CS 70 Discrete Mathematics and Probability Theory
Fall 2021 HW 13

Due: Saturday 11/27, 4:00 PM
Grace period until Tuesday 11/30, 11:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Just One Tail, Please

Let X be some random variable with finite mean and variance which is not necessarily non-negative. The *extended* version of Markov's Inequality states that for a non-negative function $\phi(x)$ which is monotonically increasing for $x > 0$ and some constant $\alpha > 0$,

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[\phi(X)]}{\phi(\alpha)}$$

Suppose $\mathbb{E}[X] = 0$, $\mathrm{Var}(X) = \sigma^2 < \infty$, and $\alpha > 0$.

(a) Use the extended version of Markov's Inequality stated above with $\phi(x) = (x+c)^2$, where c is some positive constant, to show that:

$$\mathbb{P}(X \geq \alpha) \leq \frac{\sigma^2 + c^2}{(\alpha + c)^2}$$

(b) Note that the above bound applies for all positive c, so we can choose a value of c to minimize the expression, yielding the best possible bound. Find the value for c which will minimize the RHS expression (you may assume that the expression has a unique minimum).

We can plug in the minimizing value of c you found in part (b) to prove the following bound:

$$\mathbb{P}(X \geq \alpha) \leq \frac{\sigma^2}{\alpha^2 + \sigma^2}.$$

This bound is also known as Cantelli's inequality.

(c) Recall that Chebyshev's inequality provides a two-sided bound. That is, it provides a bound on $\mathbb{P}(|X-\mathbb{E}[X]| \geq \alpha) = \mathbb{P}(X \geq \mathbb{E}[X]+\alpha) + \mathbb{P}(X \leq \mathbb{E}[X]-\alpha)$. If we only wanted to bound the probability of one of the tails, e.g. if we wanted to bound $\mathbb{P}(X \geq \mathbb{E}[X]+\alpha)$, it is tempting to just divide the bound we get from Chebyshev's by two.

  (i) Why is this not always correct in general?

  (ii) Provide an example of a random variable X (does not have to be zero-mean) and a constant α such that using this method (dividing by two to bound one tail) is not correct, that is, $\mathbb{P}(X \geq \mathbb{E}[X]+\alpha) > \frac{\mathrm{Var}(X)}{2\alpha^2}$ or $\mathbb{P}(X \leq \mathbb{E}[X]-\alpha) > \frac{\mathrm{Var}(X)}{2\alpha^2}$.

Now we see the use of the bound proven in part (b) - it allows us to bound just one tail while still taking variance into account, and does not require us to assume any property of the random variable. Note that the bound is also always guaranteed to be less than 1 (and therefore at least somewhat useful), unlike Markov's and Chebyshev's inequality!

(d) Let's try out our new bound on a simple example. Suppose X is a positively-valued random variable with $\mathbb{E}[X] = 3$ and $\mathrm{Var}(X) = 2$.

  (i) What bound would Markov's inequality give for $\mathbb{P}[X \geq 5]$?

  (ii) What bound would Chebyshev's inequality give for $\mathbb{P}[X \geq 5]$?

  (iii) What bound would Cantelli's Inequality give for $\mathbb{P}[X \geq 5]$? (*Note*: Recall that Cantelli's Inequality only applies for zero-mean random variables.)

2 Tightness of Inequalities

(a) Show by example that Markov's inequality is tight; that is, show that given some fixed $k > 0$, there exists a discrete non-negative random variable X such that $\mathbb{P}(X \geq k) = \mathbb{E}[X]/k$.

(b) Show by example that Chebyshev's inequality is tight; that is, show that given some fixed $k \geq 1$, there exists a random variable X such that $\mathbb{P}(|X-\mathbb{E}[X]| \geq k\sigma) = 1/k^2$, where $\sigma^2 = \mathrm{Var}(X)$.

3 Probabilistically Buying Probability Books

Chuck will go shopping for probability books for K hours. Here, K is a random variable and is equally likely to be 1, 2, or 3. The number of books N that he buys is random and depends on how long he shops. We are told that

$$\mathbb{P}[N=n|K=k] = \begin{cases} \frac{c}{k} & \text{for } n=1,\ldots,k \\ 0 & \text{otherwise} \end{cases}$$

for some constant c.

(a) Compute c.

(b) Find the joint distribution of K and N.

(c) Find the marginal distribution of N.

(d) Find the conditional distribution of K given that $N = 1$.

(e) We are now told that he bought at least 1 but no more than 2 books. Find the conditional mean and variance of K, given this piece of information.

(f) The cost of each book is a random variable with mean 3. What is the expectation of his total expenditure? *Hint:* Condition on events $N = 1, \dots, N = 3$ and use the total expectation theorem.

4 Law of Large Numbers

Recall that the *Law of Large Numbers* holds if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \frac{1}{n} S_n - \mathbb{E}\left[\frac{1}{n} S_n\right] \right| > \varepsilon \right) = 0.$$

In class, we saw that the Law of Large Numbers holds for $S_n = X_1 + \cdots + X_n$, where the X_i's are i.i.d. random variables. This problem explores if the Law of Large Numbers holds under other circumstances.

Packets are sent from a source to a destination node over the Internet. Each packet is sent on a certain route, and the routes are disjoint. Each route has a failure probability of $p \in (0,1)$ and different routes fail independently. If a route fails, all packets sent along that route are lost. You can assume that the routing protocol has no knowledge of which route fails.

For each of the following routing protocols, determine whether the Law of Large Numbers holds when S_n is defined as the total number of received packets out of n packets sent. Answer **Yes** if the Law of Large Number holds, or **No** if not. Give a justification of your answer. (Whenever convenient, you can assume that n is even.)

(a) **Yes** or **No**: Each packet is sent on a completely different route.

(b) **Yes** or **No**: The packets are split into $n/2$ pairs of packets. Each pair is sent together on its own route (i.e., different pairs are sent on different routes).

(c) **Yes** or **No**: The packets are split into 2 groups of $n/2$ packets. All the packets in each group are sent on the same route, and the two groups are sent on different routes.

(d) **Yes** or **No**: All the packets are sent on one route.