Homework 2

CS 70, Summer 2024

Due by Friday, July 5th at 11:59 PM

Instructions. Start each problem on a separate page. The subparts of each problem can be on the same page. Every answer should contain a calculation or reasoning. Your answers should be clear, organized, and legible—your final submission should not include scratch work or failed attempts. You must always commit to a final answer; if multiple answers are provided, the most incorrect one will be graded.

If you are completing the homework using LaTeX, you may use the templates. Homeworks must be submitted through Gradescope. See the end of the homework for submission instructions.

Sundry. Before you start writing your final homework submission, state briefly how you worked on it (e.g., if you went to office hours, how frequently you worked on it, etc.). If you worked on the assignment in a group with other students, list their names and email addresses.

1 Stable Matching Proofs

Prove or disprove each of the following statements about stable matchings and the propose-and-reject algorithm.

(a) Suppose that the propose-and-reject algorithm and the flipped propose-and-reject algorithm (where instead candidates make offers to jobs) both output the same stable matching. Then there are no other stable matchings.

(b) Suppose that in the propose-and-reject algorithm, job J never makes an offer to candidate C. Then there is no stable matching in which J is matched with C.

(c) It is possible for two candidates to both have the same optimal job.

2 Best-Case and Worst-Case Scenarios

Consider an instance of the stable matching problem with at least two jobs and at least two candidates. Prove or disprove each of the following statements.

(a) There is a set of preferences such that the propose-and-reject algorithm matches each job with its least preferred candidate.

(b) There is a set of preferences such that the propose-and-reject algorithm matches each candidate with their least preferred job.

(c) If job J is candidate C's most preferred job and candidate C is job J's most preferred candidate, then they must be matched in every stable matching.

(d) If job J is candidate C's least preferred job and candidate C is job J's least preferred candidate, then they cannot be matched in any stable matching.

3 Counting Cartesian Products

In this question, we will consider the cardinalities of the Cartesian products of countable sets.

(a) Prove that if A and B are both countable, then $A \times B$ is also countable.

(b) Let $A_1, \dots, A_n$ be a finite number of countable sets. Prove that

$$\bigtimes_{i=1}^{n} A_i = A_1 \times A_2 \times \ldots \times A_n$$

is also countable.

(c) Let $B_1, B_2, \ldots$ be a countably infinite number of finite sets for which each set B_i has at least two elements. Prove that

$$\bigtimes_{i=1}^{\infty} B_i = B_1 \times B_2 \times \ldots$$

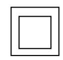
is uncountable.

4 Counting Shapes

For a two-dimensional shape $S \subseteq \mathbb{R}^2$ we call a set $\mathcal{C}$ of scaled and shifted copies of S such that no two elements in $\mathcal{C}$ intersect a *collection* of S. That is, each pair of copies of S in $\mathcal{C}$ must be disjoint for it to be a collection of S.

For example, if $S = \blacksquare$ is the filled square, the following configurations of two copies of S (drawn in different shades so that both can be seen) cannot exist in a collection of S.

However, if $S = \square$ is the empty square, then the following configuration of two copies of S is allowed in a collection of S.

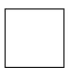


For each of the following shapes S, either prove that there exists a collection of S which is uncountable or prove that every possible collection of S is countable.

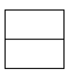
(a) S is the filled square.

$S =$ 

(b) S is the empty square.

$S =$ 

(c) S is the halved empty square.

$S =$ 

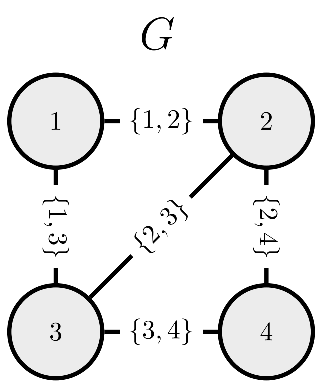
5 Edge Complement

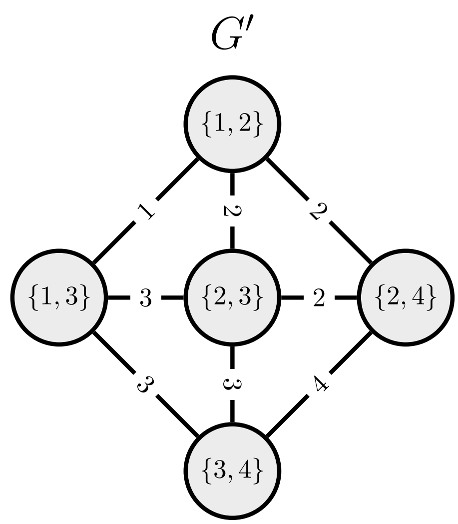
The *edge complement graph* of a graph $G = (V, E)$ is another graph $G' = (V', E')$ constructed as follows.

(1) $V' = E$. That is, the vertices of G' are the edges of G.

(2) For $i, j \in V'$, $(i, j) \in E' \iff i \cap j \neq \varnothing$. That is, two vertices in G' have an edge if and only if they share a common vertex in G.

For example, below are a graph G and its edge complement graph G'. For every edge $\{i, j\}$ in G, there is a vertex in G'. If two edges $\{i, j\}$ and $\{j, k\}$ share a vertex j in G, then the corresponding vertices have an edge connecting them in G'.





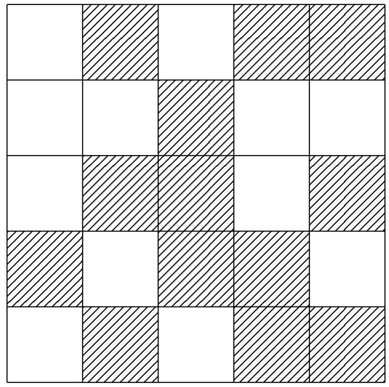
Prove or disprove each of the following statements.

(a) If a graph has an Eulerian tour, then its edge complement graph also has an Eulerian tour.

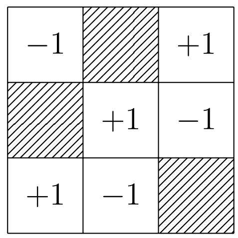
(b) If a graph's edge complement graph has an Eulerian tour, then the original graph also has an Eulerian tour.

6 Filling the Grid

Aliyah is trying to fill in the blanks of the following 5×5 grid with $+1$s and -1s such that every row sums to zero and every column sums to zero. She tried to find a solution through trial and error, but she wasn't able to find a valid solution.



As an example, Aliyah shows you a solution she has already worked out for the following 3×3 grid.



(a) Prove that if there are an odd number of blanks in a row or column of such an $n \times n$ grid, then there is no valid way of filling the grid.

Once you're done, take a moment to consider which criteria this condition reminds you of. You don't need to include these considerations in your submission.

(b) Aliyah is trying to represent the 5×5 grid as a graph to help her find a solution. She constructs an undirected graph $G = (V, E)$ with ten vertices $V = \{r_1, \ldots, r_5, c_1, \ldots, c_5\}$. The five vertices $r_1, \ldots, r_5$ represent the rows and the five vertices $c_1, \ldots, c_5$ represent the columns.

For each $i, j \in \{1, \ldots, 5\}$, Aliyah includes the edge $\{r_i, c_j\}$ in E if and only if the block of the 5×5 grid in the i^{th} row from the top and j^{th} column from the left is blank.

Draw the graph G that Aliyah constructed.

(c) Find an Eulerian tour in the graph G.

(d) When traversing the Eulerian tour, we enter each edge from one of its vertices and leave that edge from a different vertex. This gives a direction to the undirected edges.

Create a new graph G' from G with directed edges using the edge traversals in your Eulerian tour from part **(c)**.

(e) For each $v \in V$, let $\deg^-(v)$ be the in-degree of v in G' and let $\deg^+(v)$ be the out-degree of v in G'. For each $i \in \{1, \ldots, 5\}$, compute

$$\deg^-(r_i) - \deg^+(r_i) \qquad \text{and} \qquad \deg^-(c_i) - \deg^+(c_i).$$

(f) Determine a method for using your Eulerian tour to fill in the grid. Use your method to fill in Aliyah's 5×5 grid and then prove that your method works.

7 Planarity and Graph Complements

Let $G = (V, E)$ be an undirected graph. The complement of G is a graph $\bar{G} = (V, \bar{E})$, where $\bar{E} = \{(i, j) : i, j \in V, i \neq j\} \setminus E$. That is, $\bar{G}$ has the same set of vertices as G, but $\bar{G}$ has an edge between two vertices if and only if those two vertices *do not* have an edge in G.

(a) Suppose G has v vertices and e edges. Determine the number of edges $\bar{e}$ in $\bar{G}$.

(b) Prove that for any graph with at least 13 vertices, if G is planar, then $\bar{G}$ is non-planar.

(c) Now consider the converse of the claim in part **(b)**. Construct a counterexample to show that the converse does not hold.

(*Hint*: Recall that if a graph contains a copy of K_5, then it is non-planar. See if you can use this fact to construct a counterexample.)

8 Hypercubes

In this problem, we will prove some results about the hypercube graphs.

(a) Draw the one-, two-, and three-dimensional hypercubes and label the vertices using the corresponding bit strings.

(b) Prove that the edges of an n-dimensional hypercube can be colored using n colors such that no pair of edges sharing a common vertex have the same color.

(c) Prove that the n-dimensional hypercube is bipartite.

Submission. Homeworks must be submitted through Gradescope. If you are completing your homeworks on paper, please scan the pages of your homework into a PDF using any scanner or phone application such as CamScanner. **It is your responsibility to ensure that all the work on the scanned pages is legible.**

Once you upload your submission to the Gradescope assignment, you will be prompted to select pages. **It is your responsibility to correctly select the pages of your homework corresponding to each question.** If you are having difficulties scanning, uploading, or submitting your homework, post a follow-up on the main thread corresponding to this homework on Ed.