Today.

Last time:
Today.

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  Shared (and sort of kept) secrets.
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Today: Errors
   Tolerate Loss: erasure codes.

Tolerate corruption!
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In general...

Given points: \((x_1, y_1); (x_2, y_2) \cdots (x_k, y_k)\).
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Solve...

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\begin{align*}
  a_{k-1}x_1^{k-1} + \cdots + a_0 & \equiv y_1 \pmod{p} \\
  a_{k-1}x_2^{k-1} + \cdots + a_0 & \equiv y_2 \pmod{p} \\
  \vdots & \quad \vdots \\
  a_{k-1}x_k^{k-1} + \cdots + a_0 & \equiv y_k \pmod{p}
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Will this always work?
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As long as solution \textbf{exists} and it is \textbf{unique}! And...
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**Modular Arithmetic Fact:** Exactly 1 polynomial of degree \(\leq d\) with arithmetic modulo prime \(p\) contains \(d + 1\) pts.
Proof sketches.

**Property 2** A polynomial: \( P(x) = a_d x^d + \cdots + a_0 \) has \( d + 1 \) coefficients.
Proof sketches.

**Property 2** A polynomial: $P(x) = a_dx^d + \cdots a_0$ has $d + 1$ coefficients. Any set of $d + 1$ points uniquely determines the polynomial.
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Existence: Lagrange Interpolation.
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Degree \( d \), \( \Delta_i(x) \) polynomials.
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Existence: Lagrange Interpolation.
- Degree \( d \), \( \Delta_j(x) \) polynomials.
- Factors of \( (x - x_j) \) to zero out at \( x_j \neq x_i \).

Multiply by zero. My love is won.

Combine.

Uniqueness:
- Property 1 A non-zero degree \( d \) polynomial has at most \( d \) roots.

Factoring: \( P(x) \) with roots \( r_1, \ldots, r_d \)
\[
P(x) = c(x - r_0)(x - r_1)\cdots(x - r_d).
\]

Love me some contradiction!

Two polynomials: \( P(x), Q(x) \)
\[
P(x) - Q(x) \text{ has too many roots.}
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**Proof sketches.**

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Proof works for reals, rationals, and complex numbers.
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Arithmetic modulo a prime \( m \) is a **finite field** denoted by \( F_m \) or \( GF(m) \).

Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

In the rationals, the precision blows up, where in modular arithmetic, it does not.
Secret Sharing

**Modular Arithmetic Fact:** Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d + 1$ points.
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3. Share $i$ is point $(i, P(i) \mod p)$. 

**Robustness:** Any $k$ knows secret. Knowing $k$ pts, only one $P(x)$, evaluate $P(0)$.

**Secrecy:** Any $k-1$ knows nothing. Knowing $\leq k-1$ pts, any $P(0)$ is possible.

Two points make a line: the value of one point allows any y-intercept. 3 kids hand out 3 points. Any two know the line.
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n people, k is enough.

(A) The modulus needs to be at least $n + 1$.
(B) The modulus needs to be at least $k$.
(C) Use degree $k$ polynomial, hand out $n$ points.
(D) Use degree $n$ polynomial, hand out $k$ points.
(E) Use degree $k - 1$ polynomial, hand out $n$ points.
(F) The modulus needs to be at least $2^s$, where $s$ is value of secret.
(G) The modulus needs to be at least $2^s$, where $s$ is size of secret.
Secret Sharing.

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(A) The modulus needs to be at least $n + 1$.
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(A), (B), (E), (F)
Erasure Codes.

Satellite

GPS device
Erasure Codes.

Satellite

3 packet message.

GPS device
Erasure Codes.

Satellite

3 packet message.

GPS device

Lose 3 out 6 packets.
Erasure Codes.

Satellite

1 2 3

3 packet message. So send 6!

Lose 3 out 6 packets.

GPS device
Erasure Codes.

Satellite

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GPS device
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3 packet message. So send 6!

Lose 3 out 6 packets.
Erasure Codes.

Satellite

3 packet message. So send 6!

Lose 3 out 6 packets.

Gets packets 1, 1, and 3.
Solution Idea.

\( n \) packet message, channel that loses \( k \) packets.
Solution Idea.

\( n \) packet message, channel that loses \( k \) packets.
Must send \( n + k \) packets!
Solution Idea.

$n$ packet message, channel that loses $k$ packets.
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Any $n$ packets
Solution Idea.

\(n\) packet message, channel that loses \(k\) packets.
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Any \(n\) packets should allow reconstruction of \(n\) packet message.
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- Any \( n \) point values
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Solution Idea.

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Alright!
Solution Idea.

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A packet message, channel that loses $k$ packets.
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Alright!!!!!!

Use polynomials.
The Scheme

**Problem:** Want to send a message with \( n \) packets.
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A degree $n - 1$ polynomial determined by any $n$ points!
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Erasure Coding Scheme: message = $m_0, m_1 \ldots, m_{n−1}$.

1. Choose prime $p \approx 2^b$ for packet size $b$.

2. $P(x) = m_{n−1}x^{n−1} + \cdots m_0 \pmod{p}$.

3. Send $P(1), \ldots, P(n+k)$. 
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2. \( P(x) = m_{n-1}x^{n-1} + \cdots m_0 \) (mod \( p \)).

3. Send \( P(1), \ldots, P(n+k) \).

Any \( n \) of the \( n + k \) packets gives polynomial ...
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Erasure Coding Scheme: message $= m_0, m_1 \ldots, m_{n-1}$.

1. Choose prime $p \approx 2^b$ for packet size $b$.
2. $P(x) = m_{n-1}x^{n-1} + \cdots + m_0 \pmod{p}$.
3. Send $P(1), \ldots, P(n+k)$.

Any $n$ of the $n + k$ packets gives polynomial ...and message!
Erasure Codes.

Satellite

GPS device
Erasure Codes.

Satellite

GPS device

$n$ packet message.
Erasure Codes.

Satellite

$n$ packet message.

GPS device

$Lose \, k \, packets.$
Erasure Codes.

Satellite 

\[ \begin{array}{cccccccccc}
1 & 2 & \cdots & n+k \\
\end{array} \]

\[ \begin{array}{cccccccccc}
\hline
\hline
\end{array} \]

GPS device 

\( n \) packet message. So send \( n + k \)!

Lose \( k \) packets.
Erasure Codes.

Satellite

\[
\begin{align*}
1 & \quad 2 & \quad \ldots & \quad n+k \\
\hline
\text{GPS device}
\end{align*}
\]

Any \( n \) packets is enough! So send \( n + k \)!

Lose \( k \) packets.
Erasure Codes.

Lose $k$ packets. Any $n$ packets is enough! $n$ packet message. So send $n+k$!

GPS device
Erasure Codes.

$n$ packet message. So send $n+k$!

Lose $k$ packets.

Any $n$ packets is enough!
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Satellite

$1 \quad 2 \quad \cdots \quad n+k$

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GPS device

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Satellite

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GPS device

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Lose \( k \) packets.

Any \( n \) packets is enough!

\( n \) packet message. So send \( n + k \)!

Optimal.
Size: Can choose a prime between $2^{b-1}$ and $2^b$. (Lose at most 1 bit per packet.)
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There are Galois Fields $GF(2^n)$ where one loses nothing.
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– Can also run the Fast Fourier Transform.
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In practice, $O(n)$ operations with almost the same redundancy.
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Comparison with Secret Sharing: information content.
   Secret Sharing: each share is size of whole secret.
   Coding: Each packet has size $1/n$ of the whole message.
Erasure Code: Example.

Send message of 1, 4, and 4.

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$. How?

Lagrange Interpolation.
Linear System.
Work modulo 5.

$P(x) = x^2 \pmod{5}$

$P(1) = 1$, $P(2) = 4$, $P(3) = 9 = 4 \pmod{5}$

Send $(0, P(0)), \ldots, (5, P(5))$.

6 points.

Better work modulo 7 at least!

Why? $(0, P(0)) = (5, P(5)) \pmod{5}$
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Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$. 

Send Packets: $(1, 1)$, $(2, 4)$, $(3, 4)$, $(4, 7)$, $(5, 2)$, $(6, 0)$.
Example

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$. Modulo 7 to accommodate at least 6 packets.
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Notice that packets contain “x-values”.

$P(x) = 2x^2 + 4x + 2$
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\[P(x) = 2x^2 + 4x + 2\]
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Send Packets: $(1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)$

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6a_1 + 3a_0 &= 2 \pmod{7}, & 5a_1 + 4a_0 &= 0 \pmod{7} \\
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Packets: $(1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)$
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Send

Packets: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)

Notice that packets contain “x-values”.

Bad reception!

Send: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)
Bad reception!

Send: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)
Recieve: (1, 1) (2, 4), (6, 0)
Bad reception!

Send: \((1,1), (2,4), (3,4), (4,7), (5,2), (6,0)\)

Receive: \((1,1), (2,4), (6,0)\)

Reconstruct?
Bad reception!

Send: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)
Recieve: (1, 1), (2, 4), (6, 0)
Reconstruct?
Format: (i, R(i)).
Bad reception!

Send: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)  
Recieve: (1, 1), (2, 4), (6, 0)  
Reconstruct? 
Format: (i, R(i)).  
Lagrange or linear equations.
Bad reception!

Send: $(1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)$

Recieve: $(1, 1) (2, 4), (6, 0)$

Reconstruct?

Format: $(i, R(i))$.

Lagrange or linear equations.

$$P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7}$$
Bad reception!

Send: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)

Recieve: (1,1) (2,4), (6,0)

Reconstruct?

Format: $(i, R(i))$.

Lagrange or linear equations.

\[
P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7}
\]
\[
P(2) = 4a_2 + 2a_1 + a_0 \equiv 4 \pmod{7}
\]
Bad reception!

Send: (1,1), (2,4), (3,4), (4,7), (5,2), (6,0)

Recieve: (1,1) (2,4), (6,0)
   Reconstruct?

Format: (i, R(i)).

Lagrange or linear equations.

\[
P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7} \\
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\]
Bad reception!

Send: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)

Recieve: (1,1) (2,4), (6,0)
  Reconstruct?

Format: \((i, R(i))\).

Lagrange or linear equations.

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Bad reception!

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\]

Channeling Sahai
Bad reception!

Send: \((1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)\)

Receive: \((1, 1), (2, 4), (6, 0)\)
Reconstruct?

Format: \((i, R(i))\).

Lagrange or linear equations.

\[
P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7} \\
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\]

Channeling Sahai ...
Bad reception!

Send: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)

Recieve: (1, 1) (2, 4), (6, 0)
  Reconstruct?

Format: \((i, R(i))\).

Lagrange or linear equations.

\[
P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7}
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P(6) = 2a_2 + 3a_1 + a_0 \equiv 0 \pmod{7}
\]

Channeling Sahai ...

\[
P(x) = 2x^2 + 4x + 2
\]
Bad reception!

Send: (1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)

Recieve: (1, 1) (2, 4), (6, 0)
Reconstruct?

Format: \((i, R(i))\).

Lagrange or linear equations.

\[
P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7}
\]
\[
P(2) = 4a_2 + 2a_1 + a_0 \equiv 4 \pmod{7}
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Channeling Sahai ...

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Reconstruct?

Format: \((i, R(i))\).

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\[
P(6) = 2a_2 + 3a_1 + a_0 \equiv 0 \pmod{7}
\]

Channeling Sahai ...

\[
P(x) = 2x^2 + 4x + 2
\]

Message?
Bad reception!

Send: (1,1), (2,4), (3,4), (4,7), (5,2), (6,0)

Receive: (1,1) (2,4), (6,0)
Reconstruct?

Format: \((i, R(i))\).

Lagrange or linear equations.

\[
P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7}
\]
\[
P(2) = 4a_2 + 2a_1 + a_0 \equiv 4 \pmod{7}
\]
\[
P(6) = 2a_2 + 3a_1 + a_0 \equiv 0 \pmod{7}
\]

Channeling Sahai ...

\[
P(x) = 2x^2 + 4x + 2
\]

Message? \(P(1) = 1\),
Bad reception!

Send: \((1, 1), (2, 4), (3, 4), (4, 7), (5, 2), (6, 0)\)

Recieve: \((1,1)\ (2,4),\ (6,0)\)
    Reconstruct?

Format: \((i, R(i))\).

Lagrange or linear equations.

\[
P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7}
\]
\[
P(2) = 4a_2 + 2a_1 + a_0 \equiv 4 \pmod{7}
\]
\[
P(6) = 2a_2 + 3a_1 + a_0 \equiv 0 \pmod{7}
\]

Channeling Sahai ...

\[
P(x) = 2x^2 + 4x + 2
\]

Message? \( P(1) = 1,\ P(2) = 4,\)
Bad reception!

Send: \((1,1),(2,4),(3,4),(4,7),(5,2),(6,0)\)

Receive: \((1,1),(2,4),(6,0)\)

Reconstruct?

Format: \((i,R(i))\).

Lagrange or linear equations.

\[
P(1) = a_2 + a_1 + a_0 \equiv 1 \pmod{7} \\
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P(6) = 2a_2 + 3a_1 + a_0 \equiv 0 \pmod{7}
\]

Channeling Sahai ...

\[
P(x) = 2x^2 + 4x + 2
\]

Message? \(P(1) = 1, P(2) = 4, P(3) = 4\).
Questions for Review

You want to encode a secret consisting of 1,4,4.
You want to encode a secret consisting of 1,4,4.

How big should modulus be?
Questions for Review

You want to encode a secret consisting of 1, 4, 4.

How big should modulus be?
Larger than 144
Questions for Review

You want to encode a secret consisting of 1, 4, 4.

How big should modulus be?
Larger than 144 and prime!
You want to encode a secret consisting of 1, 4, 4.

How big should modulus be?
   Larger than 144 and prime!

Remember the secret, $s = 144$, must be one of the possible values.
Questions for Review

You want to encode a secret consisting of 1,4,4.

How big should modulus be?
  Larger than 144 and prime!

Remember the secret, $s = 144$, must be one of the possible values.

You want to send a message consisting of packets 1,4,2,3,0
Questions for Review

You want to encode a secret consisting of 1,4,4.

How big should modulus be?
Larger than 144 and prime!

Remember the secret, $s = 144$, must be one of the possible values.

You want to send a message consisting of packets 1,4,2,3,0 through a noisy channel that loses 3 packets.
Questions for Review

You want to encode a secret consisting of 1,4,4.

How big should modulus be?
   Larger than 144 and prime!

Remember the secret, $s = 144$, must be one of the possible values.

You want to send a message consisting of packets 1,4,2,3,0 through a noisy channel that loses 3 packets.

How big should modulus be?
You want to encode a secret consisting of 1,4,4.

How big should modulus be?
Larger than 144 and prime!

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You want to encode a secret consisting of 1,4,4.

How big should modulus be?
  Larger than 144 and prime!

Remember the secret, \( s = 144 \), must be one of the possible values.

You want to send a message consisting of packets 1,4,2,3,0 through a noisy channel that loses 3 packets.

How big should modulus be?
  Larger than 8
Questions for Review

You want to encode a secret consisting of 1, 4, 4.

How big should modulus be?
   Larger than 144 and prime!

Remember the secret, \( s = 144 \), must be one of the possible values.

You want to send a message consisting of packets 1, 4, 2, 3, 0 through a noisy channel that loses 3 packets.

How big should modulus be?
   Larger than 8 and prime!
Questions for Review

You want to encode a secret consisting of 1,4,4.
  How big should modulus be?
    Larger than 144 and prime!
    Remember the secret, \( s = 144 \), must be one of the possible values.
You want to send a message consisting of packets 1,4,2,3,0
  through a noisy channel that loses 3 packets.
  How big should modulus be?
    Larger than 8 and prime!
  The other constraint: arithmetic system can represent 0, 1, 2, 3, 4.
Questions for Review

You want to encode a secret consisting of $1, 4, 4$.

How big should modulus be?
Larger than 144 and prime!

Remember the secret, $s = 144$, must be one of the possible values.

You want to send a message consisting of packets $1, 4, 2, 3, 0$
through a noisy channel that loses 3 packets.

How big should modulus be?
Larger than 8 and prime!

The other constraint: arithmetic system can represent 0, 1, 2, 3, 4.

Send $n$ packets $b$-bit packets, with $k$ errors.
You want to encode a secret consisting of 1,4,4.

How big should modulus be?
Larger than 144 and prime!

Remember the secret, \( s = 144 \), must be one of the possible values.

You want to send a message consisting of packets 1,4,2,3,0 through a noisy channel that loses 3 packets.

How big should modulus be?
Larger than 8 and prime!

The other constraint: arithmetic system can represent 0, 1, 2, 3, 4.

Send \( n \) packets \( b \)-bit packets, with \( k \) errors.
Modulus should be larger than \( n + k \) and also larger than \( 2^b \).
Polynomials.
Polynomials.

- give Secret Sharing.
Polynomials.

- give Secret Sharing.
- give Erasure Codes.
Polynomials.

- give Secret Sharing.
- give Erasure Codes.

**Error Correction:**
Polynomials.

- give Secret Sharing.
- give Erasure Codes.

**Error Correction:**

Noisy Channel: *corrupts* $k$ packets. (rather than *loss.*)
Polynomials.

- give Secret Sharing.
- give Erasure Codes.

Error Correction:
Noisy Channel: corrupts $k$ packets. (rather than loss.)
Additional Challenge: Finding which packets are corrupt.
Error Correction

Satellite

GPS device
Error Correction

Satellite

GPS device

3 packet message.
Error Correction

Satellite

GPS device

3 packet message.

Corrupts 1 packets.
Error Correction

Satellite

1 2 3 1 2
A B C D E

3 packet message. Send 5.

Corrupts 1 packets.

GPS device
Error Correction

Satellite

A  B  C  D  E

3 packet message. Send 5.

Corrupts 1 packets.

GPS device
Error Correction

3 packet message. Send 5.

Corrupts 1 packets.
The Scheme.

**Problem:** Communicate $n$ packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.

Reed-Solomon Code:

1. Make a polynomial, $P(x)$ of degree $n-1$, that encodes message.
   ▶ $P(1) = m_1, \ldots, P(n) = m_n$.
   ▶ Comment: could encode with packets as coefficients.
2. Send $P(1), \ldots, P(n+2k)$.

After noisy channel:
Receive values $R(1), \ldots, R(n+2k)$.

Properties:
1. $P(i) = R(i)$ for at least $n+k$ points $i$,
2. $P(x)$ is unique degree $n-1$ polynomial that contains $\geq n+k$ received points.
The Scheme.

**Problem:** Communicate \( n \) packets \( m_1, \ldots, m_n \) on noisy channel that corrupts \( \leq k \) packets.

**Reed-Solomon Code:**
The Scheme.

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The Scheme.

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The Scheme.

**Problem:** Communicate $n$ packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.

**Reed-Solomon Code:**

1. Make a polynomial, $P(x)$ of degree $n - 1$, that encodes message.
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   - Comment: could encode with packets as coefficients.

2. Send $P(1), \ldots, P(n+2k)$.

**After noisy channel:** Recieve values $R(1), \ldots, R(n+2k)$. 
Problem: Communicate \( n \) packets \( m_1, \ldots, m_n \) on noisy channel that corrupts \( \leq k \) packets.

Reed-Solomon Code:

1. Make a polynomial, \( P(x) \) of degree \( n - 1 \), that encodes message.
   - \( P(1) = m_1, \ldots, P(n) = m_n \).
   - Comment: could encode with packets as coefficients.

2. Send \( P(1), \ldots, P(n + 2k) \).

After noisy channel: Recieve values \( R(1), \ldots, R(n + 2k) \).

Properties:

(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
The Scheme.

**Problem:** Communicate $n$ packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.

**Reed-Solomon Code:**

1. Make a polynomial, $P(x)$ of degree $n - 1$, that encodes message.
   - $P(1) = m_1, \ldots, P(n) = m_n$.
   - Comment: could encode with packets as coefficients.

2. Send $P(1), \ldots, P(n + 2k)$.

**After noisy channel:** Recieve values $R(1), \ldots, R(n + 2k)$.

**Properties:**
   (1) $P(i) = R(i)$ for at least $n + k$ points $i$,
   (2) $P(x)$ is unique degree $n - 1$ polynomial
Problem: Communicate \( n \) packets \( m_1, \ldots, m_n \) on noisy channel that corrupts \( \leq k \) packets.

Reed-Solomon Code:

1. Make a polynomial, \( P(x) \) of degree \( n - 1 \), that encodes message.
   - \( P(1) = m_1, \ldots, P(n) = m_n \).
   - Comment: could encode with packets as coefficients.

2. Send \( P(1), \ldots, P(n+2k) \).

After noisy channel: Recieve values \( R(1), \ldots, R(n+2k) \).

Properties:
   (1) \( P(i) = R(i) \) for at least \( n+k \) points \( i \),
   (2) \( P(x) \) is unique degree \( n - 1 \) polynomial that contains \( \geq n + k \) received points.
Properties: proof.

\[ P(x): \text{degree } n - 1 \text{ polynomial.} \]
Properties: proof.

\[ P(x): \text{degree } n - 1 \text{ polynomial.} \]

Send \[ P(1), \ldots, P(n + 2k) \]
Properties: proof.

\[ P(x) \]: degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)

Proof:
(1) Sure.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + 2k \) points.
\[ Q(x) \] agrees with \( R(i) \), \( n + 2k \) times.
\[ P(x) \] agrees with \( R(i) \), \( n + 2k \) times.
Total points contained by both: \( 2(n + 2k) \).
Total points to choose from: \( n + 2k \).
Points contained by both: \( \geq n \).
\( \geq P - H \) Collisions.
\( \Rightarrow Q(i) = P(i) \) at \( n \) points.
\( \Rightarrow Q(x) = P(x) \).
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.  
Send \( P(1), \ldots, P(n + 2k) \)  
Receive \( R(1), \ldots, R(n + 2k) \)  
At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send  \( P(1), \ldots, P(n + 2k) \)
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At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
Properties: proof.

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
Receive $R(1), \ldots, R(n + 2k)$
At most $k$ $i$'s where $P(i) \neq R(i)$.

Properties:
(1) $P(i) = R(i)$ for at least $n + k$ points $i$,
(2) $P(x)$ is unique degree $n - 1$ polynomial
Properties: proof.

\[ P(x) : \text{degree } n - 1 \text{ polynomial}. \]
Send \( P(1), \ldots, P(n + 2k) \)
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At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
that contains \( \geq n + k \) received points.
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.

Send \( P(1), \ldots, P(n+2k) \)

Receive \( R(1), \ldots, R(n+2k) \)

At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Properties:

(1) \( P(i) = R(i) \) for at least \( n+k \) points \( i \),

(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
    that contains \( \geq n+k \) received points.

Proof:
Properties: proof.

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
Receive $R(1), \ldots, R(n + 2k)$
At most $k$ i’s where $P(i) \neq R(i)$.

Properties:
(1) $P(i) = R(i)$ for at least $n + k$ points $i$,
(2) $P(x)$ is unique degree $n - 1$ polynomial
    that contains $\geq n + k$ received points.

Proof:
(1) Sure.
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
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(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
that contains \( \geq n+k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
Properties: proof.

\[ P(x) \]: degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

**Properties:**

1. \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
2. \( P(x) \) is unique degree \( n - 1 \) polynomial
   that contains \( \geq n + k \) received points.

**Proof:**

1. Sure. Only \( k \) corruptions.
2. Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
Properties: proof.

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n+2k)$
Receive $R(1), \ldots, R(n+2k)$
At most $k$ $i$'s where $P(i) \neq R(i)$.

Properties:
(1) $P(i) = R(i)$ for at least $n + k$ points $i$,
(2) $P(x)$ is unique degree $n - 1$ polynomial that contains $\geq n + k$ received points.

Proof:
(1) Sure. Only $k$ corruptions.
(2) Degree $n - 1$ polynomial $Q(x)$ consistent with $n + k$ points.
   $Q(x)$ agrees with $R(i)$, $n + k$ times.
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
    that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
    \( Q(x) \) agrees with \( R(i) \), \( n + k \) times.
    \( P(x) \) agrees with \( R(i) \), \( n + k \) times.
Properties: proof.

\[ P(x) \text{: degree } n - 1 \text{ polynomial.} \]
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
    that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
    \( Q(x) \) agrees with \( R(i), n + k \) times.
    \( P(x) \) agrees with \( R(i), n + k \) times.
    Total points contained by both: \( 2n + 2k \).
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
Receive \( R(1), \ldots, R(n+2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

Properties:

1. \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
2. \( P(x) \) is unique degree \( n - 1 \) polynomial
   that contains \( \geq n + k \) received points.

Proof:

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2. Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
   \( Q(x) \) agrees with \( R(i), n + k \) times.
   \( P(x) \) agrees with \( R(i), n + k \) times.
   Total points contained by both: \( 2n + 2k \). Pigeons.
Properties: proof.

\[ P(x) \]: degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) \( i \)'s where \( P(i) \neq R(i) \).

Properties:
1. \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
2. \( P(x) \) is unique degree \( n - 1 \) polynomial
   that contains \( \geq n + k \) received points.

Proof:
1. Sure. Only \( k \) corruptions.
2. Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
   \( Q(x) \) agrees with \( R(i) \), \( n + k \) times.
   \( P(x) \) agrees with \( R(i) \), \( n + k \) times.
   Total points contained by both: \( 2n + 2k \). \( P \) Pigeons.
   Total points to choose from : \( n + 2k \).
Properties: proof.

\( P(x) \): degree \( n - 1 \) polynomial.
Send \( P(1), \ldots, P(n+2k) \)
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Properties:
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    that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
    \( Q(x) \) agrees with \( R(i) \), \( n + k \) times.
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    Total points contained by both: \( 2n + 2k \). \( P \) \hspace{1cm} Pigeons.
    Total points to choose from \hspace{1cm} \( n + 2k \). \( H \) \hspace{1cm} Holes.
Properties: proof.

\[ P(x): \text{degree } n - 1 \text{ polynomial.} \]
Send \( P(1), \ldots, P(n+2k) \)
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Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
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    that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
    \( Q(x) \) agrees with \( R(i), n + k \) times.
    \( P(x) \) agrees with \( R(i), n + k \) times.
    Total points contained by both: \( 2n + 2k \). \( P \) \quad \text{Pigeons.}
    Total points to choose from : \( n + 2k \). \( H \) \quad \text{Holes.}
    Points contained by both : \( \geq n \).
Properties: proof.

\[ P(x) \text{: degree } n - 1 \text{ polynomial.} \]
Send \[ P(1), \ldots, P(n + 2k) \]
Receive \[ R(1), \ldots, R(n + 2k) \]
At most \( k \) i's where \( P(i) \neq R(i) \).

Properties:

(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),

(2) \( P(x) \) is unique degree \( n - 1 \) polynomial
    that contains \( \geq n + k \) received points.

Proof:

(1) Sure. Only \( k \) corruptions.

(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
    \( Q(x) \) agrees with \( R(i), n + k \) times.
    \( P(x) \) agrees with \( R(i), n + k \) times.
    Total points contained by both: \( 2n + 2k \). \( P \) Pigeons.
    Total points to choose from : \( n + 2k \). \( H \) Holes.
    Points contained by both : \( \geq n \). \( \geq P - H \) Collisions.
    \( \implies Q(i) = P(i) \) at \( n \) points.
Properties: proof.

\[ P(x) : \text{degree } n - 1 \text{ polynomial.} \]
Send \( P(1), \ldots, P(n + 2k) \)
Receive \( R(1), \ldots, R(n + 2k) \)
At most \( k \) i’s where \( P(i) \neq R(i) \).

Properties:
(1) \( P(i) = R(i) \) for at least \( n + k \) points \( i \),
(2) \( P(x) \) is unique degree \( n - 1 \) polynomial that contains \( \geq n + k \) received points.

Proof:
(1) Sure. Only \( k \) corruptions.
(2) Degree \( n - 1 \) polynomial \( Q(x) \) consistent with \( n + k \) points.
   \( Q(x) \) agrees with \( R(i) \), \( n + k \) times.
   \( P(x) \) agrees with \( R(i) \), \( n + k \) times.
   Total points contained by both: \( 2n + 2k \). \( P \) Pigeons.
   Total points to choose from : \( n + 2k \). \( H \) Holes.
   Points contained by both : \( \geq n \). \( \geq P - H \) Collisions.
   \[ \implies Q(i) = P(i) \text{ at } n \text{ points.} \]
   \[ \implies Q(x) = P(x). \]
Properties: proof.

$P(x)$: degree $n - 1$ polynomial.
Send $P(1), \ldots, P(n + 2k)$
Receive $R(1), \ldots, R(n + 2k)$
At most $k$ i’s where $P(i) \neq R(i)$.

Properties:
(1) $P(i) = R(i)$ for at least $n + k$ points $i$,
(2) $P(x)$ is unique degree $n - 1$ polynomial
    that contains $\geq n + k$ received points.

Proof:
(1) Sure. Only $k$ corruptions.
(2) Degree $n - 1$ polynomial $Q(x)$ consistent with $n + k$ points.
    $Q(x)$ agrees with $R(i)$, $n + k$ times.
    $P(x)$ agrees with $R(i)$, $n + k$ times.
    Total points contained by both: $2n + 2k$. $P$ Pigeons.
    Total points to choose from : $n + 2k$. $H$ Holes.
    Points contained by both : $\geq n$. $\geq P - H$ Collisions.
    $\implies Q(i) = P(i)$ at $n$ points.
    $\implies Q(x) = P(x)$.
Example.

Message: 3, 0, 6.
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Reed Solomon Code: $P(x) = x^2 + x + 1 \pmod{7}$ has $P(1) = 3, P(2) = 0, P(3) = 6$ modulo 7.
Example.

Message: 3, 0, 6.

Reed Solomon Code: \( P(x) = x^2 + x + 1 \) (mod 7) has
\( P(1) = 3, P(2) = 0, P(3) = 6 \) modulo 7.

Send: \( P(1) = 3, P(2) = 0, P(3) = 6, \)
Example.

Message: 3, 0, 6.

Reed Solomon Code: $P(x) = x^2 + x + 1 \pmod{7}$ has
$P(1) = 3, P(2) = 0, P(3) = 6$ modulo 7.

Send: $P(1) = 3, P(2) = 0, P(3) = 6, P(4) = 0, P(5) = 3$. 
Example.

Message: 3, 0, 6.

Reed Solomon Code: $P(x) = x^2 + x + 1 \pmod{7}$ has $P(1) = 3, P(2) = 0, P(3) = 6$ modulo 7.

Send: $P(1) = 3, P(2) = 0, P(3) = 6, P(4) = 0, P(5) = 3$.

(Aside: Message in plain text!)
Example.

Message: 3, 0, 6.

Reed Solomon Code: \( P(x) = x^2 + x + 1 \pmod{7} \) has \( P(1) = 3, P(2) = 0, P(3) = 6 \) modulo 7.

Send: \( P(1) = 3, P(2) = 0, P(3) = 6, P(4) = 0, P(5) = 3. \)

(Aside: Message in plain text!)

Receive \( R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3. \)
Example.

Message: 3, 0, 6.

Reed Solomon Code: $P(x) = x^2 + x + 1 \pmod{7}$ has $P(1) = 3, P(2) = 0, P(3) = 6$ modulo 7.

Send: $P(1) = 3, P(2) = 0, P(3) = 6, P(4) = 0, P(5) = 3$.

(Aside: Message in plain text!)

Receive $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$.

$P(i) = R(i)$ for $n+k = 3+1 = 4$ points.
Slow solution.

**Brute Force:**
For each subset of \( n + k \) points

Fit degree \( n - 1 \) polynomial, \( Q(x) \), to \( n \) of them.

Check if consistent with \( n + k \) of the total points.

If yes, output \( Q(x) \).

For subset of \( n + k \) pts where \( R(i) = P(i) \), method will reconstruct \( P(x) \).

For any subset of \( n + k \) pts,

1. there is unique degree \( n - 1 \) polynomial \( Q(x) \) that fits \( n \) of them
2. and where \( Q(x) \) is consistent with \( n + k \) points

\( \Rightarrow P(x) = Q(x) \).

Reconstructs \( P(x) \) and only \( P(x) \)!!
Slow solution.

**Brute Force:**
For each subset of $n + k$ points
  Fit degree $n - 1$ polynomial, $Q(x)$, to $n$ of them.
Slow solution.

**Brute Force:**
For each subset of \( n + k \) points
- Fit degree \( n - 1 \) polynomial, \( Q(x) \), to \( n \) of them.
- Check if consistent with \( n + k \) of the total points.
Slow solution.

Brute Force:
For each subset of $n + k$ points
   Fit degree $n - 1$ polynomial, $Q(x)$, to $n$ of them.
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Brute Force:
For each subset of $n+k$ points
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- For subset of $n+k$ pts where $R(i) = P(i)$,
  method will reconstruct $P(x)$!
Slow solution.

**Brute Force:**
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- For subset of $n + k$ pts where $R(i) = P(i)$, method will reconstruct $P(x)$!

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Slow solution.

**Brute Force:**
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  ▶ For any subset of $n + k$ pts,
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Slow solution.

**Brute Force:**
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**Slow solution.**

**Brute Force:**
For each subset of \(n + k\) points
  - Fit degree \(n - 1\) polynomial, \(Q(x)\), to \(n\) of them.
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- For subset of \(n + k\) pts where \(R(i) = P(i)\), method will reconstruct \(P(x)\)!

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  2. and where \(Q(x)\) is consistent with \(n + k\) points
      \[\implies P(x) = Q(x).\]
Slow solution.

**Brute Force:**
For each subset of $n+k$ points
- Fit degree $n-1$ polynomial, $Q(x)$, to $n$ of them.
- Check if consistent with $n+k$ of the total points.
  If yes, output $Q(x)$.

- For subset of $n+k$ pts where $R(i) = P(i)$, method will reconstruct $P(x)$!

- For any subset of $n+k$ pts,
  1. there is unique degree $n-1$ polynomial $Q(x)$ that fits $n$ of them
  2. and where $Q(x)$ is consistent with $n+k$ points
      $\implies P(x) = Q(x)$.

Reconstructs $P(x)$ and only $P(x)$!!
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

All equations..

\[ p_2 + p_1 + p_0 \equiv 3 \pmod{7} \]
\[ 4p_2 + 2p_1 + p_0 \equiv 1 \pmod{7} \]
\[ 2p_2 + 3p_1 + p_0 \equiv 6 \pmod{7} \]
\[ 2p_2 + 4p_1 + p_0 \equiv 0 \pmod{7} \]
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    4p_2 + 5p_1 + p_0 & \equiv 3 \pmod{7}
\end{align*}
\]

Assume point 1 is wrong
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

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    4p_2 + 5p_1 + p_0 & \equiv 3 \pmod{7}
\end{align*} \]

Assume point 1 is wrong and solve..
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

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4p_2 + 5p_1 + p_0 & \equiv 3 \pmod{7}
\end{align*}
\]

Assume point 1 is wrong and solve.. no consistent solution!
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

All equations..

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    p_2 + p_1 + p_0 & \equiv 3 \pmod{7} \\
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    2p_2 + 4p_1 + p_0 & \equiv 0 \pmod{7} \\
    4p_2 + 5p_1 + p_0 & \equiv 3 \pmod{7}
\end{align*}
\]

Assume point 1 is wrong and solve.. no consistent solution!
Assume point 2 is wrong
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

All equations..

\[
\begin{align*}
p_2 + p_1 + p_0 & \equiv 3 \pmod{7} \\
4p_2 + 2p_1 + p_0 & \equiv 1 \pmod{7} \\
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\end{align*}
\]

Assume point 1 is wrong and solve.. no consistent solution!
Assume point 2 is wrong and solve...
Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

All equations..

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  p_2 + p_1 + p_0 & \equiv 3 \pmod{7} \\
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  4p_2 + 5p_1 + p_0 & \equiv 3 \pmod{7}
\end{align*}
\]

Assume point 1 is wrong and solve...no consistent solution!
Assume point 2 is wrong and solve...consistent solution!
In general,

\[ P(x) = p_{n-1}x^{n-1} + \cdots + p_0 \] and receive \( R(1), \ldots, R(m = n + 2k) \).
In general,

\[ P(x) = p_{n-1}x^{n-1} + \cdots + p_0 \] and receive \( R(1), \ldots, R(m = n + 2k) \).

\[ p_{n-1} + \cdots + p_0 \equiv R(1) \pmod{p} \]
In general,

\[ P(x) = p_{n-1}x^{n-1} + \cdots p_0 \] and receive \( R(1), \ldots R(m = n + 2k) \).

\[
p_{n-1} + \cdots p_0 \equiv R(1) \pmod{p}
\]
\[
p_{n-1}2^{n-1} + \cdots p_0 \equiv R(2) \pmod{p}
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In general, 

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  p_{n-1} + \cdots p_0 & \equiv R(1) \pmod{p} \\
  p_{n-1}2^{n-1} + \cdots p_0 & \equiv R(2) \pmod{p} \\
  & \quad \quad \quad \vdots \\
  p_{n-1}i^{n-1} + \cdots p_0 & \equiv R(i) \pmod{p} \\
  & \quad \quad \quad \vdots \\
  p_{n-1}(m)^{n-1} + \cdots p_0 & \equiv R(m) \pmod{p}
\end{align*}
\]
In general, 

\[ P(x) = p_{n-1}x^{n-1} + \cdots p_0 \] 

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\[ p_{n-1}i^{n-1} + \cdots p_0 \equiv R(i) \pmod{p} \]

\[ p_{n-1}(m)^{n-1} + \cdots p_0 \equiv R(m) \pmod{p} \]

Error!!
In general..

\[ P(x) = p_{n-1}x^{n-1} + \cdots + p_0 \] and receive \( R(1), \ldots, R(m = n + 2k) \).

\[ p_{n-1} + \cdots + p_0 \equiv R(1) \pmod{p} \]
\[ p_{n-1}2^{n-1} + \cdots + p_0 \equiv R(2) \pmod{p} \]
\[ \cdots \]
\[ p_{n-1}i^{n-1} + \cdots + p_0 \equiv R(i) \pmod{p} \]
\[ \cdots \]
\[ p_{n-1}(m)^{n-1} + \cdots + p_0 \equiv R(m) \pmod{p} \]

Error!! .... Where???
In general,

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\end{align*}
\]

Error!! .... Where???
Could be anywhere!!!
In general,

\[ P(x) = p_{n-1}x^{n-1} + \cdots p_0 \] and receive \( R(1), \ldots R(m = n + 2k) \).

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&\quad \ldots \\
p_{n-1}i^{n-1} + \cdots p_0 &\equiv R(i) \pmod{p} \\
&\quad \ldots \\
p_{n-1}(m)^{n-1} + \cdots p_0 &\equiv R(m) \pmod{p}
\end{align*}
\]

Error!! .... Where???
Could be anywhere!!! ...so try everywhere.
In general,

\[ P(x) = p_{n-1}x^{n-1} + \cdots + p_0 \] and receive \( R(1), \ldots, R(m = n + 2k) \).

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p_{n-1} + \cdots + p_0 & \equiv R(1) \pmod{p} \\
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\end{align*}
\]

Error!! .... Where???
Could be anywhere!!! ...so try everywhere.
**Runtime:** \( \binom{n+2k}{k} \) possibilities.
In general,

\[ P(x) = p_{n-1}x^{n-1} + \cdots p_0 \] and receive \( R(1), \ldots R(m = n + 2k) \).

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\begin{align*}
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  \vdots \ & \vdots \\
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  \vdots \ & \vdots \\
p_{n-1}(m)^{n-1} + \cdots p_0 & \equiv R(m) \pmod{p}
\end{align*}
\]

Error!! .... Where???
Could be anywhere!!! ...so try everywhere.

**Runtime:** \( \binom{n+2k}{k} \) possibilities.

Something like \( (n/k)^k \) ...Exponential in \( k \)!
In general, 

\[ P(x) = p_{n-1}x^{n-1} + \cdots + p_0 \] and receive \( R(1), \ldots, R(m = n + 2k) \).

\[
p_{n-1} + \cdots + p_0 \equiv R(1) \pmod{p} \\
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p_{n-1}i^{n-1} + \cdots + p_0 \equiv R(i) \pmod{p} \\
\vdots \\
p_{n-1}(m)^{n-1} + \cdots + p_0 \equiv R(m) \pmod{p}
\]

Error!! .... Where???
Could be anywhere!!! ...so try everywhere.

**Runtime:** \( \binom{n+2k}{k} \) possibilities.

Something like \((n/k)^k\) ...Exponential in \(k\)!

How do we find where the bad packets are efficiently?!?!?!
Ditty...

Oh where, Oh where

Oh where, Oh where
Ditty...

Oh where, Oh where
has my little dog gone?
Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be
Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Ditty...

Oh where, Oh where has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?
Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
Ditty...

Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
have my packets gone..
Oh where, Oh where has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where have my packets gone.. wrong?
Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be
With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
have my packets gone.. wrong?
Oh where, oh where do they not fit.
Oh where, Oh where has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where have my packets gone.. wrong?
Oh where, oh where do they not fit.
With the polynomial well put
Ditty...

Oh where, Oh where has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where have my packets gone.. wrong?
Oh where, oh where do they not fit.

With the polynomial well put
But the channel a bit wrong
Oh where, Oh where
has my little dog gone?
Oh where, oh where can he be

With his ears cut short
And his tail cut long
Oh where, oh where can he be?

Oh where, Oh where
have my packets gone.. wrong?
Oh where, oh where do they not fit.

With the polynomial well put
But the channel a bit wrong
Where, oh where do we look?
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}\]
Where oh where can my bad packets be?

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}
\]
\[
(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}
\]
\[
\vdots
\]
\[
(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}
\]
Where oh where can my bad packets be?

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p} \\
(p_{n-1} 2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p} \\
\vdots \\
(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}
\]

**Idea:** Multiply equation \( i \) by 0 if and only if \( P(i) \neq R(i) \).
Where oh where can my bad packets be?

\[
(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}
\]

\[
(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2) \pmod{p}
\]

\[
\vdots
\]

\[
(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\). Zero times anything is zero!!!!!
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p} \]
\[(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p} \]
\[\vdots\]
\[(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p} \]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\). Zero times anything is zero!!!! My love is won.
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}\]

\[0 \times (p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}\]

\[\vdots\]

\[(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!!! My love is won.

All equations satisfied!!!!!
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}\]
\[(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2) \pmod{p}\]
\[\vdots\]
\[(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k) \pmod{p}\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).
Zero times anything is zero!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0?
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}\]
\[(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}\]
\[\vdots\]
\[(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}\]

Idea: Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).
Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...
Where oh where can my **bad packets** be?

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p} \\
(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p} \\
\vdots \\
(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!!! My love is won.

All equations satisfied!!!!!

But which equations should we multiply by 0? **Where oh where...??**
Where oh where can my bad packets be?

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}
\]
\[
(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}
\]
\[
\vdots
\]
\[
(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!! My love is won.

All equations satisfied!!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!!
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}\]
\[(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}\]
\[\vdots\]
\[(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).
Zero times anything is zero!!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know.
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}\]

\[(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2) \pmod{p}\]

\[\vdots\]

\[(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k) \pmod{p}\]

Idea: Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!! My love is won.

All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!
Where oh where can my **bad packets** be?

\[
(p_{n-1} + \cdots p_0) \equiv R(1) \pmod{p}
\]

\[
(p_{n-1} 2^{n-1} + \cdots p_0) \equiv R(2) \pmod{p}
\]

\[\vdots\]

\[
(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!!! My love is won.

All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

**We will use a polynomial!!!** That we don’t know. But can find!

Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)
Where oh where can my **bad** packets be?

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p} \\
(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p} \\
\vdots \\
(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).
Zero times anything is zero!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!

Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)

**Error locator polynomial:** \(E(x) = (x - e_1)\)
Where oh where can my bad packets be?

\[(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}\]
\[(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}\]
\[\vdots\]
\[(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!

Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).

**Error locator polynomial:** \(E(x) = (x - e_1)(x - e_2)\)
Where oh where can my **bad packets** be?

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p} \\
(p_{n-1} 2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p} \\
\vdots \\
(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).
Zero times anything is zero!!!! My love is won.
All equations satisfied!!!!!

But which equations should we multiply by 0? **Where oh where...??**

We will use a polynomial!!! **That we don’t know. But can find!**

Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)

**Error locator polynomial:** \(E(x) = (x - e_1)(x - e_2)\ldots\)
Where oh where can my bad packets be?

\[
\begin{align*}
(p_{n-1} + \cdots p_0) & \equiv R(1) \pmod p \\
(p_{n-1} 2^{n-1} + \cdots p_0) & \equiv R(2) \pmod p \\
& \vdots \\
(p_{n-1} (m)^{n-1} + \cdots p_0) & \equiv R(n + 2k) \pmod p
\end{align*}
\]

**Idea:** Multiply equation \( i \) by 0 if and only if \( P(i) \neq R(i) \). Zero times anything is zero!!!! My love is won. All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!

Errors at points \( e_1, \ldots, e_k \). (In diagram above, \( e_1 = 2 \).)

**Error locator polynomial:** \( E(x) = (x - e_1)(x - e_2) \cdots (x - e_k) \).
Where oh where can my bad packets be?

\[
(p_{n-1} + \cdots + p_0) \equiv R(1) \pmod{p}
\]
\[
(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2) \pmod{p}
\]
\[
\vdots
\]
\[
(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!! My love is won.

All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!

Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)

**Error locator polynomial:** \(E(x) = (x - e_1)(x - e_2)\ldots(x - e_k)\).

\(E(i) = 0\) if and only if \(e_j = i\) for some \(j\)
Where oh where can my bad packets be?

\[
E(1)(p_{n-1} + \cdots + p_0) \equiv R(1)E(1) \pmod{p}
\]
\[
E(2)(p_{n-1}2^{n-1} + \cdots + p_0) \equiv R(2)E(2) \pmod{p}
\]
\[
\vdots
\]
\[
E(m)(p_{n-1}(m)^{n-1} + \cdots + p_0) \equiv R(n+2k)E(m) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!!! My love is won.

All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!

Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)

**Error locator polynomial:** \(E(x) = (x - e_1)(x - e_2)\ldots(x - e_k)\).

\(E(i) = 0\) if and only if \(e_j = i\) for some \(j\)

Multiply equations by \(E(\cdot)\).
Where oh where can my **bad packets** be?

\[
E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p} \\
E(2)(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2)E(2) \pmod{p} \\
\vdots \\
E(m)(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k)E(m) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\). Zero times anything is zero!!!!! My love is won. All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!

Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)

**Error locator polynomial:** \(E(x) = (x - e_1)(x - e_2)\ldots(x - e_k)\).

\(E(i) = 0\) if and only if \(e_j = i\) for some \(j\)

Multiply equations by \(E(\cdot)\). (Above \(E(x) = (x-2)\).)
Where oh where can my bad packets be?

\[
E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p}
\]
\[
E(2)(p_{n-1}2^{n-1} + \cdots p_0) \equiv R(2)E(2) \pmod{p}
\]
\[\vdots\]
\[
E(m)(p_{n-1}(m)^{n-1} + \cdots p_0) \equiv R(n+2k)E(m) \pmod{p}
\]

**Idea:** Multiply equation \(i\) by 0 if and only if \(P(i) \neq R(i)\).

Zero times anything is zero!!!! My love is won.

All equations satisfied!!!!!

But which equations should we multiply by 0? Where oh where...??

We will use a polynomial!!! That we don’t know. But can find!

Errors at points \(e_1, \ldots, e_k\). (In diagram above, \(e_1 = 2\).)

**Error locator polynomial:** \(E(x) = (x - e_1)(x - e_2)\cdots(x - e_k)\).

\(E(i) = 0\) if and only if \(e_j = i\) for some \(j\)

Multiply equations by \(E(\cdot)\). (Above \(E(x) = (x-2)\).)

All equations satisfied!!
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

$$(p_2 + p_1 + p_0) \equiv (3) \pmod{7}$$

$$(4p_2 + 2p_1 + p_0) \equiv (1) \pmod{7}$$

$$(2p_2 + 3p_1 + p_0) \equiv (6) \pmod{7}$$

$$(2p_2 + 4p_1 + p_0) \equiv (0) \pmod{7}$$

$$(4p_2 + 5p_1 + p_0) \equiv (3) \pmod{7}$$
Example.

Received \( R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3 \)

Find \( P(x) = p_2 x^2 + p_1 x + p_0 \) that contains \( n + k = 3 + 1 \) points.

Plugin points...

\[
(p_2 + p_1 + p_0) \equiv (3) \pmod{7} \\
(4p_2 + 2p_1 + p_0) \equiv (1) \pmod{7} \\
(2p_2 + 3p_1 + p_0) \equiv (6) \pmod{7} \\
(2p_2 + 4p_1 + p_0) \equiv (0) \pmod{7} \\
(4p_2 + 5p_1 + p_0) \equiv (3) \pmod{7}
\]

Error locator polynomial: \( (x - 2) \).
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

\[
(1 - 2)(p_2 + p_1 + p_0) \equiv (3)(1 - 2) \quad \text{(mod 7)}
\]
\[
(2 - 2)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - 2) \quad \text{(mod 7)}
\]
\[
(3 - 2)(2p_2 + 3p_1 + p_0) \equiv (6)(3 - 2) \quad \text{(mod 7)}
\]
\[
(4 - 2)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - 2) \quad \text{(mod 7)}
\]
\[
(5 - 2)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - 2) \quad \text{(mod 7)}
\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. 
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

\[
(1 - 2)(p_2 + p_1 + p_0) \equiv (3)(1 - 2) \pmod{7}
\]

\[
(2 - 2)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - 2) \pmod{7}
\]

\[
(3 - 2)(2p_2 + 3p_1 + p_0) \equiv (6)(3 - 2) \pmod{7}
\]

\[
(4 - 2)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - 2) \pmod{7}
\]

\[
(5 - 2)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - 2) \pmod{7}
\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

$$(1 - 2)(p_2 + p_1 + p_0) \equiv (3)(1 - 2) \pmod{7}$$

$$(2 - 2)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - 2) \pmod{7}$$

$$(3 - 2)(2p_2 + 3p_1 + p_0) \equiv (6)(3 - 2) \pmod{7}$$

$$(4 - 2)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - 2) \pmod{7}$$

$$(5 - 2)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - 2) \pmod{7}$$

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial!
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

\[
(1 - 2)(p_2 + p_1 + p_0) \equiv (3)(1 - 2) \pmod{7}
\]

\[
(2 - 2)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - 2) \pmod{7}
\]

\[
(3 - 2)(2p_2 + 3p_1 + p_0) \equiv (6)(3 - 2) \pmod{7}
\]

\[
(4 - 2)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - 2) \pmod{7}
\]

\[
(5 - 2)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - 2) \pmod{7}
\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial! Do know form:
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

\[
(1 - 2)(p_2 + p_1 + p_0) \equiv (3)(1 - 2) \pmod{7}
\]
\[
(2 - 2)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - 2) \pmod{7}
\]
\[
(3 - 2)(2p_2 + 3p_1 + p_0) \equiv (6)(3 - 2) \pmod{7}
\]
\[
(4 - 2)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - 2) \pmod{7}
\]
\[
(5 - 2)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - 2) \pmod{7}
\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial! Do know form: $(x - e)$. 
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

\[
(1 - e)(p_2 + p_1 + p_0) \equiv (3)(1 - e) \pmod 7
\]
\[
(2 - e)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - e) \pmod 7
\]
\[
(3 - e)(2p_2 + 3p_1 + p_0) \equiv (3)(3 - e) \pmod 7
\]
\[
(4 - e)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - e) \pmod 7
\]
\[
(5 - e)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - e) \pmod 7
\]

Error locator polynomial: $(x - 2)$.

Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial! Do know form: $(x - e)$. 
Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$

Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

Plugin points...

$$
(1 - e)(p_2 + p_1 + p_0) \equiv (3)(1 - e) \pmod 7 \\
(2 - e)(4p_2 + 2p_1 + p_0) \equiv (1)(2 - e) \pmod 7 \\
(3 - e)(2p_2 + 3p_1 + p_0) \equiv (3)(3 - e) \pmod 7 \\
(4 - e)(2p_2 + 4p_1 + p_0) \equiv (0)(4 - e) \pmod 7 \\
(5 - e)(4p_2 + 5p_1 + p_0) \equiv (3)(5 - e) \pmod 7
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4 unknowns ($p_0, p_1, p_2$ and $e$),
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Multiply equation $i$ by $(i - 2)$. All equations satisfied!

But don’t know error locator polynomial! Do know form: $(x - e)$.

4 unknowns ($p_0, p_1, p_2$ and $e$), 5 nonlinear equations.
.. turn their heads each day,

\[
(\rho_{n-1} + \cdots \rho_0) \equiv R(1) \pmod{p}
\]

\[
\vdots
\]

\[
(\rho_{n-1} i^{n-1} + \cdots \rho_0) \equiv R(i) \pmod{p}
\]

\[
\vdots
\]

\[
(\rho_{n-1} (n+2k)^{n-1} + \cdots \rho_0) \equiv R(m) \pmod{p}
\]

...so satisfied, I'm on my way.

\( m = n + 2k \) satisfied equations, \( n + k \) unknowns. But nonlinear!

Let \( Q(x) = E(x) P(x) = a_n + k - 1 x^n + k - 1 + \cdots + a_0 \). Equations:

\[
Q(i) = R(i) E(i)
\]
..turn their heads each day,

\[
E(1)(p_{n-1} + \cdots p_0) \equiv R(1)E(1) \pmod{p} \\
\vdots \\
E(i)(p_{n-1}i^{n-1} + \cdots p_0) \equiv R(i)E(i) \pmod{p} \\
\vdots \\
E(m)(p_{n-1}(n+2k)^{n-1} + \cdots p_0) \equiv R(m)E(m) \pmod{p}
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\[
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\]

and linear in \(a_i\) and coefficients of \(E(x)\)!
Finding $Q(x)$ and $E(x)$?
Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree $k$
Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree $k$ ...

$$E(x) = x^k + b_{k-1}x^{k-1} \cdots b_0.$$
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$\implies k$ (unknown) coefficients.
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- $Q(x) = P(x)E(x)$ has degree $n + k - 1$
Finding $Q(x)$ and $E(x)$?

- $E(x)$ has degree $k$ ...

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- $Q(x) = P(x)E(x)$ has degree $n + k - 1$ ...

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Number of unknown coefficients:
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Number of unknown coefficients: $n + 2k$. 

Solving for $Q(x)$ and $E(x)$...

For all points $1, \ldots, i, n + 2k = m$,

$$Q(i) = R(i)E(i) \pmod{p}$$
Solving for $Q(x)$ and $E(x)$...

For all points $1, \ldots, i, n+2k = m$,

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Gives $n+2k$ linear equations.
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$$a_{n+k-1} + \cdots a_0 \equiv R(1)(1 + b_{k-1} \cdots b_0) \pmod{p}$$
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\vdots
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$$a_{n+k-1}(m)^{n+k-1} + \ldots a_0 \equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \ldots b_0) \pmod{p}$$
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$$\vdots$$

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..and $n + 2k$ unknown coefficients of $Q(x)$ and $E(x)$!
Solving for $Q(x)$ and $E(x)$...

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..and $n+2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$. 
Solving for $Q(x)$ and $E(x)$...and $P(x)$

For all points $1, \ldots, i, n+2k = m$,

$$Q(i) = R(i)E(i) \pmod{p}$$

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Solve for coefficients of $Q(x)$ and $E(x)$.

Find $P(x) = Q(x)/E(x)$. 
Solving for $Q(x)$ and $E(x)$...and $P(x)$

For all points $1, \ldots, i, n+2k = m,$

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Solve for coefficients of $Q(x)$ and $E(x)$.

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Example.

Received $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$
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\[ E(x) = x - b_0 \]

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\[
\begin{align*}
    a_3 + a_2 + a_1 + a_0 & \equiv 3(1 - b_0) \pmod{7} \\
    a_3 + 4a_2 + 2a_1 + a_0 & \equiv 1(2 - b_0) \pmod{7} \\
    6a_3 + 2a_2 + 3a_1 + a_0 & \equiv 6(3 - b_0) \pmod{7} \\
    a_3 + 2a_2 + 4a_1 + a_0 & \equiv 0(4 - b_0) \pmod{7} \\
    6a_3 + 4a_2 + 5a_1 + a_0 & \equiv 3(5 - b_0) \pmod{7}
\end{align*}
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    a_3 + 2a_2 + 4a_1 + a_0 & \equiv 0(4 - b_0) \pmod{7} \\
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$a_3 = 1, a_2 = 6, a_1 = 6, a_0 = 5$ and $b_0 = 2$. 
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\end{align*}
\]

$a_3 = 1, a_2 = 6, a_1 = 6, a_0 = 5$ and $b_0 = 2$.

$Q(x) = x^3 + 6x^2 + 6x + 5$. 
Example.

Received $R(1) = 3$, $R(2) = 1$, $R(3) = 6$, $R(4) = 0$, $R(5) = 3$

$Q(x) = E(x)P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$E(x) = x - b_0$

$Q(i) = R(i)E(i)$.

\[
\begin{align*}
    a_3 + a_2 + a_1 + a_0 & \equiv 3(1 - b_0) \pmod{7} \\
    a_3 + 4a_2 + 2a_1 + a_0 & \equiv 1(2 - b_0) \pmod{7} \\
    6a_3 + 2a_2 + 3a_1 + a_0 & \equiv 6(3 - b_0) \pmod{7} \\
    a_3 + 2a_2 + 4a_1 + a_0 & \equiv 0(4 - b_0) \pmod{7} \\
    6a_3 + 4a_2 + 5a_1 + a_0 & \equiv 3(5 - b_0) \pmod{7}
\end{align*}
\]

$a_3 = 1$, $a_2 = 6$, $a_1 = 6$, $a_0 = 5$ and $b_0 = 2$.

$Q(x) = x^3 + 6x^2 + 6x + 5$.

$E(x) = x - 2$. 
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c|ccccc}
\text{x} & x^3 & + & 6x^2 & + & 6x & + & 5 \\
\hline
\text{x} - 2 & \\
\end{array}
\]

What is \( x - 2 \)? Except at \( x = 2 \)?

Hole there?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c}
 & 1 \ x^2 \\
\hline
x - 2 \ ) \ x^3 + 6 \ x^2 + 6 \ x + 5 \\
\quad x^3 - 2 \ x^2 \\
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]

Message is

\[ P(1) = 3, \quad P(2) = 0, \quad P(3) = 6. \]

What is \( x - 2 \)?

Except at \( x = 2 \)?

Hole there?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{r}
1 x^2 \\
\hline
x - 2 ) x^3 + 6x^2 + 6x + 5 \\
\hline
x^3 - 2x^2 \\
\hline
1 x^2 + 6x + 5
\end{array}
\]

Message is

\[ P(1) = 3, \quad P(2) = 0, \quad P(3) = 6. \]

What is \( x - 2 \)?

Except at \( x = 2 \)?

Hole there?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[ \begin{align*}
\text{1 } x^2 & \quad + \quad 1 \ x \\
\hline
\text{x - 2) } x^3 & \quad + \quad 6 \ x^2 & \quad + \quad 6 \ x & \quad + \quad 5 \\
\text{x^3} & \quad - \quad 2 \ x^2 \\
\hline
\text{1 } x^2 & \quad + \quad 6 \ x & \quad + \quad 5 \\
\text{1 } x^2 & \quad - \quad 2 \ x
\end{align*} \]
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{r}
1 \ x^2 & + 1 \ x \\
\hline
x - 2 ) x^3 & + 6 \ x^2 & + 6 \ x & + 5 \\
x^3 & - 2 \ x^2 \\
\hline
1 \ x^2 & + 6 \ x & + 5 \\
1 \ x^2 & - 2 \ x \\
\hline
x & + 5
\end{array}
\]
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c|c}
1 & x^2 + 1 \cr \hline
x - 2 & x^3 + 6x^2 + 6x + 5 \cr x^3 - 2x^2 & \hline
1 & x^2 + 6x + 5 \cr 1 & x^2 - 2x \cr \hline
x + 5 \cr x - 2
\end{array}
\]

Message is
\[ P(x) = x^2 + x + 1 \]
\[ P(1) = 3, \quad P(2) = 0, \quad P(3) = 6. \]

What is \( x - 2 \)?

Except at \( x = 2 \)?

Hole there?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c}
1 \ x^2 + 1 \ x + 1 \\
\hline
x - 2 ) \ x^3 + 6 \ x^2 + 6 \ x + 5 \\
\hline
x^3 - 2 \ x^2 \\
\hline
1 \ x^2 + 6 \ x + 5 \\
\hline
1 \ x^2 - 2 \ x \\
\hline
x + 5 \\
\hline
x - 2 \\
\hline
----- \\
0
\end{array}
\]

Message is

\[ P(1) = 3, \ P(2) = 0, \ P(3) = 6. \]

What is \( x - 2 \)?

Except at \( x = 2 \)?

Hole there?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c}
1 \ x^2 + 1 \ x + 1 \\
\hline
x - 2 \ \\
\end{array}
\]

\[
\begin{array}{c}
x^3 + 6 \ x^2 + 6 \ x + 5 \\
\hline
x^3 - 2 \ x^2 \\
\hline
1 \ x^2 + 6 \ x + 5 \\
\hline
1 \ x^2 - 2 \ x \\
\hline
 \ x + 5 \\
 \ x - 2 \\
\hline
 \ 0 \\
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c}
\phantom{0}1 \ x^2 + 1 \ x + 1 \\
\hline
x - 2 \mid x^3 + 6x^2 + 6x + 5 \\
\phantom{0}x^3 - 2x^2 \\
\hline
1 \ x^2 + 6 \ x + 5 \\
\phantom{0}1 \ x^2 - 2 \ x \\
\hline
\phantom{0}x + 5 \\
\phantom{0}x - 2 \\
\hline
0
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c|ccccc}
& x^3 & + & 6x^2 & + & 6x & + & 5 \\
\hline
x - 2 & x^3 & - & 2x^2 & | & & \\
& & & & & & \\
& & & & & 1x^2 & + & 1x & + & 1 \\
\hline
& & & & & 1x^2 & + & 6x & + & 5 \\
& & & & & 1x^2 & - & 2x \\
\hline
& & & & & x & + & 5 \\
& & & & & x & - & 2 \\
\]

\[ x - 2 \]

\[ P(x) = x^2 + x + 1 \]
Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)

What is \( \frac{x-2}{x-2} \)?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{r}
1 \quad x^2 + 1 \quad x + 1 \\
\hline
x - 2 \quad ) \quad x^3 + 6 \quad x^2 + 6 \quad x + 5 \\
\quad x^3 - 2 \quad x^2 \\
\hline
\quad 1 \quad x^2 + 6 \quad x + 5 \\
\quad 1 \quad x^2 - 2 \quad x \\
\hline
\quad x + 5 \\
\quad x - 2 \\
\hline
\quad 0
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)
What is \( \frac{x-2}{x-2} \)? 1
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[ \begin{array}{c}
1 \ x^2 + 1 \ x + 1 \\
\hline
x - 2 \big) x^3 + 6x^2 + 6x + 5 \\
\quad x^3 - 2x^2 \\
\hline
1 \ x^2 + 6 \ x + 5 \\
1 \ x^2 - 2 \ x \\
\hline
x + 5 \\
x - 2 \\
\hline
0
\end{array} \]

\[ P(x) = x^2 + x + 1 \]
Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)

What is \( \frac{x-2}{x-2} \)? 1
Except at \( x = 2 \)?
Example: finishing up.

\[ Q(x) = x^3 + 6x^2 + 6x + 5. \]
\[ E(x) = x - 2. \]

\[
\begin{array}{c}
1.6.1 + 1.6 + 1 \\
\hline
1.6^2 + 6x + 5 \\
1.6^2 - 2x \\
\hline
x + 5 \\
x - 2 \\
\hline
0
\end{array}
\]

\[ P(x) = x^2 + x + 1 \]
Message is \( P(1) = 3, P(2) = 0, P(3) = 6. \)

What is \( \frac{x-2}{x-2} \)?

Except at \( x = 2 \)? Hole there?
Error Correction: Berlekamp-Welsh

Message: \( m_1, \ldots, m_n \).

**Sender:**

1. Form degree \( n-1 \) polynomial \( P(x) \) where \( P(i) = m_i \).
2. Send \( P(1), \ldots, P(n+2k) \).

**Receiver:**

1. Receive \( R(1), \ldots, R(n+2k) \).
2. Solve \( n+2k \) equations, \( Q(i) = E(i)R(i) \) to find \( Q(x) = E(x)P(x) \) and \( E(x) \).
3. Compute \( P(x) = Q(x)/E(x) \).
4. Compute \( P(1), \ldots, P(n) \).
Check your understanding.

You have error locator polynomial!
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor?
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values?
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Check your understanding.

You have error locator polynomial!

Where oh where have my packets gone wrong?

Factor? Sure.
Check all values? Sure.
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Efficiency?
You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Efficiency? Sure.
You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Efficiency? Sure. Only $n+2k$ values.
Check your understanding.

You have error locator polynomial!
Where oh where have my packets gone wrong?
Factor? Sure.
Check all values? Sure.
Efficiency? Sure. Only $n + 2k$ values.
    See where it is 0.
Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?
Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?

**Existence:** there is a $P(x)$ and $E(x)$ that satisfy equations.
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hfill (1)
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)

**Proof:**

Can cross divide at $n$ points.
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$ \hspace{1cm} (1)

**Proof:**
We claim

...
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$
\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).
$$

(1)

**Proof:**

We claim

$$
Q'(x)E(x) = Q(x)E'(x) \text{ on } n + 2k \text{ values of } x.
$$

(2)
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$

(1)

**Proof:**

We claim

$$Q'(x)E(x) = Q(x)E'(x)$$

on $n + 2k$ values of $x$. (2)

Equation 2 implies 1:
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  

\(1\)

**Proof:**

We claim

$$Q'(x)E(x) = Q(x)E'(x)$$
on \(n+2k\) values of \(x\).  

\(2\)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)

**Proof:**
We claim

$$Q'(x)E(x) = Q(x)E'(x)$$ on $n + 2k$ values of $x$. \hspace{1cm} (2)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n + 2k - 1$ and agree on $n + 2k$ points.
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  

(1)

**Proof:**
We claim

$$Q'(x)E(x) = Q(x)E'(x)$$  on $n+2k$ values of $x$.  

(2)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$

and agree on $n+2k$ points

$E(x)$ and $E'(x)$ have at most $k$ zeros each.
**Unique solution for \( P(x) \)**

**Uniqueness:** any solution \( Q'(x) \) and \( E'(x) \) have

\[
\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).
\]  
(1)

**Proof:**

We claim

\[
Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x.
\]  
(2)

Equation 2 implies 1:

\( Q'(x)E(x) \) and \( Q(x)E'(x) \) are degree \( n+2k-1 \)
and agree on \( n+2k \) points
\( E(x) \) and \( E'(x) \) have at most \( k \) zeros each.
Can cross divide at \( n \) points.
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$

(1)

**Proof:**

We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x.$$  

(2)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$ and agree on $n+2k$ points.

$E(x)$ and $E'(x)$ have at most $k$ zeros each.

Can cross divide at $n$ points.

$$\Rightarrow \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} \text{ equal on } n \text{ points.}$$
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x). \quad (1)$$

**Proof:**

We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x. \quad (2)$$

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$
and agree on $n+2k$ points

$E(x)$ and $E'(x)$ have at most $k$ zeros each.
Can cross divide at $n$ points.

$$\implies \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} \text{ equal on } n \text{ points.}$$

Both degree $\leq n-1$
Unique solution for $P(x)$

Uniqueness: any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)

Proof:
We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x.$$ \hspace{1cm} (2)

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$
and agree on $n+2k$ points.
$E(x)$ and $E'(x)$ have at most $k$ zeros each.
Can cross divide at $n$ points.

$$\Rightarrow \quad \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} \text{ equal on } n \text{ points.}$$

Both degree $\leq n-1 \quad \Rightarrow \quad \text{Same polynomial!}$
Unique solution for $P(x)$

**Uniqueness:** any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x).$$  \hspace{1cm} (1)

**Proof:**

We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x. \hspace{1cm} (2)$$

Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$

and agree on $n+2k$ points

$E(x)$ and $E'(x)$ have at most $k$ zeros each.

Can cross divide at $n$ points.

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} \text{ equal on } n \text{ points.}$$

Both degree $\leq n-1 \implies$ Same polynomial!
Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n + 2k$ values of $x$. 
Last bit.

Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof:
Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n + 2k$ values of $x$.

Proof: Construction implies that
Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n + 2k$ values of $x$.

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

Cross multiplying gives equality in fact for these points. Points to polynomials, have to deal with zeros!

Example: dealing with $x^2 - 2x - 2$ at $x = 2$. 

Last bit.
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n+2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n+2k\} \).
Last bit.

**Fact:** \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

**Proof:** Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \).
Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n + 2k$ values of $x$.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$. 

Cross multiplying gives equality in fact for these points.

Points to polynomials, have to deal with zeros!

Example: dealing with $x - 2$ at $x = 2$. 
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i)
\]
\[
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots, n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[ \Rightarrow Q(i)E'(i) = Q'(i)E(i) \] holds when \( E(i) \) or \( E'(i) \) are zero.
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i)  \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[ \Rightarrow Q(i)E'(i) = Q'(i)E(i) \] holds when \( E(i) \) or \( E'(i) \) are zero.

When \( E'(i) \) and \( E(i) \) are not zero
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[ \implies Q(i)E'(i) = Q'(i)E(i) \]

holds when \( E(i) \) or \( E'(i) \) are zero.

When \( E'(i) \) and \( E(i) \) are not zero

\[
\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).
\]
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n + 2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots n + 2k\} \).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[ \implies Q(i)E'(i) = Q'(i)E(i) \text{ holds when } E(i) \text{ or } E'(i) \text{ are zero.} \]

When \( E'(i) \) and \( E(i) \) are not zero

\[
\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).
\]

Cross multiplying gives equality in fact for these points.
Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n + 2k$ values of $x$.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.

$$\implies Q(i)E'(i) = Q'(i)E(i)$$
holds when $E(i)$ or $E'(i)$ are zero.

When $E'(i)$ and $E(i)$ are not zero

$$\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).$$

Cross multiplying gives equality in fact for these points. $$\Box$$
Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n + 2k$ values of $x$.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n + 2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.

$$\implies Q(i)E'(i) = Q'(i)E(i)$$ holds when $E(i)$ or $E'(i)$ are zero.

When $E'(i)$ and $E(i)$ are not zero

$$\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).$$

Cross multiplying gives equality in fact for these points. \[ \square \]

Points to polynomials, have to deal with zeros!
Fact: \( Q'(x)E(x) = Q(x)E'(x) \) on \( n+2k \) values of \( x \).

Proof: Construction implies that

\[
Q(i) = R(i)E(i) \\
Q'(i) = R(i)E'(i)
\]

for \( i \in \{1, \ldots, n+2k\}\).

If \( E(i) = 0 \), then \( Q(i) = 0 \). If \( E'(i) = 0 \), then \( Q'(i) = 0 \).

\[
\implies Q(i)E'(i) = Q'(i)E(i)
\]

holds when \( E(i) \) or \( E'(i) \) are zero.

When \( E'(i) \) and \( E(i) \) are not zero

\[
\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).
\]

Cross multiplying gives equality in fact for these points.

Points to polynomials, have to deal with zeros!

Example: dealing with \( \frac{x-2}{x-2} \) at \( x = 2 \).
Yaay!!

Berlekamp-Welsh algorithm decodes correctly when $k$ errors!
Poll

Say you sent a message of length 4, encoded as $P(x)$ where one sends packets $P(1),...P(8)$.

You receive packets $R(1),...R(8)$.

Packets 1 and 4 are corrupted.

(A) $R(1) \neq P(1)$
(B) The degree of $P(x)E(x) = 3 + 2 = 5$.
(C) The degree of $E(x)$ is 2.
(D) The number of coefficients of $P(x)$ is 4.
(E) The number of coefficients of $P(x)Q(x)$ is 6.
Say you sent a message of length 4, encoded as $P(x)$ where one sends packets $P(1), \ldots, P(8)$.

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(E) is false.

(A) $E(x) = (x - 1)(x - 4)$
(B) The number of coefficients in $E(x)$ is 2.
(C) The number of unknown coefficients in $E(x)$ is 2.
(D) $E(x) = (x - 1)(x - 2)$
(E) $R(4) \neq P(4)$
(F) The degree of $R(x)$ is 5.
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(D) $E(x) = (x - 1)(x - 2)$
(E) $R(4) \neq P(4)$
(F) The degree of $R(x)$ is 5.

(A), (C), (E). (F) doesn’t type check!
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets?
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode?
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

How many packets? \( n + k \)
How to encode? With polynomial, \( P(x) \).
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
Of degree?

Reed-Solomon codes.
Welsh-Berlekamp Decoding.
Perfection!
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

How many packets? \( n + k \)
How to encode? With polynomial, \( P(x) \).
Of degree? \( n - 1 \)
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
Of degree? $n - 1$
Recover?
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

How many packets? \( n + k \)
How to encode? With polynomial, \( P(x) \).
Of degree? \( n - 1 \)
Recover? Reconstruct \( P(x) \) with any \( n \) points!
Communicate $n$ packets, with $k$ erasures.

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Summary. Error Correction.

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Communicate $n$ packets, with $k$ errors.

How many packets? $n + 2k$
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
Of degree? $n - 1$
Recover? Reconstruct $P(x)$ with any $n$ points!

Communicate $n$ packets, with $k$ errors.

How many packets? $n + 2k$
Why?
Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
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Communicate $n$ packets, with $k$ errors.

How many packets? $n + 2k$
Why?
  $k$ changes to make diff. messages overlap
Summary. Error Correction.

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Summary. Error Correction.

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Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.
   How many packets? $n + k$
   How to encode? With polynomial, $P(x)$.
   Of degree? $n - 1$
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Communicate $n$ packets, with $k$ errors.
   How many packets? $n + 2k$
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Summary. Error Correction.

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Nonlinear equations.
Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$.
Linear equations.
Polynomial division!
Reed-Solomon codes.
Welsh-Berlekamp Decoding.
Perfection!
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Communicate $n$ packets, with $k$ erasures.

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How to encode? With polynomial, $P(x)$. Of degree? $n - 1$.
Recover?
  Reconstruct error polynomial, $E(X)$, and $P(x)$!
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

- How many packets? \( n + k \)
- How to encode? With polynomial, \( P(x) \).
- Of degree? \( n - 1 \)
- Recover? Reconstruct \( P(x) \) with any \( n \) points!

Communicate \( n \) packets, with \( k \) errors.

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Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
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Summary. Error Correction.

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Reconstruct $E(x)$ and $Q(x) = E(x)P(x)$. Linear Equations.
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
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Polynomial division!
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

- How many packets? \( n + k \)
- How to encode? With polynomial, \( P(x) \).
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Communicate \( n \) packets, with \( k \) errors.

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  - Reconstruct error polynomial, \( E(X) \), and \( P(x) \)!
    - **Nonlinear equations.**
  - Reconstruct \( E(x) \) and \( Q(x) = E(x)P(x) \). Linear Equations.
  - Polynomial division! \( P(x) = Q(x)/E(x) \)!
Summary. Error Correction.

Communicate \( n \) packets, with \( k \) erasures.

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- Of degree? \( n - 1 \)
- Recover? Reconstruct \( P(x) \) with any \( n \) points!

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Reed-Solomon codes.
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

How many packets? $n + k$
How to encode? With polynomial, $P(x)$.
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Reed-Solomon codes. Welsh-Berlekamp Decoding.
Summary. Error Correction.

Communicate $n$ packets, with $k$ erasures.

- How many packets? $n + k$
- How to encode? With polynomial, $P(x)$.
- Of degree? $n - 1$
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  - Polynomial division! $P(x) = Q(x)/E(x)$!

Reed-Solomon codes. Welsh-Berlekamp Decoding. Perfection!
Cool.

Really Cool!