Today.

Finish up counting.
Today.

Finish up counting.
Countability.
Some Practice.

How many orderings of letters of CAT?

3 ways to choose first letter, 2 ways for second, 1 for last.

$3 \times 2 \times 1 = 3!$ orderings

How many orderings of the letters in ANAGRAM?

Ordered, except for A!

$7!$ total orderings of 7 letters.

$3!$ total “extra counts” or orderings of three A’s?

Total orderings?

$7! \div 3!$ orderings per “unordered object”.

How many orderings of MISSISSIPPI?

4 S’s, 4 I’s, 2 P’s.

11 letters total.

$11!$ ordered objects.

$4! \times 4! \times 2!$ ordered objects per “unordered object”.

$11! \div 4! \div 4! \div 2!$. 
Some Practice.

How many orderings of letters of CAT?
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3 ways to choose first letter, 2 ways for second, 1 for last.
⇒ $3 \times 2 \times 1$
Some Practice.

How many orderings of letters of CAT?
3 ways to choose first letter, 2 ways for second, 1 for last.
\[ \Rightarrow 3 \times 2 \times 1 = 3! \text{ orderings} \]
Some Practice.

How many orderings of letters of CAT?
3 ways to choose first letter, 2 ways for second, 1 for last.

$$\Rightarrow \ 3 \times 2 \times 1 = 3! \text{ orderings}$$

How many orderings of the letters in ANAGRAM?
Some Practice.

How many orderings of letters of CAT?
3 ways to choose first letter, 2 ways for second, 1 for last.
⇒ 3 × 2 × 1 = 3! orderings

How many orderings of the letters in ANAGRAM?
Ordered,
Some Practice.

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total orderings of 7 letters.
Some Practice.

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\[\text{total orderings of 7 letters. } 7!\]
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Total orderings? \( \frac{7!}{3!} \)
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4 S’s, 4 I’s, 2 P’s.
11 letters total.
11! ordered objects.
\[ 4! \times 4! \times 2! \text{ ordered objects per “unordered object”} \]
\[ \Rightarrow 11! / (4! \times 4! \times 2!) \]
Some Practice.

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4 S’s, 4 I’s, 2 P’s.
11 letters total.

11! ordered objects.
4! × 4! × 2! ordered objects per “unordered object”

\[ \Rightarrow \frac{11!}{4!4!2!}. \]
Summary.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \).
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**First rule:** \( n_1 \times n_2 \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
Summary.

**First rule:** $n_1 \times n_2 \cdots \times n_3$.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: \( \frac{n!}{(n-k)!} \).

"n choose k" One-to-one rule: equal in number if one-to-one correspondence. Pause Bijection! Sample $k$ times from $n$ objects with replacement and order doesn't matter: \( k + n - 1 \frac{n!}{(n-k)!} \).
Summary.

**First rule:** \( n_1 \times n_2 \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
Sample without replacement: \( \frac{n!}{(n-k)!} \)

**Second rule:** when order doesn’t matter (sometimes) can divide...
Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$.

Second rule: when order doesn’t matter (sometimes) can divide...

Sample without replacement and order doesn’t matter: \binom{n}{k} = \frac{n!}{(n-k)!k!}.
“$n$ choose $k$”
Summary.

First rule: \( n_1 \times n_2 \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
Sample without replacement: \( \frac{n!}{(n-k)!} \).

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One-to-one rule: equal in number if one-to-one correspondence. pause  Bijection!
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First rule: \( n_1 \times n_2 \cdots \times n_3 \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
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Sample without replacement and order doesn’t matter: \( \binom{n}{k} = \frac{n!}{(n-k)!k!} \).
“n choose k”

One-to-one rule: equal in number if one-to-one correspondence.

Sample \( k \) times from \( n \) objects with replacement and order doesn’t matter: \( \binom{k+n-1}{n-1} \).
Sampling...

Sample $k$ items out of $n$
Sampling...

Sample $k$ items out of $n$

Without replacement:

Order matters:

Without replacement: $n \times n \times \ldots \times n = n^k$

Order does not matter:

Second rule: divide by number of orders

$\Rightarrow n! \div (n-k)!$.

$n\choose k$

With Replacement.

Order matters:

$n \times n \times \ldots \times n = n^k$

Order does not matter:

Second rule

Problem: depends on how many of each item we chose!

Different number of unordered elts map to each unordered elt.

Unordered elt: 1, 2, 3

3! ordered elts map to it.

Unordered elt: 1, 2, 2

3! 2! ordered elts map to it.

How do we deal with this mess?
Sampling...

Sample \( k \) items out of \( n \)

Without replacement:
  Order matters:

With Replacement.
  Order matters:
Sampling...

Sample $k$ items out of $n$

Without replacement:
Order matters: $n \times$

Order does not matter: Second Rule: divide by number of orders

Problem: depends on how many of each item we chose!
Different number of unordered elts map to each unordered elt.

How do we deal with this mess?
Sampling...

Sample $k$ items out of $n$

Without replacement:
  Order matters: $n \times n - 1 \times n - 2 \ldots$
Sampling...

Sample $k$ items out of $n$

Without replacement:
- Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1$
Sampling...

Sample $k$ items out of $n$

Without replacement:
  Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
  Order does not matter:

With Replacement.
  Order matters:
  Order does not matter:
Sampling...

Sample $k$ items out of $n$

Without replacement:
- Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$
- Order does not matter:
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With Replacement.
- Order matters:
- Order does not matter:
  - Second rule ???
Sampling...

Sample $k$ items out of $n$

Without replacement:
  Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
  
  Order does not matter:
  Second Rule: divide by number of orders – “$k!$”

With Replacement.

Order matters: $n \times n \times n \times \ldots \times n$ = $n^k$

Order does not matter:
Second rule: ???
Sampling...

Sample \( k \) items out of \( n \)

Without replacement:
- Order matters: \( n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!} \)
- Order does not matter:
  - Second Rule: divide by number of orders – “\( k! \)”
    \[ \Rightarrow \frac{n!}{(n-k)!k!} \]
Sampling...

Sample $k$ items out of $n$

Without replacement:
  Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
  Order does not matter:
    Second Rule: divide by number of orders – “$k!””
    \[ \frac{n!}{(n-k)!k!} \cdot \]
    “$n$ choose $k$”
Sampling...

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- Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
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    \[ \Rightarrow \frac{n!}{(n-k)!k!} \cdot \]
  
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With Replacement.
Sampling...

Sample $k$ items out of $n$

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  Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$
  Order does not matter:
    Second Rule: divide by number of orders – “$k!”$
    \[\rightarrow \frac{n!}{(n-k)!k!}.\]
    “$n$ choose $k$”

With Replacement.
  Order matters: $n$
Sampling...

Sample $k$ items out of $n$

Without replacement:
  Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
  Order does not matter:
    Second Rule: divide by number of orders – “$k!$”
    \[ \implies \frac{n!}{(n-k)!k!} \cdot \]
    “$n$ choose $k$”

With Replacement.
  Order matters: $n \times n$
Sampling...

Sample $k$ items out of $n$

Without replacement:
- Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
- Order does not matter:
  - Second Rule: divide by number of orders – “$k!$”

  $\implies \frac{n!}{(n-k)!k!} \cdot$

  “$n$ choose $k$”

With Replacement.
- Order matters: $n \times n \times \ldots n$
Sampling...

Sample $k$ items out of $n$

Without replacement:
- Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
- Order does not matter:
  - Second Rule: divide by number of orders – “$k!”$ 
  \[ \implies \frac{n!}{(n-k)!k!} \cdot \]
  “$n$ choose $k”$ 

With Replacement.
- Order matters: $n \times n \times \ldots n = n^k$
Sampling...

Sample $k$ items out of $n$

Without replacement:
- Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
- Order does not matter:
  
  Second Rule: divide by number of orders — “$k!””

\[
\Rightarrow \frac{n!}{(n-k)!k!}.
\]

“$n$ choose $k$”

With Replacement.
- Order matters: $n \times n \times \ldots n = n^k$
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Sampling...

Sample \( k \) items out of \( n \)

Without replacement:
- Order matters: \( n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!} \)
- Order does not matter:
  - Second Rule: divide by number of orders – “\( k! \)”
    \[ \frac{n!}{(n-k)!k!} \]
  - “\( n \) choose \( k \)”

With Replacement.
- Order matters: \( n \times n \times \ldots n = n^k \)
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Sampling...

Sample $k$ items out of $n$

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- Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$
- Order does not matter:
  - Second Rule: divide by number of orders – “$k!”$
  
  $$\frac{n!}{(n-k)!k!} \cdot \text{“n choose } k\text{”}$$

With Replacement.
- Order matters: $n \times n \times \ldots n = n^k$
- Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!
Different number of unordered elts map to each unordered elt.

Unordered elt: 1, 2, 3
3!

ordered elts map to it.

Unordered elt: 1, 2, 2
3!
2!

ordered elts map to it.

How do we deal with this mess???
Sampling...

Sample $k$ items out of $n$

Without replacement:
   Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
   Order does not matter:
      Second Rule: divide by number of orders – “$k!”$
       $\Rightarrow \frac{n!}{(n-k)!k!} \cdot$
      “$n$ choose $k$”

With Replacement.
   Order matters: $n \times n \times \ldots n = n^k$
   Order does not matter: Second rule ???
Sampling...

Sample \( k \) items out of \( n \)

Without replacement:

- Order matters: \( n \times (n-1) \times (n-2) \cdots \times (n-k+1) = \frac{n!}{(n-k)!} \)
- Order does not matter:
  - Second Rule: divide by number of orders – “\( k! \)”
    \[ \implies \frac{n!}{(n-k)!k!} \cdot \]
  - “\( n \) choose \( k \)”

With Replacement.

- Order matters: \( n \times n \times \cdots n = n^k \)
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Problem: depends on how many of each item we chose!
Sampling...

Sample \( k \) items out of \( n \)

Without replacement:
- Order matters: \( n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!} \)
- Order does not matter:
  - Second Rule: divide by number of orders – “\( k! \)”
    \[ \frac{n!}{(n-k)!} \cdot \frac{1}{k!} \]
    “\( n \) choose \( k \)”

With Replacement.
- Order matters: \( n \times n \times \ldots n = n^k \)
- Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!
Different number of unordered elts map to each unordered elt.
Sampling...

Sample $k$ items out of $n$

Without replacement:
- Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
- Order does not matter:
  - Second Rule: divide by number of orders – “$k!”$  
    \[
    \frac{n!}{(n-k)!k!} \cdot \text{“} n \text{ choose } k \text{”} \]

With Replacement.
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Different number of unordered elts map to each unordered elt.
Sampling...

Sample $k$ items out of $n$

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- Order does not matter:
  - Second Rule: divide by number of orders – “$k!”$ 
    \[ \frac{n!}{(n-k)!k!} . \]
  - “$n$ choose $k$”

With Replacement.
- Order matters: $n \times n \times \ldots n = n^k$
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Unordered elt: 1, 2, 3
Sampling...

Sample $k$ items out of $n$

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  Order does not matter:
    Second Rule: divide by number of orders – “$k!”$
    $\Rightarrow \frac{n!}{(n-k)!k!} \cdot$
    “$n$ choose $k”$ 

With Replacement.
  Order matters: $n \times n \times \cdots n = n^k$
  Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!
  Different number of unordered elts map to each unordered elt.

Unordered elt: 1, 2, 3  3! ordered elts map to it.
Sampling...

Sample $k$ items out of $n$

Without replacement:
- Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
- Order does not matter:
  - Second Rule: divide by number of orders – “$k!”
  
  $\Rightarrow \frac{n!}{(n-k)!k!} \cdot$

“$n$ choose $k”$

With Replacement.
- Order matters: $n \times n \times \ldots n = n^k$
- Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!

Different number of unordered els map to each unordered elt.

Unordered elt: 1,2,3 3! ordered els map to it.
Unordered elt: 1,2,2
Sampling...

Sample $k$ items out of $n$

Without replacement:
Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
Order does not matter:
Second Rule: divide by number of orders – “$k$!”

$$\Rightarrow \frac{n!}{(n-k)!k!} \cdot \text{“}n \text{ choose } k\text{”}$$

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Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!

Different number of unordered elts map to each unordered elt.

Unordered elt: 1, 2, 3 \hspace{1cm} 3! ordered elts map to it.
Unordered elt: 1, 2, 2 \hspace{1cm} \frac{3!}{2!} ordered elts map to it.
Sampling...

Sample $k$ items out of $n$

Without replacement:
  Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
  Order does not matter:
    Second Rule: divide by number of orders – “$k!””
    $$\longrightarrow \frac{n!}{(n-k)!k!} \cdot “n \ choose \ k”$$

With Replacement.
  Order matters: $n \times n \times \ldots n = n^k$
  Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!
  Different number of unordered elts map to each unordered elt.

Unordered elt: $1, 2, 3$  \hspace{1em} \text{3! ordered elts map to it.}$
Unordered elt: $1, 2, 2$  \hspace{1em} 3! \hspace{0.5em} \text{2! ordered elts map to it.}$
Sampling...

Sample $k$ items out of $n$

Without replacement:
- Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$
- Order does not matter:
  - Second Rule: divide by number of orders – “$k!””

  \[
  \implies \frac{n!}{(n-k)!k!} \cdot \quad \text{“$n$ choose $k$”}
  \]

With Replacement.
- Order matters: $n \times n \times \ldots n = n^k$
- Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!
Different number of unordered elts map to each unordered elt.

Unordered elt: $1, 2, 3$ \quad $3!$ ordered elts map to it.
Unordered elt: $1, 2, 2$ \quad $\frac{3!}{2!}$ ordered elts map to it.

How do we deal with this
Sampling...

Sample $k$ items out of $n$

Without replacement:
- Order matters: $n \times n - 1 \times n - 2 \ldots \times n - k + 1 = \frac{n!}{(n-k)!}$
- Order does not matter:
  
  Second Rule: divide by number of orders – “$k!””

  \[
  \Rightarrow \frac{n!}{(n-k)!k!}.
  \]
  
  “$n$ choose $k$”

With Replacement.
- Order matters: $n \times n \times \ldots n = n^k$
- Order does not matter: Second rule ???

Problem: depends on how many of each item we chose!

Different number of unordered elts map to each unordered elt.

Unordered elt: $1, 2, 3$ \[3!\] ordered elts map to it.
Unordered elt: $1, 2, 2$ \[\frac{3!}{2!}\] ordered elts map to it.

How do we deal with this mess??
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice($2^5$), divide out order
Splitting up some money....

How many ways can Bob and Alice split 5 dollars? For each of 5 dollars pick Bob or Alice ($2^5$), divide out order ???
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice($2^5$), divide out order ???
5 dollars for Bob and 0 for Alice:
How many ways can Bob and Alice split 5 dollars? For each of 5 dollars pick Bob or Alice \(2^5\), divide out order ???

5 dollars for Bob and 0 for Alice:
one ordered set: \((B, B, B, B, B)\).
Splitting up some money....

How many ways can Bob and Alice split 5 dollars? For each of 5 dollars pick Bob or Alice\(2^5\), divide out order ???

5 dollars for Bob and 0 for Alice: one ordered set: \((B, B, B, B, B)\).

4 for Bob and 1 for Alice:
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice($2^5$), divide out order ???

5 dollars for Bob and 0 for Alice:
one ordered set: $(B, B, B, B, B)$.

4 for Bob and 1 for Alice:
5 ordered sets: $(A, B, B, B, B)$; $(B, A, B, B, B)$; ...
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice\(2^5\), divide out order ???

5 dollars for Bob and 0 for Alice:
one ordered set: \((B, B, B, B, B)\).

4 for Bob and 1 for Alice:
5 ordered sets: \((A, B, B, B, B)\); \((B, A, B, B, B)\); ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.
How many ways can Bob and Alice split 5 dollars? 
For each of 5 dollars pick Bob or Alice ($2^5$), divide out order ???

5 dollars for Bob and 0 for Alice:
one ordered set: $(B, B, B, B, B)$.

4 for Bob and 1 for Alice:
5 ordered sets: $(A, B, B, B, B)$; $(B, A, B, B, B)$; ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.
$(B, B, B, B, B)$:
$(A, B, B, B, B)$:
$(A, A, B, B, B)$:
How many ways can Bob and Alice split 5 dollars? For each of 5 dollars pick Bob or Alice \(2^5\), divide out order ???

5 dollars for Bob and 0 for Alice:

one ordered set: \((B, B, B, B, B)\).

4 for Bob and 1 for Alice:

5 ordered sets: \((A, B, B, B, B)\); \((B, A, B, B, B)\); ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.

\((B, B, B, B, B)\):

\((A, B, B, B, B)\):

\((A, A, B, B, B)\):

and so on.
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice ($2^5$), divide out order ???

5 dollars for Bob and 0 for Alice:
one ordered set: $(B, B, B, B, B)$.

4 for Bob and 1 for Alice:
5 ordered sets: $(A, B, B, B, B)$; $(B, A, B, B, B)$; ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.
$(B, B, B, B, B)$:
$(A, B, B, B, B)$:
$(A, A, B, B, B)$:
and so on.
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice \(2^5\), divide out order

5 dollars for Bob and 0 for Alice:
one ordered set: \((B, B, B, B, B)\).

4 for Bob and 1 for Alice:
5 ordered sets: \((A, B, B, B, B)\); \((B, A, B, B, B)\); ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.
\((B, B, B, B, B)\): 1: \((B, B, B, B, B)\)
\((A, B, B, B, B)\):
\((A, A, B, B, B)\):
and so on.
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?  
For each of 5 dollars pick Bob or Alice \(2^5\), divide out order ???

5 dollars for Bob and 0 for Alice:  
one ordered set: \((B, B, B, B, B)\).

4 for Bob and 1 for Alice:  
5 ordered sets: \((A, B, B, B, B), (B, A, B, B, B)\); ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.  
\((B, B, B, B, B)\):  1: \((B,B,B,B,B)\)  
\((A, A, B, B, B)\):
and so on.

\[...\]
Splitting up some money....

How many ways can Bob and Alice split 5 dollars?
For each of 5 dollars pick Bob or Alice ($2^5$), divide out order ???

5 dollars for Bob and 0 for Alice:
one ordered set: $(B, B, B, B, B)$.

4 for Bob and 1 for Alice:
5 ordered sets: $(A, B, B, B, B) ; (B, A, B, B, B)$; ...

“Sorted” way to specify, first Alice’s dollars, then Bob’s.

$(B, B, B, B, B): 1: (B,B,B,B,B)$
and so on.

Second rule of counting is no good here!
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.
How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).
Splitting 5 dollars.

How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Five dollars are five stars: ★★★★★.
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.  
Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Five dollars are five stars: ⭐⭐⭐⭐⭐.
Alice: 2, Bob: 1, Eve: 2.
How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Five dollars are five stars: ★★★★★.
Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: ★★|★|★★.
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Five dollars are five stars: ⋆ ⋆ ⋆ ⋆ ⋆.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: ⋆ ⋆ | ⋆ | ⋆ ⋆.
Alice: 0, Bob: 1, Eve: 4.
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).

Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.

Five dollars are five stars: ⋆ ⋆ ⋆ ⋆ ⋆.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: ⋆ ⋆ | ⋆ | ⋆ ⋆.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: | ⋆ | ⋆ ⋆ ⋆.
Splitting 5 dollars.

How many ways can Alice, Bob, and Eve split 5 dollars.
Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Five dollars are five stars: \(*\, *\, *\, *\, *\).
Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: \(\ast\, \ast\, |\, \ast\, \ast\).
Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: \(|\, \ast\, \ast\, \ast\, \ast\).
Each split “is” a sequence of stars and bars.
How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).

Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.

Five dollars are five stars: \(\ast \ast \ast \ast \ast\).

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: \(\ast \ast | \ast | \ast \ast\).

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: \(| \ast | \ast \ast \ast \ast\).

Each split “is” a sequence of stars and bars.
Each sequence of stars and bars “is” a split.
Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.  
Alice gets 3, Bob gets 1, Eve gets 1: (A, A, A, B, E).
Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.
Five dollars are five stars: ⭐⭐⭐⭐⭐.
Alice: 2, Bob: 1, Eve: 2.  
Stars and Bars: ⭐⭐|⭐|⭐⭐.
Alice: 0, Bob: 1, Eve: 4.  
Stars and Bars: |⭐|⭐⭐⭐⭐.
Each split “is” a sequence of stars and bars.  
Each sequence of stars and bars “is” a split.
How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1: \((A, A, A, B, E)\).

Separate Alice’s dollars from Bob’s and then Bob’s from Eve’s.

Five dollars are five stars: ******.

Alice: 2, Bob: 1, Eve: 2.
Stars and Bars: ******.

Alice: 0, Bob: 1, Eve: 4.
Stars and Bars: | ******.

Each split “is” a sequence of stars and bars.
Each sequence of stars and bars “is” a split.

**Counting Rule:** if there is a one-to-one mapping between two sets they have the same size!
Stars and Bars.

How many different 5 star and 2 bar diagrams?
Stars and Bars.

How many different 5 star and 2 bar diagrams?
| ⋆ | ⋆ ⋆ ⋆ ⋆ ⋆ |
Stars and Bars.

How many different 5 star and 2 bar diagrams?

\[ \star | \star \star \star \star. \]

7 positions in which to place the 2 bars.

- - - - - - -
Stars and Bars.

How many different 5 star and 2 bar diagrams?
| ⋆ | ⋆ ⋆ ⋆ ⋆ ⋆.

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4
Stars and Bars.

How many different 5 star and 2 bar diagrams?

| ⋆ | ⋆ ⋆ ⋆ ⋆ |

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4
| ⋆ | ⋆ ⋆ ⋆ ⋆ |

Bars in second and seventh position.

7^2 ways to do so and 7^2 ways to split 5 dollars among 3 people.
How many different 5 star and 2 bar diagrams?

\[
\begin{array}{c|c|c|c|c|c|c}
| & \star & | & \star \star \star \star \star \\
\end{array}
\]

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4

\[
\begin{array}{c|c|c|c|c|c|c}
| & \star & | & \star \star \star \star \star \\
| & \star & | & \star \star \star \star \star \\
\end{array}
\]

Bars in first and third position.
How many different 5 star and 2 bar diagrams?

| ⋆ | ⋆ ⋆ ⋆ ⋆.

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4

| ⋆ | ⋆ ⋆ ⋆ ⋆.

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
Stars and Bars.

How many different 5 star and 2 bar diagrams?

| ⋆ | ⋆ ⋆ ⋆ ⋆.

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4

| ⋆ | ⋆ ⋆ ⋆ ⋆.

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0

⋆ | ⋆ ⋆ ⋆ ⋆ |.
Stars and Bars.

How many different 5 star and 2 bar diagrams?

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4
Bars in first and third position.

Alice: 1; Bob 4; Eve: 0
Bars in second and seventh position.
How many different 5 star and 2 bar diagrams?

| ⋆ | ⋆ ⋆ ⋆ ⋆ |

7 positions in which to place the 2 bars.

Alice: 0; Bob 1; Eve: 4

| ⋆ | ⋆ ⋆ ⋆ ⋆ |

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0

⋆ | ⋆ ⋆ ⋆ ⋆ |

Bars in second and seventh position.

\binom{7}{2} \text{ ways to do so and}
Stars and Bars.

How many different 5 star and 2 bar diagrams?

| ⋆ | ⋆ ⋆ ⋆ ⋆ |

7 positions in which to place the 2 bars.

- - - - - - -

Alice: 0; Bob 1; Eve: 4

| ⋆ | ⋆ ⋆ ⋆ ⋆ |

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0

⋆ | ⋆ ⋆ ⋆ ⋆ |

Bars in second and seventh position.

\( \binom{7}{2} \) ways to do so and \( \binom{7}{2} \) ways to split 5 dollars among 3 people.
Stars and Bars.

Ways to add up $n$ numbers to sum to $k$?
Stars and Bars.

Ways to add up \( n \) numbers to sum to \( k \)? or

“\( k \) from \( n \) with replacement where order doesn’t matter.”
Stars and Bars.

Ways to add up $n$ numbers to sum to $k$? or

“$k$ from $n$ with replacement where order doesn’t matter.”

In general, $k$ stars $n - 1$ bars.

\[
\begin{align*}
\star & \star | \cdots | \star \star.
\end{align*}
\]
Stars and Bars.

Ways to add up \( n \) numbers to sum to \( k \)? or

“ \( k \) from \( n \) with replacement where order doesn’t matter.”

In general, \( k \) stars \( n - 1 \) bars.

\[
\star \star | \star | \cdots | \star \star.
\]

\( n + k - 1 \) positions from which to choose \( n - 1 \) bar positions.
Stars and Bars.

Ways to add up $n$ numbers to sum to $k$? or

“$k$ from $n$ with replacement where order doesn’t matter.”

In general, $k$ stars $n-1$ bars.

\[
\begin{array}{c|c|\cdots|c}
\star & \star & \cdots & \star \\
\end{array}
\]

$n + k - 1$ positions from which to choose $n-1$ bar positions.

\[
\binom{n+k-1}{n-1}
\]
Ways to add up $n$ numbers to sum to $k$? or

“$k$ from $n$ with replacement where order doesn’t matter.”

In general, $k$ stars $n - 1$ bars.

```
* * | * | · · · | * *
```

$n + k - 1$ positions from which to choose $n - 1$ bar positions.

\[
\binom{n+k-1}{n-1}
\]

Or: $k$ unordered choices from set of $n$ possibilities with replacement.

**Sample with replacement where order doesn’t matter.**
Counting basics.

First rule: \( n_1 \times n_2 \cdots \times n_3 \).
Counting basics.

**First rule:** \( n_1 \times n_2 \cdots \times n_k \).

\( k \) Samples with replacement from \( n \) items: \( n^k \).
Counting basics.

**First rule:** $n_1 \times n_2 \cdots \times n_3$.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$.
Counting basics.

**First rule:** $n_1 \times n_2 \cdots \times n_3$.

$k$ Samples with replacement from $n$ items: $n^k$.
Sample without replacement: $\frac{n!}{(n-k)!}$

**Second rule:** when order doesn’t matter divide..when possible.
Counting basics.

**First rule:** \( n_1 \times n_2 \cdots \times n_3. \)

\( k \) Samples with replacement from \( n \) items: \( n^k. \)

Sample without replacement: \( \frac{n!}{(n-k)!} \)

**Second rule:** when order doesn’t matter divide..when possible.

Sample without replacement and order doesn’t matter: \( \binom{n}{k} = \frac{n!}{(n-k)!k!} \cdot \)

“\( n \) choose \( k \)”
Counting basics.

**First rule:** \( n_1 \times n_2 \cdots \times n_3. \)

\( k \) Samples with replacement from \( n \) items: \( n^k. \)
Sample without replacement: \( \frac{n!}{(n-k)!} \)

**Second rule:** when order doesn’t matter divide..when possible.
Sample without replacement and order doesn’t matter: \( \binom{n}{k} = \frac{n!}{(n-k)!k!}. \)
“\( n \) choose \( k \)”

**One-to-one rule:** equal in number if one-to-one correspondence.
Counting basics.

First rule: $n_1 \times n_2 \cdots \times n_3$.  

$k$ Samples with replacement from $n$ items: $n^k$.  
Sample without replacement: $\frac{n!}{(n-k)!}$.

Second rule: when order doesn’t matter divide..when possible.  
Sample without replacement and order doesn’t matter:  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

“One choose $k$”

One-to-one rule: equal in number if one-to-one correspondence.  
Sample with replacement and order doesn’t matter: $\binom{k+n-1}{n-1}$.
Bijection: sums to 'k' → stars and bars.

\[ S = \{(n_1, n_2, n_3) : n_1 + n_2 + n_3 = 5\} \]
Bijection: sums to 'k' $\rightarrow$ stars and bars.

$S = \{(n_1, n_2, n_3) : n_1 + n_2 + n_3 = 5\}$

$T = \{s \in \{|', \ast'\} : |s| = 7, \text{number of bars in } s = 2\}$
Bijection: sums to 'k' \( \rightarrow \) stars and bars.

\[
S = \{(n_1, n_2, n_3) : n_1 + n_2 + n_3 = 5\}
\]

\[
T = \{s \in \{|', ', '*'\} : |s| = 7, \text{number of bars in } s = 2\}
\]

\[
f((n_1, n_2, n_3)) = \ast^{n_1} ' '|\ast^{n_2} '|' \ast^{n_3}
\]
Bijection: sums to ‘k’ → stars and bars.

\[ S = \{ (n_1, n_2, n_3) : n_1 + n_2 + n_3 = 5 \} \]
\[ T = \{ s \in \{ '||', '★' \} : |s| = 7, \text{number of bars in } s = 2 \} \]
\[ f((n_1, n_2, n_3)) = ★^{n_1} || ★^{n_2} || ★^{n_3} \]
Bijection:
argument: unique \((n_1, n_2, n_3)\) from any \(s\).
Bijection: sums to 'k' → stars and bars.

\[ S = \{(n_1, n_2, n_3) : n_1 + n_2 + n_3 = 5\} \]
\[ T = \{s \in \{′|′, ′⋆′\} : |s| = 7, \text{number of bars in } s = 2\} \]
\[ f((n_1, n_2, n_3)) = ⋆^{n_1} ′|′ ⋆^{n_2} ′|′ ⋆^{n_3} \]

Bijection:
  argument: unique \((n_1, n_2, n_3)\) from any \(s\).

\(|S| = |T| = \binom{7}{2}.\)
Mark what's correct.
(A) ways to split n dollars among k: \( \binom{n+k-1}{k-1} \)
(B) ways to split k dollars among n: \( \binom{k+n-1}{n-1} \)
(C) ways to split 5 dollars among 3: \( \binom{7}{5} \)
(D) ways to split 5 dollars among 3: \( \binom{5+3-1}{3-1} \)
Mark what's correct.
(A) ways to split n dollars among k: \( \binom{n+k-1}{k-1} \)
(B) ways to split k dollars among n: \( \binom{k+n-1}{n-1} \)
(C) ways to split 5 dollars among 3: \( \binom{7}{5} \)
(D) ways to split 5 dollars among 3: \( \binom{5+3-1}{3-1} \)
All correct.
Balls in bins.

“$k$ Balls in $n$ bins” ≡ “$k$ samples from $n$ possibilities.”
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”
“indistinguishable balls” $\equiv$ “order doesn’t matter”
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”
“indistinguishable balls” $\equiv$ “order doesn’t matter”
“only one ball in each bin” $\equiv$ “without replacement”
Balls in bins.

“$k$ Balls in $n$ bins” ≡ “$k$ samples from $n$ possibilities.”
“indistinguishable balls” ≡ “order doesn’t matter”
“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”

“indistinguishable balls” $\equiv$ “order doesn’t matter”

“only one ball in each bin” $\equiv$ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
Balls in bins.

“$k$ Balls in $n$ bins” ≡ “$k$ samples from $n$ possibilities.”
“indistinguishable balls” ≡ “order doesn’t matter”
“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement

Example: 5 digit numbers.
Balls in bins.

“$k$ Balls in $n$ bins” ≡ “$k$ samples from $n$ possibilities.”

“indistinguishable balls” ≡ “order doesn’t matter”

“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement

   Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
Balls in bins.

“$k$ Balls in $n$ bins” ≡ “$k$ samples from $n$ possibilities.”

“indistinguishable balls” ≡ “order doesn’t matter”

“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
  Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”

“indistinguishable balls” $\equiv$ “order doesn’t matter”

“only one ball in each bin” $\equiv$ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
   Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
   Example: Poker hands.
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”

“indistinguishable balls” $\equiv$ “order doesn’t matter”

“only one ball in each bin” $\equiv$ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
   Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
   Example: Poker hands.

5 indistinguishable balls into 3 bins
Balls in bins.

“$k$ Balls in $n$ bins” $\equiv$ “$k$ samples from $n$ possibilities.”

“indistinguishable balls” $\equiv$ “order doesn’t matter”

“only one ball in each bin” $\equiv$ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
   Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
   Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
“k Balls in n bins” ≡ “k samples from n possibilities.”
“indistinguishable balls” ≡ “order doesn’t matter”
“only one ball in each bin” ≡ “without replacement”

5 balls into 10 bins
5 samples from 10 possibilities with replacement
   Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
   Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
   Dividing 5 dollars among Alice, Bob and Eve.
Mark what's correct.

k Balls in n bins.

dis == distinguishable
unique = one ball in each bin.

(A) dis => $n^k$
(B) dis, unique => $n!/(n - k)!$
(C) indis, unique => $\binom{n}{k}$
(D) dis, => $n!/(n - k)!$
(E) indis, => $\binom{n+k-1}{k-1}$
(F) dis, unique => $\binom{n}{k}$
Mark what's correct.

k Balls in n bins.

dis == distinguishable
unique = one ball in each bin.

(A) dis => \( n^k \)
(B) dis, unique => \( n!/(n-k)! \)
(C) indis, unique => \( \binom{n}{k} \)
(D) dis, => \( n!/(n-k)! \)
(E) indis, => \( \binom{n+k-1}{k-1} \)
(F) dis, unique => \( \binom{n}{k} \)
Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?

\[ \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3}. \]
Two indistinguishable jokers in 54 card deck. How many 5 card poker hands? **Sum rule: Can sum over disjoint sets.**
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.
Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers

\[
\binom{52}{5}
\]
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers “exclusive” or One Joker

\[
\binom{52}{5} + \binom{52}{4}
\]
Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule**: Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]
Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck.
Sum Rule

Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands?

Wait a minute! Same as choosing 5 cards from 54 or \( \binom{54}{5} \).

**Theorem:** \( \binom{54}{5} = \binom{52}{5} + 2 \times \binom{52}{4} + \binom{52}{3} \).

**Algebraic Proof:** Why? Just why? Especially on Tuesday! Already have a combinatorial proof.
Two indistinguishable jokers in 54 card deck. How many 5 card poker hands?

**Sum rule:** Can sum over disjoint sets.

No jokers “exclusive” or One Joker “exclusive” or Two Jokers

\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
\]

Two distinguishable jokers in 54 card deck. How many 5 card poker hands?

\[
\binom{52}{5} + 
\]
Sum Rule

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\[
\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.
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Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? **Choose 4 cards plus one of 2 jokers!**

\[
\binom{52}{5} + 2 \times \binom{52}{4} + \binom{54}{5}.
\]
Sum Rule

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...
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**Theorem:** \(\binom{54}{5}\)
**Sum Rule**

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**Algebraic Proof:** Why?
Sum Rule

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Two distinguishable jokers in 54 card deck. How many 5 card poker hands? **Choose 4 cards plus one of 2 jokers!**

\[
\binom{52}{5} + 2 \cdot \binom{52}{4} + \binom{52}{3}
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\binom{54}{5}
\]

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**Algebraic Proof:** Why? Just why?
Sum Rule

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**Algebraic Proof:** Why? Just why? Especially on Tuesday!
Already have a **combinatorial proof.**
Combinatorial Proofs.

**Theorem:** \( \binom{n}{k} = \binom{n}{n-k} \)
Combinatorial Proofs.

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)?
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Combinatorial Proofs.

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How many subsets of size \( k \)?
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Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n-k \)
Combinatorial Proofs.

**Theorem:** $\binom{n}{k} = \binom{n}{n-k}$

**Proof:** How many subsets of size $k$? $\binom{n}{k}$

How many subsets of size $k$?
Choose a subset of size $n-k$
and what’s left out
Combinatorial Proofs.

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

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Choose a subset of size \( n - k \)
and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

How many subsets of size \( k \)?
Choose a subset of size \( n - k \)
and what’s left out is a subset of size \( k \).
Choosing a subset of size \( k \) is same
as choosing \( n - k \) elements to not take.
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: How many subsets of size \( k \)? \( \binom{n}{k} \)

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Choose a subset of size \( n-k \)
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\[ \Rightarrow \binom{n}{n-k} \] subsets of size \( k \).
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\( \implies \binom{n}{n-k} \) subsets of size \( k \).
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x)\cdots(1 + x)$.

Foil (4 terms): $2^n$ terms: choose 1 or $x$ from each term $(1 + b)$.

Simplify: collect all terms corresponding to $x^k$.

Coefficient of $x^k$: choose $k$ terms with $x$ in product.

Pascal’s rule $\Rightarrow n + 1 \choose k = n \choose k + n \choose k - 1$. 
Pascal’s Triangle

\[
\begin{array}{cccc}
0 \\
1 & 1 \\
\end{array}
\]

Row \( n \): coefficients of \((1 + x)^n\) = \((1 + x)(1 + x)(1 + x) \cdots (1 + x)\).

Foil (4 terms):

\[2^n \text{ terms: choose 1 or } x \text{ from each term } (1 + b).\]

Simplify: collect all terms corresponding to \(x^k\).

Coefficient of \(x^k\) \(n\): choose \(k\) terms with \(x\) in product.

Pascal’s rule

\[
\begin{align*}
\binom{n+1}{k} &= \binom{n}{k} + \binom{n}{k-1}.
\end{align*}
\]
Pascal’s Triangle

\[
\begin{array}{cccccc}
0 \\
1 & 1 \\
1 & 2 & 1 \\
\end{array}
\]

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Pascal’s rule \( \Rightarrow \) \( n+1 \choose k \) = \( n \choose k \) + \( n \choose k-1 \)
Pascal’s Triangle

0
1 1
1 2 1
1 3 3 1

Row $n$: coefficients of $(1 + x)^n$.

Foil (4 terms) on steroids: $2^n$ terms: choose 1 or $x$ from each term $(1 + b)$.

Simplify: collect all terms corresponding to $x^k$.

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Pascal’s rule $\Rightarrow n_{k+1} = n_k + n_{k-1}$.
Pascal’s Triangle

\[
\begin{array}{cccccc}
0 \\
1 & 1  \\
1 & 2 & 1  \\
1 & 3 & 3 & 1  \\
1 & 4 & 6 & 4 & 1  \\
\end{array}
\]

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

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Simplify: collect all terms corresponding to $x^k$.

Coefficient of $x^k$: choose $k$ terms with $x$ in product.

Pascal’s rule $n+1 \binom{k}{n} = n \binom{k}{n-1} + \binom{k}{n-1}$. 
Pascal’s Triangle

Row $n$: coefficients of $(1 + x)^n = (1 + x)(1 + x) \cdots (1 + x)$.

\[
\begin{array}{cccccc}
0 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]
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Foil (4 terms) on steroids:

$2^n$ terms:
Pascal’s Triangle

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Foil (4 terms) on steroids:

$2^n$ terms: choose 1 or $x$ from each term $(1 + b)$. 

```
0
1 1
1 2 1
1 3 3 1
1 4 6 4 1
```
Pascal’s Triangle

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Simplify: collect all terms corresponding to $x^k$. 

\[ \begin{array}{ccccccccccc} 
0 & & & & & & & & & & \\
 & 1 & & & & & & & & & 1 \\
 & 1 & & & & & & & & & 2 & 1 \\
 & & & 1 & & & & & & & 3 & 3 & 1 \\
 & & & & & 1 & & & & & 4 & 6 & 4 & 1 \\
\end{array} \]
Pascal’s Triangle

\[
\begin{array}{cccc}
0 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
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Simplify: collect all terms corresponding to \(x^k\).
Coefficient of \(x^k \binom{n}{k}\): choose \(k\) terms with \(x\) in product.
Pascal’s Triangle

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\begin{array}{cccc}
0 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
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Simplify: collect all terms corresponding to \(x^k\).

- Coefficient of \(x^k\): choose \(k\) terms with \(x\) in product.

\[
\binom{n}{k}
\]

- \(\binom{0}{0}\)
- \(\binom{1}{0} \binom{1}{1}\)
Pascal’s Triangle

\[
\begin{array}{cccccc}
& & & & & \\
& & & & 1 & 1 \\
& & 1 & 2 & 1 \\
& 1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
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\[
\begin{array}{ccc}
\binom{0}{0} & \binom{1}{1} & \binom{2}{2} \\
\end{array}
\]
Pascal’s Triangle

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Simplify: collect all terms corresponding to $x^k$.
Coefficient of $x^k \binom{n}{k}$: choose $k$ terms with $x$ in product.

$$
\begin{align*}
\binom{0}{0} & \quad \binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} & \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} & \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}
\end{align*}
$$

Pascal’s rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$. 

Combinatorial Proofs.

**Theorem:** \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

**Proof:** How many size \( k \) subsets of \( n+1 \)?
Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

Proof: How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).
Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

Proof: How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

How many size \( k \) subsets of \( n+1 \)?

Combinatorial Proofs.
Combinatorial Proofs.

**Theorem:** \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

**Proof:** How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

How many size \( k \) subsets of \( n+1 \)?

How many contain the first element?

How many don't contain the first element?

Sum Rule: size of union of disjoint sets of objects.

Without and with first element \( \rightarrow \) disjoint.

So, \( \binom{n}{k} - 1 + \binom{n}{k} = \binom{n+1}{k} \).
Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

Proof: How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

How many size \( k \) subsets of \( n+1 \)?

How many contain the first element?

Chose first element,
Combinatorial Proofs.

Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

Proof: How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

How many size \( k \) subsets of \( n+1 \)?
How many contain the first element?
  Chose first element, need \( k-1 \) more from remaining \( n \) elements.
Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

Proof: How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

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\[ \Rightarrow \binom{n}{k-1} \]
**Theorem:** \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

**Proof:** How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

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Chose first element, need \( k-1 \) more from remaining \( n \) elements.
\( \implies \binom{n}{k-1} \)
Combinatorial Proofs.

**Theorem:** $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

**Proof:** How many size $k$ subsets of $n+1$? $\binom{n+1}{k}$.

How many size $k$ subsets of $n+1$?
How many contain the first element?
Chose first element, need $k-1$ more from remaining $n$ elements.
$\implies \binom{n}{k-1}$

How many don’t contain the first element?
**Theorem:** \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

**Proof:** How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

How many size \( k \) subsets of \( n+1 \)?

How many contain the first element?

Chose first element, need \( k - 1 \) more from remaining \( n \) elements. \[ \implies \binom{n}{k-1} \]

How many don’t contain the first element?

Need to choose \( k \) elements from remaining \( n \) elts.
Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

Proof: How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

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How many contain the first element?
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How many don’t contain the first element?
  Need to choose \( k \) elements from remaining \( n \) elts.
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\[ \implies \binom{n}{k} \]
Theorem: \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \).

Proof: How many size \( k \) subsets of \( n+1 \)? \( \binom{n+1}{k} \).

How many size \( k \) subsets of \( n+1 \)?
How many contain the first element?
Chose first element, need \( k-1 \) more from remaining \( n \) elements.
\[ \Rightarrow \binom{n}{k-1} \]

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**Combinatorial Proof.**

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\[ \{1, \ldots, i, \ldots, n\} \]

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Add them up to get the total number of subsets of size \( k \).
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Add them up to get the total number of subsets of size \( k \) which is also \( \binom{n+1}{k} \).

\[ \square \]
Binomial Theorem: $x = 1$

**Theorem:** $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$
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Construct a subset with sequence of $n$ choices:
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Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$

Used to reason about all subsets
  by adding number of subsets of size 1, 2, 3,...
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![Venn Diagram](image-url)
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![Venn Diagram](attachment://venn_diagram.png)

- In $T$ $\implies$ $|T|$
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\[
\begin{align*}
\text{In } T. & \quad \implies |T| \\
\text{In } S. & \quad \implies + |S| \\
\text{Elements in } S \cap T \text{ are counted twice.}
\end{align*}
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**Inclusion/Exclusion Rule:**
For any $S$ and $T$, $|S \cup T| = |S| + |T| - |S \cap T|$.

- Elements in $S \cap T$ are counted twice.
- Subtract. $\implies -|S \cap T|$
Simple Inclusion/Exclusion

**Sum Rule:** For disjoint sets $S$ and $T$, $|S \cup T| = |S| + |T|$
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**Diagram:**
- In $T$ implies $|T|$.
- In $S$ implies $+ |S|$.
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$|S \cup T| = |S| + |T| - |S \cap T|$
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**Example:** How many 10-digit phone numbers have 7 as their first or second digit?
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$S =$ phone numbers with 7 as first digit.$|S| = 10^9$
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    \[ |S \cup T| = |S| + |T| - |S \cap T| \]

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S =$ phone numbers with 7 as first digit. $|S| = 10^9$

$T =$ phone numbers with 7 as second digit.
Simple Inclusion/Exclusion

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$S \cap T =$ phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.  

**Simple Inclusion/Exclusion**

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**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S =$ phone numbers with 7 as first digit. $|S| = 10^9$

$T =$ phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T =$ phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.  

Inclusion/Exclusion

$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|.$$
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|. \]

Idea: For \( n = 3 \) how many times is an element counted?
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \cdots (-1)^n |A_1 \cap \cdots A_n|. \]

Idea: For \( n = 3 \) how many times is an element counted? Consider \( x \in A_i \cap A_j \).

\[ x \text{ counted once for } |A_i| \text{ and once for } |A_j|. \]
\[ x \text{ subtracted from count once for } |A_i \cap A_j|. \]
\[ \text{Total: } 2 - 1 = 1. \]

Consider \( x \in A_1 \cap A_2 \cap A_3 \).
\[ x \text{ counted once in each term: } |A_1|, |A_2|, |A_3|. \]
\[ x \text{ subtracted once in terms: } |A_1 \cap A_3|, |A_1 \cap A_2|, |A_2 \cap A_3|. \]
\[ x \text{ added once in } |A_1 \cap A_2 \cap A_3|. \]
\[ \text{Total: } 3 - 3 + 1 = 1. \]

Formulaically: \( x \) is in intersection of three sets.
\[ 3 \text{ for terms of form } |A_i|. \]
\[ 3 \cdot 2 \text{ for terms of form } |A_i \cap A_j|. \]
\[ 3 \cdot 3 \text{ for terms of form } |A_i \cap A_j \cap A_k|. \]
\[ \text{Total: } 3 - 3 + 3 = 1. \]
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots \cap A_n|. \]

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- \( 3 \cdot 3 \cdot 2 \) for terms of form \( |A_i \cap A_j \cap A_k| \).

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Inclusion/Exclusion

\[|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|.\]

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Total: \(2 - 1 = 1\).
|A_1 \cup \cdots \cup A_n| = 
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$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|.$$  

Idea: For $n = 3$ how many times is an element counted?

Consider $x \in A_i \cap A_j$.
- $x$ counted once for $|A_i|$ and once for $|A_j|$.
- $x$ subtracted from count once for $|A_i \cap A_j|$.

Total: $2 - 1 = 1$.

Consider $x \in A_1 \cap A_2 \cap A_3$
- $x$ counted once in each term: $|A_1|, |A_2|, |A_3|$.
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Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots \cap A_n|. \]

Idea: For \( n = 3 \) how many times is an element counted?

Consider \( x \in A_i \cap A_j \).

- \( x \) counted once for \( |A_i| \) and once for \( |A_j| \).
- \( x \) subtracted from count once for \( |A_i \cap A_j| \).

Total: \( 2 - 1 = 1 \).

Consider \( x \in A_1 \cap A_2 \cap A_3 \).

- \( x \) counted once in each term: \( |A_1|, |A_2|, |A_3| \).
- \( x \) subtracted once in terms: \( |A_1 \cap A_3|, |A_1 \cap A_2|, |A_2 \cap A_3| \).
- \( x \) added once in \( |A_1 \cap A_2 \cap A_3| \).
Inclusion/Exclusion

|A_1 \cup \cdots \cup A_n| = 
\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|.

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Consider \( x \in A_i \cap A_j \).
- \( x \) counted once for \(|A_i|\) and once for \(|A_j|\).
- \( x \) subtracted once from count once for \(|A_i \cap A_j|\).
Total: \( 2 - 1 = 1 \).

Consider \( x \in A_1 \cap A_2 \cap A_3 \)
- \( x \) counted once in each term: \(|A_1|, |A_2|, |A_3|\).
- \( x \) subtracted once in terms: \(|A_1 \cap A_3|, |A_1 \cap A_2|, |A_2 \cap A_3|\).
- \( x \) added once in \(|A_1 \cap A_2 \cap A_3|\).
Total: \( 3 - 3 + 1 = 1 \).
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|. \]

Idea: For \( n = 3 \) how many times is an element counted?
Consider \( x \in A_i \cap A_j \).
- \( x \) counted once for \( |A_i| \) and once for \( |A_j| \).
- \( x \) subtracted once for \( |A_i \cap A_j| \).
Total: \( 2 -1 = 1 \).

Consider \( x \in A_1 \cap A_2 \cap A_3 \)
- \( x \) counted once in each term: \( |A_1|, |A_2|, |A_3| \).
- \( x \) subtracted once in terms: \( |A_1 \cap A_3|, |A_1 \cap A_2|, |A_2 \cap A_3| \).
- \( x \) added once in \( |A_1 \cap A_2 \cap A_3| \).
Total: \( 3 - 3 + 1 = 1 \).

Formulaically: \( x \) is in intersection of three sets.
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \]
\[ \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|. \]

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- \( x \) subtracted once in terms: \( |A_1 \cap A_3|, |A_1 \cap A_2|, |A_2 \cap A_3| \).
- \( x \) added once in \( |A_1 \cap A_2 \cap A_3| \).

Total: \( 3 - 3 + 1 = 1 \).

Formulaically: \( x \) is in intersection of three sets.

- 3 for terms of form \( |A_i| \), \( \binom{3}{2} \) for terms of form \( |A_i \cap A_j| \).
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|. \]

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- 3 for terms of form \( |A_i| \), \( \binom{3}{2} \) for terms of form \( |A_i \cap A_j| \).
- \( \binom{3}{3} \) for terms of form \( |A_i \cap A_j \cap A_k| \).
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|. \]

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\( x \) counted once in each term: \( |A_1|, |A_2|, |A_3| \).
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\( x \) added once in \( |A_1 \cap A_2 \cap A_3| \).
Total: \( 3 - 3 + 1 = 1 \).

Formulaically: \( x \) is in intersection of three sets.
3 for terms of form \( |A_i| \), \( \binom{3}{2} \) for terms of form \( |A_i \cap A_j| \).
\( \binom{3}{3} \) for terms of form \( |A_i \cap A_j \cap A_k| \).
Total: \( \binom{3}{1} - \binom{3}{2} + \binom{3}{3} = 1 \).
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots \cap A_n|. \]
Inclusion/Exclusion

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Idea: how many times is each element counted?

Element \( x \) in \( m \) sets: \( x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m} \).
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots \cap A_n|. \]

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Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|. \]

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Total: \( \binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1} \binom{m}{m} \).
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \cdots - (-1)^n |A_1 \cap \cdots \cap A_n|. \]

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Inclusion/Exclusion

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Counted \( \binom{m}{i} \) times in \( i \)th summation.

Total: \( \binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1} \binom{m}{m} \).

Binomial Theorem:

\( (x + y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \cdots \binom{m}{m} y^m. \)
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots \cap A_n|. \]

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Proof: \( m \) factors in product: \( (x + y)(x + y) \cdots (x + y) \).
Inclusion/Exclusion

\[
|A_1 \cup \cdots \cup A_n| = \\
\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|.
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Total: \(\binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1} \binom{m}{m}\).

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\]

Proof: \(m\) factors in product: \((x + y)(x + y) \cdots (x + y)\).

Get a term \(x^{m-i} y^i\) by choosing \(i\) factors to use for \(y\).
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots \cap A_n|. \]

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Proof: \( m \) factors in product: \( (x+y)(x+y)\cdots(x+y) \).

Get a term \( x^{m-i} y^i \) by choosing \( i \) factors to use for \( y \).

are \( \binom{m}{i} \) ways to choose factors where \( y \) is provided.
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots \cap A_n|. \]

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Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|. \]

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Get a term \( x^{m-i} y^i \) by choosing \( i \) factors to use for \( y \).

are \( \binom{m}{i} \) ways to choose factors where \( y \) is provided.

For \( x = 1, y = -1 \),
Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|. \]

Idea: how many times is each element counted?

Element \( x \) in \( m \) sets: \( x \in A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}. \)

Counted \( \binom{m}{i} \) times in \( i \)th summation.

Total: \( \binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1} \binom{m}{m}. \)

Binomial Theorem:
\[(x + y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \cdots \binom{m}{m} y^m.\]

Proof: \( m \) factors in product: \( (x + y)(x + y) \cdots (x + y). \)

Get a term \( x^{m-i} y^i \) by choosing \( i \) factors to use for \( y \).

are \( \binom{m}{i} \) ways to choose factors where \( y \) is provided.

For \( x = 1, y = -1, \)
\[ 0 = (1 - 1)^m = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \cdots + (-1)^m \binom{m}{m}. \]
Inclusion/Exclusion

$$|A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|.$$ 

Idea: how many times is each element counted?

Element $x$ in $m$ sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.

Counted $\binom{m}{i}$ times in $i$th summation.

Total: $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1} \binom{m}{m}$.

Binomial Theorem:

$$(x + y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \cdots \binom{m}{m} y^m.$$ 

Proof: $m$ factors in product: $(x + y)(x + y) \cdots (x + y)$.

Get a term $x^{m-i} y^i$ by choosing $i$ factors to use for $y$.

are $\binom{m}{i}$ ways to choose factors where $y$ is provided.

For $x = 1$, $y = -1$,

$$0 = (1 - 1)^m = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \cdots + (-1)^{m-1} \binom{m}{m}$$ 

$$\implies 1 = \binom{m}{0} = \binom{m}{1} - \binom{m}{2} \cdots + (-1)^{m-1} \binom{m}{m}.$$
### Inclusion/Exclusion

\[ |A_1 \cup \cdots \cup A_n| = \sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| - \cdots (-1)^n |A_1 \cap \cdots \cap A_n|. \]

Idea: how many times is each element counted?

Element \( x \) in \( m \) sets: \( x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m} \).

Counted \( \binom{m}{i} \) times in \( i \)th summation.

**Total:** \( \binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1} \binom{m}{m} \).

**Binomial Theorem:**

\[(x + y)^m = \binom{m}{0} x^m + \binom{m}{1} x^{m-1} y + \binom{m}{2} x^{m-2} y^2 + \cdots \binom{m}{m} y^m.\]

Proof: \( m \) factors in product: \((x + y)(x + y)\cdots(x + y)\). Get a term \( x^{m-i} y^i \) by choosing \( i \) factors to use for \( y \). are \( \binom{m}{i} \) ways to choose factors where \( y \) is provided.

For \( x = 1, y = -1 \),

\[
0 = (1 - 1)^m = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \cdots + (-1)^m \binom{m}{m}
\]

\[
\implies 1 = \binom{m}{0} - \binom{m}{1} + \binom{m}{2} \cdots + (-1)^{m-1} \binom{m}{m}.
\]

Each element counted once!
Summary.

First Rule of counting:
Summary.

First Rule of counting: Objects from a sequence of choices:
Summary.

First Rule of counting: Objects from a sequence of choices:
\( n_i \) possibilities for \( i \)th choice:
Summary.

First Rule of counting: Objects from a sequence of choices:

\[ n_i \text{ possibilities for } i\text{th choice} : n_1 \times n_2 \times \cdots \times n_k \text{ objects.} \]
Summary.

First Rule of counting: Objects from a sequence of choices:
\[ n_1 \times n_2 \times \cdots \times n_k \text{ objects.} \]

Second Rule of counting:
Summary.

First Rule of counting: Objects from a sequence of choices:

\( n_i \) possibilities for \( i \)th choice \( : n_1 \times n_2 \times \cdots \times n_k \) objects.

Second Rule of counting: If order does not matter.
Summary.

First Rule of counting: Objects from a sequence of choices:
   \( n_i \) possibilities for \( i \)th choice : \( n_1 \times n_2 \times \cdots \times n_k \) objects.

Second Rule of counting: If order does not matter.
   Count with order:
Summary.

First Rule of counting: Objects from a sequence of choices: 
\[ n_i \] possibilities for \( i \)th choice : \( n_1 \times n_2 \times \cdots \times n_k \) objects.

Second Rule of counting: If order does not matter.
Count with order: Divide number of orderings.
Summary.

First Rule of counting: Objects from a sequence of choices: 
\( n_i \) possibilities for \( i \)th choice : \( n_1 \times n_2 \times \cdots \times n_k \) objects.

Second Rule of counting: If order does not matter. 
Count with order: Divide number of orderings. Typically: \( \binom{n}{k} \).
Summary.

First Rule of counting: Objects from a sequence of choices: $n_i$ possibilities for $i$th choice: $n_1 \times n_2 \times \cdots \times n_k$ objects.

Second Rule of counting: If order does not matter.
   Count with order: Divide number of orderings. Typically: $\binom{n}{k}$.

Stars and Bars:
Summary.

First Rule of counting: Objects from a sequence of choices: 
\( n_i \) possibilities for \( i \)th choice : \( n_1 \times n_2 \times \cdots \times n_k \) objects.

Second Rule of counting: If order does not matter.
Count with order: Divide number of orderings. Typically: \( \binom{n}{k} \).

Stars and Bars: Sample \( k \) objects with replacement from \( n \).
Summary.

First Rule of counting: Objects from a sequence of choices:

\[ n_i \text{ possibilities for } i^{\text{th}} \text{ choice } : n_1 \times n_2 \times \cdots \times n_k \text{ objects.} \]

Second Rule of counting: If order does not matter.

Count with order: Divide number of orderings. Typically: \( \binom{n}{k} \).

Stars and Bars: Sample \( k \) objects with replacement from \( n \).

Order doesn’t matter:
Summary.

First Rule of counting: Objects from a sequence of choices:
\[ n_i \text{ possibilities for } i\text{th choice : } n_1 \times n_2 \times \cdots \times n_k \text{ objects.} \]

Second Rule of counting: If order does not matter.
Count with order: Divide number of orderings. Typically: \( \binom{n}{k} \).

Stars and Bars: Sample \( k \) objects with replacement from \( n \).
Order doesn’t matter: Typically: \( \binom{n+k-1}{n-1} = \binom{n+k-1}{k} \).
Summary.

First Rule of counting: Objects from a sequence of choices:
\[ n_i \text{ possibilities for } i\text{th choice : } n_1 \times n_2 \times \cdots \times n_k \text{ objects.} \]

Second Rule of counting: If order does not matter.
Count with order: Divide number of orderings. Typically: \( \binom{n}{k} \).

Stars and Bars: Sample \( k \) objects with replacement from \( n \).
Order doesn’t matter: Typically: \( \binom{n+k-1}{n-1} = \binom{n+k-1}{k} \).

Inclusion/Exclusion: two sets of objects.
Summary.

First Rule of counting: Objects from a sequence of choices:

\[ n_i \text{ possibilities for } i\text{th choice : } n_1 \times n_2 \times \cdots \times n_k \text{ objects.} \]

Second Rule of counting: If order does not matter.

Count with order: Divide number of orderings. Typically: \( \binom{n}{k} \).

Stars and Bars: Sample \( k \) objects with replacement from \( n \).

Order doesn’t matter: Typically: \( \binom{n+k-1}{n-1} = \binom{n+k-1}{k} \).

Inclusion/Exclusion: two sets of objects.

Add number of each subtract intersection of sets.
Summary.

First Rule of counting: Objects from a sequence of choices:
\[ n_i \text{ possibilities for } i\text{th choice : } n_1 \times n_2 \times \cdots \times n_k \text{ objects.} \]

Second Rule of counting: If order does not matter.
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Stars and Bars: Sample \( k \) objects with replacement from \( n \).
\[ \text{Order doesn’t matter: Typically: } \binom{n+k-1}{n-1} = \binom{n+k-1}{k}. \]

Inclusion/Exclusion: two sets of objects.
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Poll: How big is infinity?

(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) Pairs of natural numbers.
Poll: How big is infinity?

Mark what’s true.
(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) pairs of natural numbers $>>$ natural numbers.
Two sets are the same size?
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(A) Bijection between the sets.
(B) Count the objects and get the same number. same size.
(C) Counting to infinity is hard.
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(A), (B).
Two sets are the same size?

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(A), (B).
(C)?
Countable.

How to count?
Countable.

How to count?
0,
Countable.

How to count?

0, 1,
Countable.

How to count?
0, 1, 2,
Countable.

How to count?
0, 1, 2, 3,
Countable.

How to count?
0, 1, 2, 3, …
Countable.

How to count?

0, 1, 2, 3, ...  
The Counting numbers.
How to count?
0, 1, 2, 3, ... 
The Counting numbers.
The natural numbers! $N$
How to count?
0, 1, 2, 3, ... 

The Counting numbers.
The natural numbers! \( N \)

Definition: \( S \) is **countable** if there is a bijection between \( S \) and some subset of \( N \).
Countable.

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0, 1, 2, 3, …
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Definition: $S$ is **countable** if there is a bijection between $S$ and some subset of $N$.

If the subset of $N$ is finite, $S$ has finite **cardinality**.
If the subset of $N$ is infinite, $S$ is **countably infinite**.
Enumerating a set implies countable.
Corollary: Any subset $T$ of a countable set $S$ is countable.
Countably infinite subsets.

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Enumerate $T$ as follows:
Get next element, $x$, of $S$, 

\[ \text{output only if } x \in T. \]
Countably infinite subsets.

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Corollary: Any subset \( T \) of a countable set \( S \) is countable.

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$\mathbb{Z}^+$ is countable.
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All countably infinite sets have the same cardinality.
Enumeration example.

All binary strings.
All binary strings.
\[ B = \{0, 1\}^*. \]
All binary strings.
\[ B = \{0, 1\}^* . \]
\[ B = \{\phi, \]
All binary strings.
\[ B = \{0, 1\}^* \]
\[ B = \{\emptyset, 0, \ldots\} \]
\[ \text{never get to 1.} \]
Enumeration example.

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Enumeration example.

All binary strings.

\[ B = \{0, 1\}^* . \]

\[ B = \{\phi, 0, 1, 00, \ldots\} . \]

\(\phi\) is empty string.

For any string, it appears at some position in the list. If \(n\) bits, it will appear before position \(2^n + 1\).

Should be careful here.
All binary strings.

$B = \{0, 1\}^*$. 

$B = \{\phi, 0, 1, 00, 01, 10, 11, ...\}$. 

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Enumeration example.

All binary strings.
\[ B = \{0, 1\}^*. \]

\[ B = \{\emptyset, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}. \]
\(\emptyset\) is empty string.
All binary strings.

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Should be careful here.

$B = \{\phi; , 0, 00, 000, 0000, \ldots\}$

Never get to 1.
More fractions?

Enumerate the rational numbers in order...
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0, ..., 1/2, ..
More fractions?

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0, ..., 1/2,..

Where is 1/2 in list?
More fractions?

Enumerate the rational numbers in order...
0, ..., 1/2, ...
Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...
More fractions?

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Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...

A thing about fractions:
More fractions?

Enumerate the rational numbers in order...
0, ..., \(\frac{1}{2}, \ldots\)

Where is \(\frac{1}{2}\) in list?

After \(\frac{1}{3}\), which is after \(\frac{1}{4}\), which is after \(\frac{1}{5}\)...

A thing about fractions:
any two fractions has another fraction between it.
More fractions?

Enumerate the rational numbers in order...
0, ..., 1/2, ..

Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...

A thing about fractions:
any two fractions has another fraction between it.
Can’t even get to “next” fraction!
More fractions?

Enumerate the rational numbers in order...
0, . . . , 1/2, . . .
Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...
A thing about fractions:
any two fractions has another fraction between it.
Can’t even get to “next” fraction!
Can’t list in “order”.
Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
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E.g.: (1, 2), (100, 30), etc.
Pairs of natural numbers.

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For finite sets \( S_1 \) and \( S_2 \),
Pairs of natural numbers.

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For finite sets $S_1$ and $S_2$, then $S_1 \times S_2$
Consider pairs of natural numbers: $N \times N$
E.g.: $(1, 2)$, $(100, 30)$, etc.

For finite sets $S_1$ and $S_2$,
then $S_1 \times S_2$ has size $|S_1| \times |S_2|$. 
Pairs of natural numbers.

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Pairs of natural numbers.

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So, \( N \times N \) is countably infinite
Consider pairs of natural numbers: $N \times N$
E.g.: $(1, 2), (100, 30)$, etc.

For finite sets $S_1$ and $S_2$,
then $S_1 \times S_2$
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So, $N \times N$ is countably infinite squared
Pairs of natural numbers.

Consider pairs of natural numbers: \( N \times N \)
E.g.: \((1, 2), (100, 30), \) etc.

For finite sets \( S_1 \) and \( S_2 \),
then \( S_1 \times S_2 \)
has size \( |S_1| \times |S_2| \).

So, \( N \times N \) is countably infinite squared ???
Pairs of natural numbers.

Enumerate in list:

- (0, 0)
- (1, 0)
- (0, 1)
- (2, 0)
- (1, 1)
- (0, 2)

The pair \((a, b)\), is in first 
\[a + b + 1\] elements of list!

(i.e., "triangle").

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:

(0, 0),

(1, 0),

(0, 1),

(2, 0),

(1, 1),

(0, 2),

......
Pairs of natural numbers.

Enumerate in list:
\[(0,0), (1,0), \ldots\]
Pairs of natural numbers.

Enumerate in list:
(0, 0), (1, 0), (0, 1),
Pairs of natural numbers.

Enumerate in list:

\[(0,0), (1,0), (0,1), (2,0), \ldots\]

The pair \((a, b)\), is in first \(\approx (a + b + 1)(a + b) / 2\) elements of list!

(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:
(0, 0), (1, 0), (0, 1), (2, 0), (1, 1),
Pairs of natural numbers.

Enumerate in list:
(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), ……

The pair \((a, b)\), is in first \(\approx \frac{(a + b + 1)(a + b)}{2}\) elements of list!

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:
(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), …

The pair \((a, b)\) is in first \(\approx (a + b + 1)(a + b) / 2\) elements of list!

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:

(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), ……

The pair (a, b), is in first ≈ (a + b + 1)(a + b)/2 elements of list!

(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:
(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), ……

The pair \((a, b)\), is in first \(\approx (a + b + 1)(a + b)/2\) elements of list!

Countably infinite.
Same size as the natural numbers!!
Pairs of natural numbers.

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Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

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Pairs of natural numbers.

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The pair \((a, b)\), is in first \((a + b + 1)(a + b) / 2\) elements of list!

(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

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(i.e., "triangle").

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:
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Pairs of natural numbers.

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Pairs of natural numbers.

Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots$

The pair $(a,b)$ is in first $\approx (a+b+1)(a+b)/2$ elements of list! (i.e., “triangle”).
Pairs of natural numbers.

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(i.e., “triangle”).

Countably infinite.

Same size as the natural numbers!!
Poll.

Enumeration to get bijection with naturals?
Enumeration to get bijection with naturals?

(A) Integers: First all negatives, then positives.
(B) Integers: By absolute value, break ties however.
(C) Pairs of naturals: by sum of values, break ties however.
(D) Pairs of naturals: by value of first element.
(E) Pairs of integers: by sum of values, break ties.
(F) Pairs of integers: by sum of absolute values, break ties.
Poll.

Enumeration to get bijection with naturals?

(A) Integers: First all negatives, then positives.
(B) Integers: By absolute value, break ties however.
(C) Pairs of naturals: by sum of values, break ties however.
(D) Pairs of naturals: by value of first element.
(E) Pairs of integers: by sum of values, break ties.
(F) Pairs of integers: by sum of absolute values, break ties.

(B), (C), (F).
Rationals?

Positive rational number.
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
Rationals?

Positive rational number.
Lowest terms: $\frac{a}{b}$
\[ a, b \in N \]
Rationals?

Positive rational number.
Lowest terms: $a/b$
$a, b \in N$
with $gcd(a, b) = 1$. 

Infinite subset of $N \times N$.
Countably infinite!
All rational numbers?
Negative rationals are countable.
(Same size as positive rationals.)
Put all rational numbers in a list.
First negative, then nonegative
???
No!
Repeatedly and alternatively take one from each list.
Interleave Streams in 61A
The rationals are countably infinite.
Rationals?

Positive rational number.
Lowest terms: $a/b$

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First negative, then nonnegative
Rationals?

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$a, b \in N$
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Interleave Streams in 61A
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Lowest terms: \( a/b \)
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Infinite subset of \( N \times N \).

Countably infinite!

All rational numbers?

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Put all rational numbers in a list.

First negative, then nonegative ??? No!

Repeatedly and alternatively take one from each list.

Interleave Streams in 61A

The rationals are countably infinite.
Real numbers are same size as integers?
Are the set of reals countable?
The reals.

Are the set of reals countable?

Let's consider the reals [0, 1].
The reals.

Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$\frac{1}{2}, \pi, \frac{1}{e}, \frac{1}{2} - \frac{1}{e}, \ldots$
The reals.

Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$.500000000...$ (1/2)
The reals.

Are the set of reals countable?

Let's consider the reals \([0,1]\).

Each real has a decimal representation.

- \[0.500000000\ldots\] (1/2)
- \[0.785398162\ldots\]
The reals.

Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$0.500000000...$ (1/2)

$0.785398162...$ $\pi/4$
Are the set of reals countable?

Let's consider the reals $[0,1]$.

Each real has a decimal representation.

$.500000000...$ (1/2)

$.785398162...$ $\pi/4$

$.367879441...$
The reals.

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Let's consider the reals \([0, 1]\).

Each real has a decimal representation.

- .500000000... \((1/2)\)
- .785398162... \(\pi/4\)
- .367879441... \(1/e\)
The reals.

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Lets consider the reals [0, 1].
Each real has a decimal representation.
.500000000... (1/2)
.785398162... \(\pi/4\)
.367879441... 1/e
.632120558...
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Each real has a decimal representation.

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- $0.785398162...$ $\pi/4$
- $0.367879441...$ $1/e$
- $0.632120558...$ $1 - 1/e$
The reals.

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$.345212312...$
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

.500000000... (1/2)
.785398162... $\pi/4$
.367879441... $1/e$
.632120558... $1 - 1/e$
.345212312... Some real number
The reals.

Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

- $0.500000000\ldots$ (1/2)
- $0.785398162\ldots \pi/4$
- $0.367879441\ldots 1/e$
- $0.632120558\ldots 1 - 1/e$
- $0.345212312\ldots$ Some real number
Diagonalization.

If countable, there a listing, $L$ contains all reals.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

Construct "diagonal" number:

```
1.
2.
3.
4.
...
```

Diagonal Number:

```
Digit $i$ is 7 if number $i$'s $i$th digit is not 7 and 6 otherwise.
```

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0, 1]$ is not countable!!
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example
0: \( .5000000000... \)
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...

Construct “diagonal” number:

7 7 6 7 7 ...

Diagonal Number:

Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

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0: .500000000...
1: .785398162...
2: .367879441...
Diagonalization.

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0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
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Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: $.500000000...$
1: $.785398162...$
2: $.367879441...$
3: $.632120558...$
4: $.345212312...$

:
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

...

Construct “diagonal” number:
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .7
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .776
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .7767
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example:

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .500000000\ldots \)
1: \( .785398162\ldots \)
2: \( .367879441\ldots \)
3: \( .632120558\ldots \)
4: \( .345212312\ldots \)

\[ \vdots \]

Construct “diagonal” number: \( .77677\ldots \)
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \( .500000000 \ldots \)
1: \( .785398162 \ldots \)
2: \( .367879441 \ldots \)
3: \( .632120558 \ldots \)
4: \( .345212312 \ldots \)

Construct “diagonal” number: \( .77677 \ldots \)

Diagonal Number:
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677...

Diagonal Number: Digit $i$ is 7 if number $i$’s $i$th digit is not 7
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \[0.500000000\ldots\]

1: \[0.785398162\ldots\]

2: \[0.367879441\ldots\]

3: \[0.632120558\ldots\]

4: \[0.345212312\ldots\]

:\

Construct “diagonal” number: \[0.77677\ldots\]

Diagonal Number: Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \(.5000000000\ldots\)
1: \(.785398162\ldots\)
2: \(.367879441\ldots\)
3: \(.632120558\ldots\)
4: \(.345212312\ldots\)

Construct “diagonal” number: \(.77677\ldots\)

Diagonal Number: Digit \( i \) is 7 if number \( i \)’s \( i \)th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .500000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

:\:

Construct “diagonal” number: \( .77677... \)

Diagonal Number: Digit \( i \) is 7 if number \( i \)’s \( i \)th digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonal number not in list.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: $0.5000000000...$
1: $0.785398162...$
2: $0.367879441...$
3: $0.632120558...$
4: $0.345212312...$

Construct “diagonal” number: $0.77677...$

**Diagonal Number:** Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list! Diagonal number not in list.

Diagonal number is real.
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .500000000\ldots \)
1: \( .785398162\ldots \)
2: \( .367879441\ldots \)
3: \( .632120558\ldots \)
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Diagonal Number: Digit \( i \) is 7 if number \( i \)’s \( i \)th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: .500000000...
1: .785398162...
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...

Construct “diagonal” number: .77677…

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Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset \([0,1]\) is not countable!!
All reals?

 Subset $[0, 1]$ is not countable!!
All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
All reals?

Subset \([0, 1]\) is not countable!!

What about all reals?
No.
All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.
All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.
If reals are countable then so is $[0, 1]$. 
Diagonalization.

1. Assume that a set $S$ can be enumerated.
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2. Consider an arbitrary list of all the elements of $S$. 

Diagonalization.
Diagonalization.

1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$.
3. Use the diagonal from the list to construct a new element $t$.

4. Show that $t$ is different from all elements in the list $⇒ t$ is not in the list.
5. Show that $t$ is in $S$.
6. Contradiction.
Diagonalization.

1. Assume that a set $S$ can be enumerated.
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Diagonalization.

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5. Show that $t$ is in $S$.
6. Contradiction.
Another diagonalization.

The set of all subsets of $\mathbb{N}$. 
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$,
Another diagonalization.

The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: $\{0\}$, $\{0,\ldots,7\}$,
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0, \ldots, 7\} \),
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \{0\}, \{0, \ldots, 7\},
evens,
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0,\ldots,7\}$, evens, odds, 
Another diagonalization.

The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: $\{0\}$, $\{0,\ldots,7\}$, evens, odds, primes,
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \), \( \{0, \ldots, 7\} \),
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The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes,

Assume is countable.
Another diagonalization.

The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $\mathbb{N}$. 

==Theorem:== The set of all subsets of $\mathbb{N}$ is not countable. (The set of all subsets of $S$, is the powerset of $\mathbb{N}$.)


Another diagonalization.

The set of all subsets of \(N\).

Example subsets of \(N\): \(\{0\}\), \(\{0, \ldots, 7\}\), evens, odds, primes,

Assume is countable.

There is a listing, \(L\), that contains all subsets of \(N\).
Define a diagonal set, \(D\):

\[
\text{If } i \text{th set in } L \text{ does not contain } i, \quad i \in D
\]

\[
\text{otherwise } i \not\in D
\]

\(D\) is different from \(i\)th set in \(L\) for every \(i\).

\(\Rightarrow \) \(D\) is not in the listing.

\(D\) is a subset of \(N\).

\(L\) does not contain all subsets of \(N\).

Contradiction.

Theorem: The set of all subsets of \(N\) is not countable.

(The set of all subsets of \(S\), is the powerset of \(N\).)
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$. 

Theorem: The set of all subsets of $N$ is not countable.

(The set of all subsets of $S$, is the powerset of $N$.)
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Another diagonalization.

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Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.
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$D$ is different from $i$th set in $L$ for every $i$.
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\}, \{0, \ldots , 7\}, \) evens, odds, primes,

Assume is countable.

There is a listing, \( L \), that contains all subsets of \( N \).

Define a diagonal set, \( D \):
If \( i \)th set in \( L \) does not contain \( i \), \( i \in D \).
otherwise \( i \notin D \).

\( D \) is different from \( i \)th set in \( L \) for every \( i \).
\( \implies D \) is not in the listing.

\( D \) is a subset of \( N \).
\( L \) does not contain all subsets of \( N \).

Contradiction.

Theorem: The set of all subsets of \( N \) is not countable.
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Another diagonalization.

The set of all subsets of $N$.

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Another diagonalization.

The set of all subsets of $N$.

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Contradiction.

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\[\implies D\text{ is not in the listing.}\]

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**Theorem:** The set of all subsets of $N$ is not countable.
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$:  
\{0\}, \{0, \ldots, 7\}, 
evens, odds, primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.
otherwise $i \not\in D$.

$D$ is different from $i$th set in $L$ for every $i$.
$\implies D$ is not in the listing.

$D$ is a subset of $N$.

$L$ does not contain all subsets of $N$.

Contradiction.

Theorem: The set of all subsets of $N$ is not countable.  
(The set of all subsets of $S$, is the powerset of $N$.)
Poll: diagonalization Proof.

Mark parts of proof.
Poll: diagonalization Proof.

Mark parts of proof.

(A) Integers are larger than naturals cuz obviously.
(B) Integers are countable cuz, interleaving bijection.
(C) Reals are uncountable cuz obviously!
(D) Reals can’t be in a list: diagonal number not on list.
(E) Powerset in list: diagonal set not in list.
Poll: diagonalization Proof.

Mark parts of proof.

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(E) Powerset in list: diagonal set not in list.

(B), (C)?, (D), (E)
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals. First of Hilbert’s problems!
Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?
Cardinalities of uncountable sets?

Cardinality of \([0, 1]\) smaller than all the reals?

\[ f : R^+ \rightarrow [0, 1]. \]
Cardinalities of uncountable sets?

Cardinality of \([0, 1]\) smaller than all the reals?

\[ f : R^+ \rightarrow [0, 1]. \]

\[ f(x) = \begin{cases} 
  x + \frac{1}{2} & 0 \leq x \leq 1/2 \\
  \frac{1}{4x} & x > 1/2
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One to one.
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If both in $[0, 1/2]$,
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f(x) = \begin{cases} 
\frac{1}{4x} & x > 1/2 \\
x + \frac{1}{2} & 0 \leq x \leq 1/2
\end{cases}
\]

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If one is in $[0, 1/2]$ and one isn’t,
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Bijection!
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If one is in \([0, 1/2]\) and one isn’t, different ranges \(\implies f(x) \neq f(y)\).

Bijection!

\([0, 1]\) is same cardinality as nonnegative reals!
Rao is freaked out.

Are real numbers even real?
Rao is freaked out.

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Almost all real numbers can’t be described.
Rao is freaked out.

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\( \pi \)?
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$\pi$?
The ratio of the perimeter of a circle to its diameter.
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e? Transendental number.

\[ \lim_{n \to \infty} (1 + 1/n)^n. \]
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\[
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\]

$\sqrt{2}$? Algebraic number.
A solution of $x^2 = 2$. 

Really, rationals seem fine for... say... calculus. 

\[
\lim_{n \to \infty} \sum_{i=1}^{\infty} (b-a) f(x_i), \\
\text{where } x_i = a + i \times \frac{b-a}{n}.
\]

So why real numbers?

$\int_a^b f(x) \, dx$ is beautiful, succinct notation for a beautiful, succinct, powerful idea. 

What’s the idea?

Area.
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\[ \lim_{n \to \infty} \sum_{i=0}^{n} \frac{(b-a)}{n} f(x_i), \text{ where } x_i = a + i \times (b-a)/n. \]
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\[ \int_{a}^{b} f(x)dx \text{ is beautiful, succint notation} \]
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So why real numbers?

\[ \int_{a}^{b} f(x)dx \text{ is beautiful, succinct notation for a beautiful, succinct, powerful idea.} \]

What’s the idea?
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So why real numbers?

\(\int_{a}^{b} f(x)dx\) is beautiful, succinct notation for a beautiful, succinct, powerful idea.

What’s the idea? Area.
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.
Resolution of hypothesis?
Resolution of hypothesis?

Gödel. 1940.
Can’t use math!
Resolution of hypothesis?

Gödel. 1940.
Can’t use math!
If math doesn’t contain a contradiction.
Resolution of hypothesis?

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Is the statement above true?
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Is the statement above true?

The barber shaves every person who does not shave themselves.
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Who shaves the barber?
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Uh oh....
The barber shaves every person who does not shave themselves.
The Barber!

The barber shaves every person who does not shave themselves.

(A) Barber not Mark. Barber shaves Mark.
(B) Mark shaves the Barber.
(C) Barber doesn’t shave themself.
(D) Barber shaves themself.

Its all true.
It’s all a problem.
The barber shaves every person who does not shave themselves.

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There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

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Recall: powerset of the naturals is not countable.
Resolution of hypothesis?

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