Today.

Polynomials.

Secret Sharing.

Correcting for loss or even corruption.

Secret Sharing.

Share secret among n people.

Secrecy: Any $k-1$ knows nothing.
Roubustness: Any k knows secret.
Efficient: minimize storage.

The idea of the day.

Two points make a line.
Lots of lines go through one point.

Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1}x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

$P(x)$ **contains** point (a,b) if $b = P(a)$.

Polynomials over reals: $a_1, \ldots, a_d \in \Re$, use $x \in \Re$.

Polynomials $P(x)$ with arithmetic modulo p: [1] $a_i \in \{0, \ldots, p-1\}$ and

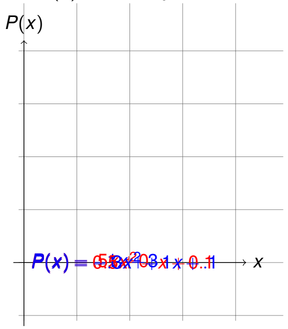
$$P(x) = a_d x^d + a_{d-1}x^{d-1} \cdots + a_0 \pmod p,$$

for $x \in \{0, \ldots, p-1\}$.

[1] A field is a set of elements with addition and multiplication operations, with inverses. $GF(p) = (\{0, \ldots, p-1\}, + \pmod p, * \pmod p)$.

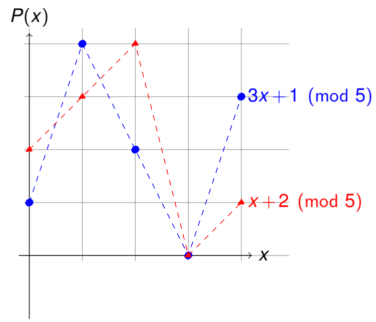
Polynomial: $P(x) = a_d x^4 + \cdots + a_0$

Line:$P(x) = a_1 x + a_0 = mx + b$



Parabola: $P(x) = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c$

Polynomial: $P(x) = a_d x^4 + \cdots + a_0 \pmod p$



Finding an intersection.
$x+2 \equiv 3x+1 \pmod 5$
$\implies 2x \equiv 1 \pmod 5 \implies x \equiv 3 \pmod 5$
3 is multiplicative inverse of 2 modulo 5.
Good when modulus is prime!!

Two points make a line.

Fact: Exactly 1 degree $\leq d$ polynomial contains $d+1$ points. [2]

Two points specify a line. Three points specify a parabola.

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

[2] Points with different x values.

Poll.

Two points determine a line.
What facts below tell you this?

Say points are $(x_1, y_1), (x_2, y_2)$**.**

(A) Line is $y = mx + b$.
(B) Plug in a point gives an equation: $y_1 = mx_1 + b$
(C) The unknowns are m and b.
(D) If equations have unique solution, done.

All true.

Flow Poll.

Why solution? Why unique?

(A) Solution cuz: $m = (y_2 - y_1)/(x_2 - x_1), b = y_1 - m(x_1)$
(B) Unique cuz, only one line goes through two points.
(C) Try: $(m'x + b') - (mx + b) = (m' - m)x + (b - b') = ax + c \neq 0$.
(D) Either $ax_1 + c \neq 0$ or $ax_2 + c \neq 0$.
(E) Contradiction.

Flow poll. (All true. (B) is not a proof, it is restatement.)

Notation: two points on a line.

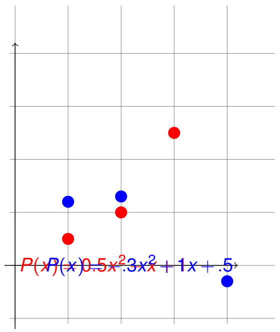
Polynomial: $a_n x^n + \cdots + a_0$.

Consider line: $mx + b$

(A) $a_1 = m$
(B) $a_1 = b$
(C) $a_0 = m$
(D) $a_0 = b$.

(A) and (D)

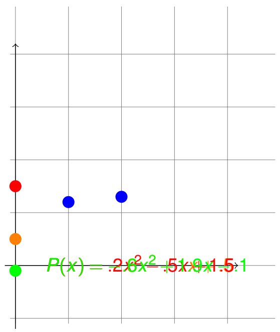
3 points determine a parabola.



Fact: Exactly 1 degree $\leq d$ polynomial contains $d+1$ points. [3]

[3] Points with different x values.

2 points not enough.



There is $P(x)$ contains blue points and *any* $(0, y)$!

Modular Arithmetic Fact and Secrets

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

Shamir's k out of n Scheme:
Secret $s \in \{0, \ldots, p-1\}$

1. Choose $a_0 = s$, and random $a_1, \ldots, a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Roubustness: Any k shares gives secret.
Knowing k pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
Secrecy: Any $k-1$ shares give nothing.
Knowing $\leq k-1$ pts $\implies$ any $P(0)$ is possible.

Poll:example.

The polynomial from the scheme: $P(x) = 2x^2 + 1x + 3 \pmod 5$.
What is true for the secret sharing scheme using $P(x)$?

(A) The secret is "2".
(B) The secret is "3".
(C) A share could be $(1,5)$ cuz $P(1) = 5$
(D) A share could be $(2,4)$
(E) A share could be $(0,3)$

From $d+1$ points to degree d polynomial?

For a line, $a_1x + a_0 = mx + b$ contains points $(1,3)$ and $(2,4)$.

$$\begin{aligned} P(1) = m(1)+b &\equiv m+b \equiv 3 \pmod 5 \\ P(2) = m(2)+b &\equiv 2m+b \equiv 4 \pmod 5 \end{aligned}$$

Subtract first from second..

$$\begin{aligned} m+b &\equiv 3 \pmod 5 \\ m &\equiv 1 \pmod 5 \end{aligned}$$

Backsolve: $b \equiv 2 \pmod 5$. Secret is 2.

And the line is...

$$x + 2 \mod 5.$$

Quadratic

For a quadratic polynomial, $a_2x^2 + a_1x + a_0$ hits $(1,2);(2,4);(3,0)$.
Plug in points to find equations.

$$\begin{aligned} P(1) = a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ P(2) = 4a_2 + 2a_1 + a_0 &\equiv 4 \pmod 5 \\ P(3) = 4a_2 + 3a_1 + a_0 &\equiv 0 \pmod 5 \end{aligned}$$

$$\begin{aligned} a_2 + a_1 + a_0 &\equiv 2 \pmod 5 \\ 3a_1 + 2a_0 &\equiv 1 \pmod 5 \\ 4a_1 + 2a_0 &\equiv 2 \pmod 5 \end{aligned}$$

Subtracting 2nd from 3rd yields: $a_1 = 1$.
$a_0 = (2 - 4(a_1))2^{-1} = (-2)(2^{-1}) = (3)(3) = 9 \equiv 4 \pmod 5$
$a_2 = 2 - 1 - 4 \equiv 2 \pmod 5$.

So polynomial is $2x^2 + 1x + 4 \pmod 5$

In general..

Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

Solve...

$$\begin{aligned} a_{k-1}x_1^{k-1} + \cdots + a_0 &\equiv y_1 \pmod p \\ a_{k-1}x_2^{k-1} + \cdots + a_0 &\equiv y_2 \pmod p \\ &\cdot \\ &\cdot \\ a_{k-1}x_k^{k-1} + \cdots + a_0 &\equiv y_k \pmod p \end{aligned}$$

Will this always work?

As long as solution **exists** and it is **unique!** And...

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

Another Construction: Interpolation!

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,2);(2,4);(3,0)$.

Find $\Delta_1(x)$ polynomial contains $(1,1);(2,0);(3,0)$.

Try $(x-2)(x-3) \pmod 5$.

Value is 0 at 2 and 3. Value is 2 at 1. Not 1! Doh!!
So "Divide by 2" or multiply by 3.
$\Delta_1(x) = (x-2)(x-3)(3) \pmod 5$ contains $(1,1);(2,0);(3,0)$.

$\Delta_2(x) = (x-1)(x-3)(4) \pmod 5$ contains (1,0);(2,1);(3,0).

$\Delta_3(x) = (x-1)(x-2)(3) \pmod 5$ contains (1,0);(2,0);(3,1).

But wanted to hit $(1,2);(2,4);(3,0)$!

$P(x) = 2\Delta_1(x) + 4\Delta_2(x) + 0\Delta_3(x)$ works.

Same as before?

...after a lot of calculations... $P(x) = 2x^2 + 1x + 4 \mod 5$.

The same as before!

Fields.. .

Flowers, and grass, oh so nice.

Set and two commutative operations: addition and multiplication with additive/multiplicative identities and inverses expect for additive identity has no mulitplicative inverse.

E.g., Reals, rationals, complex numbers.
Not E.g., the integers, matrices.

We will work with polynomials with arithmetic modulo p.

Addition is cool. Inherited from integers and integer division (remainders).
Multiplicative inverses due to $gcd(x,p) = 1$, forall $x \in \{1, \ldots, p-1\}$

Delta Polynomials: Concept.

For set of x-values, $x_1, \ldots, x_{d+1}$.

$$\Delta_i(x) = \begin{cases} 1, & \text{if } x = x_i. \\ 0, & \text{if } x = x_j \text{ for } j \neq i. \\ ?, & \text{otherwise.} \end{cases} \quad (1)$$

Given $d+1$ points, use Δ_i functions to go through points? $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$.

Will $y_1\Delta_1(x)$ contain (x_1, y_1)?

Will $y_2\Delta_2(x)$ contain (x_2, y_2)?

Does $y_1\Delta_1(x) + y_2\Delta_2(x)$ contain (x_1, y_1)? and (x_2, y_2)?

See the idea? Function that contains all points?

$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x) \ldots + y_{d+1}\Delta_{d+1}(x)$.

There exists a polynomial...

Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

Proof of at least one polynomial:
Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_{d+1}, y_{d+1})$.

$$\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)} = \prod_{j \neq i}(x - x_j) \prod_{j \neq i}(x_i - x_j)^{-1}$$

Numerator is 0 at $x_j \neq x_i$.

"Denominator" makes it 1 at x_i.

And..

$$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x) + \cdots + y_{d+1}\Delta_{d+1}(x).$$

hits points $(x_1, y_1); (x_2, y_2) \cdots (x_{d+1}, y_{d+1})$. Degree d polynomial!

Construction proves the existence of a polynomial!

Poll

Mark what's true.

(A) $\Delta_1(x_1) = y_1$
(B) $\Delta_1(x_1) = 1$
(C) $\Delta_1(x_2) = 0$
(D) $\Delta_1(x_3) = 1$
(E) $\Delta_2(x_2) = 1$
(F) $\Delta_2(x_1) = 0$

(B), (C), and (E)

Example.

$\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$.

Degree 1 polynomial, $P(x)$, that contains $(1,3)$ and $(3,4)$?

Work modulo 5.

$\Delta_1(x)$ contains $(1,1)$ and $(3,0)$.

$\Delta_1(x) = \frac{(x-3)}{1-3} = \frac{x-3}{-2} = (x-3)(-2)^{-1}$
$\Delta_1(x) = (x-3)(1-3)^{-1} = (x-3)(-2)^{-1}$
$= 2(x-3) = 2x - 6 = 2x + 4 \pmod 5$.

For a quadratic, $a_2x^2 + a_1x + a_0$ hits $(1,3); (2,4); (3,0)$.

Work modulo 5.

Find $\Delta_1(x)$ polynomial contains $(1,1); (2,0); (3,0)$.

$\Delta_1(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2} = (2)^{-1}(x-2)(x-3) = 3(x-2)(x-3)$
$= 3x^2 + 3 \pmod 5$

Put the delta functions together.

In general.

Given points: $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

$$\Delta_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)} = \prod_{j \neq i}(x - x_j) \prod_{j \neq i}(x_i - x_j)^{-1}$$

Numerator is 0 at $x_j \neq x_i$.

Denominator makes it 1 at x_i.

And..

$$P(x) = y_1\Delta_1(x) + y_2\Delta_2(x) + \cdots + y_k\Delta_k(x).$$

hits points $(x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)$.

Construction proves the existence of the polynomial!

Uniqueness.

Uniqueness Fact. At most one degree d polynomial hits $d+1$ points.

Roots fact: Any nontrivial degree d polynomial has at most d roots.

Non-zero line (degree 1 polynomial) can intersect $y = 0$ at only one x.

A parabola (degree 2), can intersect $y = 0$ at only two x's.

Proof:
Assume two different polynomials $Q(x)$ and $P(x)$ hit the points.

$R(x) = Q(x) - P(x)$ has $d+1$ roots and is degree d.
Contradiction.

$\square$

Must prove **Roots fact.**

Polynomial Division.

Divide $4x^2 - 3x + 2$ by $(x-3)$ modulo 5.

```
              4 x + 4 r 4
         -----------------
  x - 3 ) 4x^2 - 3 x + 2
          4x^2 - 2x
          ----------
                 4x + 2
                 4x - 2
                 -------
                      4
```

$4x^2 - 3x + 2 \equiv (x-3)(4x+4) + 4 \pmod 5$

In general, divide $P(x)$ by $(x-a)$ gives $Q(x)$ and remainder r.

That is, $P(x) = (x-a)Q(x) + r$

Only d roots.

Lemma 1: $P(x)$ has root a iff $P(x)/(x-a)$ has remainder 0: $P(x) = (x-a)Q(x)$.

Proof: $P(x) = (x-a)Q(x) + r$.
Plugin a: $P(a) = r$.
It is a root if and only if $r = 0$. □

Lemma 2: $P(x)$ has d roots; $r_1, \ldots, r_d$ then
$P(x) = c(x-r_1)(x-r_2)\cdots(x-r_d)$.
Proof Sketch: By induction.

Induction Step: $P(x) = (x-r_1)Q(x)$ by Lemma 1. $Q(x)$ has smaller degree so use the induction hypothesis. □

$d+1$ roots implies degree is at least $d+1$.

Roots fact: Any degree d polynomial has at most d roots.

Finite Fields

Proof works for reals, rationals, and complex numbers.

..but not for integers, since no multiplicative inverses.

Arithmetic modulo a prime p has multiplicative inverses..

..and has only a finite number of elements.

Good for computer science.

Arithmetic modulo a prime m is a **finite field** denoted by F_m or $GF(m)$.

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

Secret Sharing

Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

Shamir's k out of n Scheme:
Secret $s \in \{0, \ldots, p-1\}$

1. Choose $a_0 = s$, and randomly $a_1, \ldots, a_{k-1}$.
2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Roubustness: Any k knows secret.
Knowing k pts, only one $P(x)$, evaluate $P(0)$.
Secrecy: Any $k-1$ knows nothing.
Knowing $\leq k-1$ pts, any $P(0)$ is possible.

Minimality.

Need $p > n$ to hand out n shares: $P(1) \ldots P(n)$.

For b-bit secret, must choose a prime $p > 2^b$.

Theorem: There is always a prime between n and $2n$.
Chebyshev said it,
And I say it again,
There is always a prime
Between n and 2n.

Working over numbers within 1 bit of secret size. **Minimality.**

With k shares, reconstruct polynomial, $P(x)$.

With $k-1$ shares, any of p values possible for $P(0)$!

(Almost) any b-bit string possible!

(Almost) the same as what is missing: one $P(i)$.

Runtime.

Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $k-1$ polynomial n times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of k equations using $\log p$-bit arithmetic.

A bit more counting.

What is the number of degree d polynomials over $GF(m)$?

- m^{d+1}: $d+1$ coefficients from $\{0,\ldots,m-1\}$.
- m^{d+1}: $d+1$ points with y-values from $\{0,\ldots,m-1\}$

Infinite number for reals, rationals, complex numbers!

Summary

Two points make a line.

Compute solution: m,b.
Unique:
Assume two solutions, show they are the same.

Today: $d+1$ points make a unique degree d polynomial.

Cuz:
Solution: lagrange interpolation.
Unique:
Roots fact: Factoring sez $(x-r)$ is root.
Induction, says only d roots.
Apply: $P(x)$, $Q(x)$ degree d.
$P(x)-Q(x)$ is degree $d \implies d$ roots.
$P(x)=Q(x)$ on $d+1$ points $\implies P(x)=Q(x)$.

Secret Sharing:
k points on degree $k-1$ polynomial is great!
Can hand out n points on polynomial as shares.