CS 70 Discrete Mathematics and Probability Theory
Fall 2021

HW 11

1 Exploring Concepts

The following snippet illustrates a concept about polynomials.

Polynomial?
Interpolation gives one!
Few roots says; No more!

Yet another concept.

Lo! Delta sub i.
Behold! One at x sub i.
Sub j? Such empty.

And finally.

Polynomial in x.
A zero at r.
Cleft by x minus r.
And nothing remains.

1. You should write a poem or snippet to illustrate a concept from this weeks content. We are certain you can do better, but no pressure. It's all good.
2. In terms of the staff using your content for fun and "profit". Do you wish to (1) allow us to share to the class without atttribution (2) allow us to share to the class with attribution or (3) please, please do not share!

This problem will be sampled but only adds to the numerator not the "out of" (or denominator). Also, it will be generously graded and worth 10 points. If we feel this is useful to students (based on the connection to concepts in the course), we may continue it in future problem sets.

2 Will I Get My Package?

A delivery guy in some company is out delivering n packages to n customers, where $n \in \mathbb{N}$, $n > 1$. Not only does he hand each customer a package uniformly at random from the remaining packages, he opens the package before delivering it with probability $1/2$. Let X be the number of customers who receive their own packages unopened.

(a) Compute the expectation $\mathbb{E}(X)$.

(b) Compute the variance $\mathrm{Var}(X)$.

Solution:

(a) Define

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th customer gets his/her package unopened,} \\ 0, & \text{otherwise.} \end{cases}$$

By linearity of expectation, $\mathbb{E}(X) = \mathbb{E}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathbb{E}(X_i)$. We have

$$\mathbb{E}(X_i) = \mathbb{P}[X_i = 1] = \frac{1}{2n},$$

since the ith customer will get his/her own package with probability $1/n$ and it will be unopened with probability $1/2$ and the delivery guy opens the packages independently.
Hence,

$$\mathbb{E}(X) = n \cdot \frac{1}{2n} = \boxed{\frac{1}{2}}.$$

(b) To calculate $\mathrm{Var}(X)$, we need to know $\mathbb{E}(X^2)$.
By linearity of expectation:

$$\mathbb{E}(X^2) = \mathbb{E}\big((X_1 + X_2 + \cdots + X_n)^2\big) = \mathbb{E}\Big(\sum_{i,j} X_i X_j\Big) = \sum_{i,j} \mathbb{E}(X_i X_j).$$

Then we consider two cases, either $i = j$ or $i \neq j$.
Hence $\sum_{i,j} \mathbb{E}(X_i X_j) = \sum_i \mathbb{E}(X_i^2) + \sum_{i \neq j} \mathbb{E}(X_i X_j)$.

$$\mathbb{E}(X_i^2) = \mathbb{E}(X_i) = \frac{1}{2n}$$

for all i. To find $\mathbb{E}(X_i X_j)$, we need to calculate $\mathbb{P}[X_i X_j = 1]$.

$$\mathbb{P}[X_i X_j = 1] = \mathbb{P}[X_i = 1]\mathbb{P}[X_j = 1 \mid X_i = 1] = \frac{1}{2n} \cdot \frac{1}{2(n-1)}$$

since if customer i has received his/her own package, customer j has $n-1$ choices left.
Hence,

$$\mathbb{E}(X^2) = n \cdot \frac{1}{2n} + n \cdot (n-1) \cdot \frac{1}{2n} \cdot \frac{1}{2(n-1)} = \frac{3}{4},$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{3}{4} - \frac{1}{4} = \boxed{\frac{1}{2}}.$$

3 Diversify Your Hand

You are dealt 5 cards from a standard 52 card deck. Let X be the number of distinct values in your hand. For instance, the hand (A, A, A, 2, 3) has 3 distinct values.

(a) Calculate $E[X]$.

(b) Calculate $Var[X]$.

Solution:

(a) Let X_i be the indicator of the ith value appearing in your hand. Then, $X = X_1 + X_2 + ... + X_{13}$ (Let 13 correspond to K, 12 correspond to Q, 11 correspond to J). By linearity of expectation then, $E[X] = \sum_{i=1}^{13} E[X_i]$. We can calculate $\mathbb{P}[X_i = 1]$ by taking the complement, $1 - Pr[X_i = 0]$, or 1 minus the probability that the card does not appear in your hand. This is $1 - \frac{\binom{48}{5}}{\binom{52}{5}}$. Then, $E[X] = 13\mathbb{P}[X_1 = 1] = 13(1 - \frac{\binom{48}{5}}{\binom{52}{5}})$.

(b) To calculate variance, since the indicators are not independent, we have to use the formula $E[X^2] = \sum_{i=j} E[X_i^2] + \sum_{i \neq j} E[X_i X_j]$.

$\sum_{i=j} E[X_i^2] = \sum_{i=j} E[X_i] = 13(1 - \frac{\binom{48}{5}}{\binom{52}{5}})$

To calculate $\mathbb{P}[X_i X_j = 1]$, we note that $\mathbb{P}[X_i X_j = 1] = 1 - \mathbb{P}[X_i = 0] - \mathbb{P}[X_j = 0] + \mathbb{P}[X_i = 0, X_j = 0]$.

$\sum_{i \neq j} E[X_i X_j] = 13 \cdot 12\mathbb{P}[X_i X_j = 1] = 13 \cdot 12(1 - \mathbb{P}[X_i = 0] - \mathbb{P}[X_j = 0] + \mathbb{P}[X_i = 0, X_j = 0])$
$= 156(1 - 2\frac{\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{44}{5}}{\binom{52}{5}})$

Putting it all together, we have $Var[X] = E[X^2] - E[X]^2 = 13(1 - \frac{\binom{48}{5}}{\binom{52}{5}}) + 156(1 - 2\frac{\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{44}{5}}{\binom{52}{5}}) - (13(1 - \frac{\binom{48}{5}}{\binom{52}{5}}))^2$

4 Fishy Computations

Assume for each part that the random variable can be modelled by a Poisson distribution.

(a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?

(b) Suppose that on average, you go to Fisherman's Wharf twice a year. What is the probability that you will go at most once in 2018?

(c) Suppose that in March, on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?

Solution:

(a) Let X be the number of salmon the fisherman catches per week. $X \sim \text{Poiss}(20 \text{ salmon/week})$, so

$$\mathbb{P}[X = 7 \text{ salmon/week}] = \frac{20^7}{7!}e^{-20} \approx 5.23 \cdot 10^{-4}.$$

(b) Similarly $X \sim \text{Poiss}(2)$, so

$$\mathbb{P}[X \leq 1] = \frac{2^0}{0!}e^{-2} + \frac{2^1}{1!}e^{-2} \approx 0.41.$$

(c) Let X_1 be the number of sailing boats on the next day, and X_2 be the number of sailing boats on the day after next. Now, we can model sailing boats on day i as a Poisson distribution $X_i \sim \text{Poiss}(\lambda = 5.7)$. Let Y be the number of boats that sail in the next two days. We are interested in $Y = X_1 + X_2$. We know that the sum of two independent Poisson random variables is Poisson (from Theorem 19.5 in lecture notes). Thus, we have $Y \sim \text{Poiss}(\lambda = 5.7 + 5.7 = 11.4)$.

$$\begin{aligned}
\mathbb{P}[Y \geq 3] &= 1 - \mathbb{P}[Y < 3] \\
&= 1 - \mathbb{P}[Y = 0 \cup Y = 1 \cup Y = 2] \\
&= 1 - (\mathbb{P}[Y = 0] + \mathbb{P}[Y = 1] + \mathbb{P}[Y = 2]) \\
&= 1 - \left(\frac{11.4^0}{0!}e^{-11.4} + \frac{11.4^1}{1!}e^{-11.4} + \frac{11.4^2}{2!}e^{-11.4}\right) \\
&\approx 0.999.
\end{aligned}$$

5 Unreliable Servers

A Google competitor owns a warehouse consisting of a large number of servers (a server farm). On any given day, each server in the farm is equally likely to go down or to stay online, independently of all other servers, and independently of what happens on any number of other days. On average, 4 servers go down in the cluster per day.

(a) What is an appropriate distribution to model the number of servers that crash on any given day for a certain cluster? What is its parameter?

(b) Compute the expected value and variance of the number of crashed servers on a given day for a certain cluster.

(c) Compute the probability that fewer than 3 servers crashed on a given day for a certain cluster.

(d) Compute the probability at least 3 servers crashed on a given day for a certain cluster.

Solution:

(a) Because each server goes down independently of the other servers, and with the same probability, the number of servers that crash on a given day follows a binomial distribution Binom(n, p), where n is the number of servers and p is the probability of each individual server crashing on any given day. Since on average, 4 servers crash per day, we have $p = \frac{4}{n}$. We are given that the number of servers in the cluster is large, so $n \gg p$ and we can model the number of servers that crash as a Poisson distribution with $\lambda = 4$.

(b) Recall that the expectation and variance of a Poisson distribution with parameter λ are both equal to λ and in this case $\lambda = 4$.

(c) To compute the probability that fewer than 3 servers went down, we must add the probabilities that 0 servers go down, 1 server goes down, and the probability that 2 servers go down. The PMF of the Poisson distribution is

$$\mathbb{P}[X = i] = \frac{\lambda^i}{i!} e^{-\lambda}.$$

Thus

$$\mathbb{P}[X = 0 \text{ or } X = 1 \text{ or } X = 2] = e^{-4} + 4e^{-4} + \frac{4^2}{2} e^{-4} = e^{-4} + 4e^{-4} + 8e^{-4} = 13e^{-4}.$$

(d) $1 - \mathbb{P}[\text{fewer than 3 servers crashed}] = 1 - 13e^{-4}$.

6 Subset Card Game

Jonathan and Yiming are playing a card game. Jonathan has $k > 2$ cards, and each card has a real number written on it. Jonathan tells Yiming (truthfully), that the sum of the card values is 0, and that the sum of squares of the values on the cards is 1. Specifically, if the card values are $c_1, c_2, \ldots, c_k$, then we have $\sum_{i=1}^{k} c_i = 0$ and $\sum_{i=1}^{k} c_i^2 = 1$.

The cards are then going to be dealt randomly in the following fashion: for each card in the deck, a fair coin is flipped. If the coin lands heads, then the card goes to Yiming, and if the coin lands tails, the card goes to Jonathan. Note that it is possible for either player to end up with no cards/all the cards.

Calculate $\mathbb{E}[S]$ and $\text{Var}(S)$, where S is the sum of value of cards in Yiming's hand. The answer should not include a summation.

Solution: Let I_i be the indicator random variable indicating whether or not card i goes to Yiming. We have $S = \sum_{i=1}^{k} c_i I_i$ as the value of Yiming's hand. Then, we see that $\mathbb{E}[S] = \sum_{i=1}^{k} c_i \cdot \frac{1}{2} = 0$ and

$$\begin{aligned} \text{Var}(S) &= \sum_{i=1}^{k} \text{Var}(c_i I_i) \qquad \text{(due to independence) of } I_i \\ &= \sum_{i=1}^{k} c_i^2 \, \text{Var}(I_i) \end{aligned}$$

We know that I_i is a Bernoulli random variable, so its variance is $\frac{1}{4}$. Thus, we see that $\text{Var}(S) = \frac{1}{4}$.

7 Optimal Gambling

Jonathan has a coin that may be biased, but he doesn't think so. You disagree with him though, and he challenges you to a bet. You start off with X_0 dollars. You and Jonathan then play multiple rounds, and each round, you bet an amount of money of your choosing, and then coin is tossed. Jonathan will match your bet, no matter what, and if the coin comes up heads, you win and you take both yours and Jonathan's bet, and if it comes up tails, then you lose your bet.

(a) Now suppose you actually secretly know that the bias of the coin is $\frac{1}{2} < p < 1$! You use the following strategy: on each round, you will bet a fraction q of the money you have at the start of the round. Let's say you play n rounds. What is the probability that you win exactly k of the rounds? What is the amount of money you would have if you win exactly k rounds? [*Hint*: Does the order in which you win the games affect your profit?]

(b) Let X_n denote the amount of money you have on round n. X_0 represents your initial assets and is a constant value. Show that $\mathbb{E}[X_n] = ((1-p)(1-q)+p(1+q))^n X_0$.

You may use the binomial theorem in your answer:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{(n-k)}$$

[*Hint*: Try computing a sum over the number of rounds you win out of the n rounds you play - use your answers from the previous part.]

(c) What value of q will maximize $\mathbb{E}[X_n]$? For this value of q, what is the distribution of X_n? Can you predict what will happen as $n \to \infty$? [*Hint*: Under this betting strategy, what happens if you ever lose a round?]

(d) The problem with the previous approach is that we were too concerned about expected value, so our gambling strategy was too extreme. Let's start over: again we will use a gambling strategy in which we bet a fraction q of our money at each round. Express X_n in terms of n, q, X_0, and W_n, where W_n is the number of rounds you have won up until round n. [*Hint*: Does the order in which you win the games affect your profit?]

(e) By the law of large numbers, what does W_n/n converge to as $n \to \infty$? Using this fact, what does $(\ln X_n)/n$ converge to as $n \to \infty$?

(f) The rationale behind $(\ln X_n)/n$ is that if $(\ln X_n)/n \to c$, where c is a constant, then that means for large n, X_n is roughly e^{cn}. Therefore, c is the asymptotic growth rate of your fortune! Find the value of q that maximizes your asymptotic growth rate. (*Hint*: Use calculus!)

(g) Using the value of q you found in the previous part, compute $\mathbb{E}[X_n]$.

(h) Say Jonathan wishes to estimate the bias of the coin - he may want to use the value W_n/n as his estimate. What is the expectation of this value? What is a bound on the variance of this value? (Your bound should not include p.)

Solution:

(a) The number of rounds we win out of n rounds played is described by a binomial distribution with n trials and probability of success p. So the probability we win exactly k rounds is $\binom{n}{k}p^k(1-p)^{n-k}$.

You win k times and each time you win, your fortune is multiplied by $1+q$; you lose $n-k$ times, and each time you lose, your fortune is multiplied by $1-q$. Therefore, the amount of money you will have is $X_0(1+q)^k(1-q)^{n-k}$.

(b) As seen in the previous part, X_n can take on n different values, described by $X_0(1+q)^k(1-q)^{n-k}$ for $1 \le k \le n$. We also know the probability that X_n takes each of those values, and so can compute expectation directly from the definition:

$$\begin{aligned}\mathbb{E}[X_n] &= \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\cdot X_0(1+q)^k(1-q)^{n-k}\\ &= X_0\sum_{k=0}^{n}\binom{n}{k}(p(1+q))^k((1-p)(1-q))^{n-k}\\ &= ((1-p)(1-q)+p(1+q))^n X_0\end{aligned}$$

(c) We want $(1-p)(1-q)+p(1+q)$ to be as large as possible. Note that this is linear in q, and the coefficient for q is $p-(1-p) > 0$. Hence, we should take q to be as large as possible, which is 1 (you cannot bet more money than you actually have).

For this value of q, note that on each round you either double your money or go broke. Hence, the distribution is:

$$X_n = \begin{cases} 2^n X_0, & \text{with probability } p^n \\ 0, & \text{with probability } 1-p^n \end{cases}$$

Uh-oh. As $n \to \infty$, the probability that you are broke approaches 1. The issue here is that your expected fortune grows large, but the probability that you are rich grows vanishingly small. In general, $X_n \to 0$ as $n \to \infty$ does not necessarily imply that $\mathbb{E}[X_n] \to 0$, which is what we see here.

(d) As in part (a): you win W_n times and each time you win, your fortune is multiplied by $1+q$; you lose $n-W_n$ times, and each time you lose, your fortune is multiplied by $1-q$. Therefore,

$$X_n = X_0(1-q)^{n-W_n}(1+q)^{W_n}.$$

(e) W_n/n converges to $\mathbb{E}[W_n/n] = np/n = p$. One has

$$\begin{aligned}\frac{\ln X_n}{n} &= \frac{\ln X_0(1-q)^{n-W_n}(1+q)^{W_n}}{n}\\ &= \frac{1}{n}\cdot(\ln X_0 + (n-W_n)\ln(1-q) + W_n\ln(1+q)\\ &= \frac{\ln X_0}{n} + \left(1-\frac{W_n}{n}\right)\ln(1-q) + \frac{W_n}{n}\ln(1+q)\\ &\xrightarrow[n\to\infty]{} (1-p)\ln(1-q) + p\ln(1+q).\end{aligned}$$

(f) We can use calculus to optimize c:

$$\frac{\mathrm{d}}{\mathrm{d}q}((1-p)\ln(1-q)+p\ln(1+q)) = -\frac{1-p}{1-q}+\frac{p}{1+q}.$$

Set the derivative to 0:

$$\frac{p}{1+q} = \frac{1-p}{1-q} \implies p-pq = 1+q-p-pq \implies q = 2p-1.$$

This means $q = 2p-1$ is a critical point.

Taking the second derivative, we have that:

$$\frac{\mathrm{d}}{\mathrm{d}q}((1-p)\ln(1-q)+p\ln(1+q)) = -\frac{1-p}{(1-q)^2}-\frac{p}{(1+q)^2} < 0 \quad (\text{for } 0 \le p \le 1)$$

This implies that the expression we are trying to maximize is strictly concave with respect to q as long as $0 \le p \le 1$ (which we know to be true). Thus it has a unique maximum at a critical point, making $q = 2p-1$ the maximizing value.

This is known as the Kelly betting criterion. Notice the farther from $\frac{1}{2}$ that p is, the more that you bet. If $p = 1$, you would all-in every time. The CS 70 course staff is not responsible for any losses you incur with this betting strategy, but we do think it's pretty cool that you can analyze optimal gambling with the methods in this course.

(g) We can plug in $q = 2p-1$ from our previous result:

$$\mathbb{E}[X_n] = ((1-p)(1-q)+p(1+q))^n X_0 = 2^n(p^2+(1-p)^2)^n X_0.$$

(h) Expectation:

$$\mathbb{E}\left[\frac{W_n}{n}\right] = \frac{\mathbb{E}[W_n]}{n} = \frac{np}{n} = p$$

Since the expectation is equal to the value we are estimating, we call $\frac{W_n}{n}$ an unbiased estimator.

We see that

$$\mathrm{Var}\,(W_n/n) = \frac{1}{n^2}\mathrm{Var}\,(W_n) = \frac{p(1-p)}{n} \le \frac{1}{4n}$$

Here we make use of the fact $p(1-p)$ is at most $\frac{1}{4}$ (attained when $p = \frac{1}{2}$).