$).

(a) What is the probability that $Y = k$ for a given k? How about $\mathbb{P}[Z = k]$? *Hint*: You can use the identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

(b) State the name and parameters of the distribution of Y and Z.

(c) Prove that Y and Z are independent. In particular, prove that for every pair of values y, z, we have $\mathbb{P}[Y = y, Z = z] = \mathbb{P}[Y = y]\mathbb{P}[Z = z]$.

6 Swaps and Cycles

We'll say that a permutation $\pi = (\pi(1), \ldots, \pi(n))$ contains a *swap* if there exist $i, j \in \{1, \ldots, n\}$ so that $\pi(i) = j$ and $\pi(j) = i$m where $i \neq j$.

(a) What is the expected number of swaps in a random permutation?

(b) In the same spirit as above, we'll say that π contains a *s-cycle* if there exist $i_1, \ldots, i_s \in \{1, \ldots, n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_s) = i_1$. Compute the expectation of the number of s-cycles.