Poll: How big is infinity?

(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) Pairs of natural numbers.
Poll: How big is infinity?

Mark what’s true.
(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) pairs of natural numbers >> natural numbers.
Two sets are the same size?
Two sets are the same size?

(A) Bijection between the sets.
(B) Count the objects and get the same number. same size.
(C) Counting to infinity is hard.
Two sets are the same size?

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(A), (B).
Two sets are the same size?

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(A), (B).
(C)?
Countable.

How to count?
Countable.

How to count?
0,
How to count?

0, 1,
Countable.

How to count?
0, 1, 2,
How to count?
0, 1, 2, 3,
How to count?
0, 1, 2, 3, …
Countable.

How to count?
0, 1, 2, 3, …
The Counting numbers.
Countable.

How to count?
0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! $\mathbb{N}$
Countable.

How to count?
0, 1, 2, 3, …
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The natural numbers! \( N \)

Definition: \( S \) is **countable** if there is a bijection between \( S \) and some subset of \( N \).
Countable.

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0, 1, 2, 3, …
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If the subset of \( N \) is finite, \( S \) has finite **cardinality**.
Countable.

How to count?
0, 1, 2, 3, …

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Definition: $S$ is **countable** if there is a bijection between $S$ and some subset of $N$.

If the subset of $N$ is finite, $S$ has finite **cardinality**.

If the subset of $N$ is infinite, $S$ is **countably infinite**.
Countably infinite subsets.

Enumerating a set implies countable.  
Corollary: Any subset $T$ of a countable set $S$ is countable.
Countably infinite subsets.

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Corollary: Any subset $T$ of a countable set $S$ is countable.  
Enumerate $T$ as follows:  
Get next element, $x$, of $S$,
Countably infinite subsets.

Enumerating a set implies countable. 
Corollary: Any subset $T$ of a countable set $S$ is countable.

Enumerate $T$ as follows:
Get next element, $x$, of $S$,
output only if $x \in T$. 

Implications:
$\mathbb{Z}^+$ is countable.
It is infinite since the list goes on.
There is a bijection with the natural numbers.
So it is countably infinite.
All countably infinite sets have the same cardinality.
Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset $T$ of a countable set $S$ is countable.

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Enumeration example.

All binary strings.
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\( B = \{0, 1\}^* \).
All binary strings.
$B = \{0, 1\}^*$.  
$B = \{\phi, \}$
All binary strings.

$B = \{0, 1\}^*$. 

$B = \{\phi, 0,$
Enumeration example.

All binary strings. 
\( B = \{0, 1\}^* \).

\( B = \{\emptyset, 0, 1, \ldots\} \). 
\( \emptyset \) is empty string. For any string, it appears at some position in the list. If \( n \) bits, it will appear before position \( 2^{n+1} \). 

Should be careful here.
All binary strings.
\[ B = \{0, 1\}^* \]
\[ B = \{\phi, 0, 1, 00,\ldots\} \]
\[ \phi \text{ is empty string.} \]
For any string, it appears at some position in the list.
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Enumeration example.

All binary strings.

$B = \{0, 1\}^*.$

$B = \{\emptyset, 0, 1, 00, 01, 10, 11, \ldots\}.$
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\(B = \{\emptyset, 0, 00, 000, 0000, \ldots\}\)
Enumeration example.

All binary strings.
$B = \{0, 1\}^*$. 

$B = \{\phi, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$. 

$\phi$ is empty string.

For any string, it appears at some position in the list. If $n$ bits, it will appear before position $2^{n+1}$.

Should be careful here.

$B = \{\phi; 0, 00, 000, 0000, \ldots\}$

Never get to 1.
More fractions?

Enumerate the rational numbers in order...
More fractions?

Enumerate the rational numbers in order...

0, ..., 1/2, ..
Enumerate the rational numbers in order...

0, ..., 1/2,..

Where is 1/2 in list?
More fractions?

Enumerate the rational numbers in order...
0, ..., 1/2, ...

Where is 1/2 in list?

After 1/3, which is after 1/4, which is after 1/5...
More fractions?

Enumerate the rational numbers in order...
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Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...
A thing about fractions:
More fractions?

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A thing about fractions:
any two fractions has another fraction between it.
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0, ..., 1/2, ...
Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...
A thing about fractions:
any two fractions has another fraction between it.
Can’t even get to “next” fraction!
More fractions?

Enumerate the rational numbers in order...
0, ..., 1/2, ...

Where is 1/2 in list?
After 1/3, which is after 1/4, which is after 1/5...

A thing about fractions:
any two fractions has another fraction between it.
Can’t even get to “next” fraction!
Can’t list in “order”.

Pairs of natural numbers.

Consider pairs of natural numbers: $N \times N$
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For finite sets \( S_1 \) and \( S_2 \),
Consider pairs of natural numbers: \( N \times N \)

E.g.: (1, 2), (100, 30), etc.

For finite sets \( S_1 \) and \( S_2 \), then \( S_1 \times S_2 \)
Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.
For finite sets $S_1$ and $S_2$, then $S_1 \times S_2$
has size $|S_1| \times |S_2|$. 

Pairs of natural numbers.
Consider pairs of natural numbers: $N \times N$
E.g.: $(1, 2), (100, 30), \text{ etc.}$

For finite sets $S_1$ and $S_2$, 
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Consider pairs of natural numbers: \( N \times N \)
E.g.: (1, 2), (100, 30), etc.

For finite sets \( S_1 \) and \( S_2 \),
then \( S_1 \times S_2 \)
has size \( |S_1| \times |S_2| \).

So, \( N \times N \) is countably infinite
Consider pairs of natural numbers: $N \times N$
E.g.: $(1,2)$, $(100,30)$, etc.

For finite sets $S_1$ and $S_2$, then $S_1 \times S_2$
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So, $N \times N$ is countably infinite squared
Consider pairs of natural numbers: $N \times N$
E.g.: (1, 2), (100, 30), etc.

For finite sets $S_1$ and $S_2$, then $S_1 \times S_2$
has size $|S_1| \times |S_2|$.

So, $N \times N$ is countably infinite squared ????
Pairs of natural numbers.

Enumerate in list:

(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2),......

The pair \((a, b)\), is in first \(\approx (a + b + 1)(a + b)/2\) elements of list!

Countably infinite.

Same size as the natural numbers!!
Pairs of natural numbers.

Enumerate in list:

\((0,0), (1,0), (0,1), (2,0), (1,1), (0,2), \ldots\)

The pair \((a, b)\), is in first \(\approx (a+b+1)(a+b)/2\) elements of list!

(i.e., "triangle").

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The pair \((a, b)\), is in first \(\approx (a + b + 1)(a + b) / 2\) elements of list!

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Enumeration to get bijection with naturals?
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(A) Integers: First all negatives, then positives.
(B) Integers: By absolute value, break ties however.
(C) Pairs of naturals: by sum of values, break ties however.
(D) Pairs of naturals: by value of first element.
(E) Pairs of integers: by sum of values, break ties.
(F) Pairs of integers: by sum of absolute values, break ties.
Enumeration to get bijection with naturals?

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(B), (C), (F).
Rationals?

Positive rational number.

\[ \frac{a}{b}, \quad a, b \in \mathbb{N} \text{ with } \gcd(a, b) = 1 \]

Infinite subset of \( \mathbb{N} \times \mathbb{N} \).

Countably infinite!

All rational numbers?

Negative rationals are countable. (Same size as positive rationals.)

Put all rational numbers in a list. First negative, then nonegative? No!

Repeatedly and alternatively take one from each list. Interleave Streams in 61A

The rationals are countably infinite.
Rationals?

Positive rational number.
Lowest terms: \( a/b \)
Rationals?

Positive rational number.
Lowest terms: \( \frac{a}{b} \)
\( a, b \in N \)
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Interleave Streams in 61A
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Interleave Streams in 61A

The rationals are countably infinite.
Real numbers..

Real numbers are same size as integers?
Are the set of reals countable?
The reals.

Are the set of reals countable?
Let's consider the reals $[0, 1]$. 
Are the set of reals countable?

Let's consider the reals \([0, 1]\).

Each real has a decimal representation.
The reals.

Are the set of reals countable?

Let's consider the reals [0, 1].

Each real has a decimal representation.

.5000000000...
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

$.5000000000... \ (1/2)$
Are the set of reals countable?

Let's consider the reals [0, 1].

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.500000000... (1/2)
.785398162...
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\(0.500000000\ldots\) (1/2)
\(0.785398162\ldots\) \(\pi/4\)
The reals.

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.345212312...
Are the set of reals countable?

Let's consider the reals $[0, 1]$.

Each real has a decimal representation.

- $0.500000000\ldots$ (1/2)
- $0.785398162\ldots \pi/4$
- $0.367879441\ldots 1/e$
- $0.632120558\ldots 1 - 1/e$
- $0.345212312\ldots$ Some real number
The reals.

Are the set of reals countable?

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- $0.345212312...$ Some real number
Diagonalization.

If countable, there a listing, $L$ contains all reals.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example 0: .500000000...

Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...

Diagonal number for a list differs from every number in list!
Diagonal number not in list.
Diagonal number is real.
Contradiction!
Subset $[0, 1]$ is not countable!!
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...

Construct "diagonal" number:

...77677...

Diagonal Number:

Digit $i$ is 7 if number $i$'s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!

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3: .632120558...
Diagonalization.

If countable, there is a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct "diagonal" number:

.77677...

Diagonal Number: Digit $i$ is 7 if number $i$'s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!

Diagonal number not in list.

Diagonal number is real.

Contradiction!

Subset $[0, 1]$ is not countable!!
Diagonalization.

If countable, there is a listing, \( L \) contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

\[ \vdots \]
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \( .500000000 \ldots \)
1: \( .785398162 \ldots \)
2: \( .367879441 \ldots \)
3: \( .632120558 \ldots \)
4: \( .345212312 \ldots \)

Construct “diagonal” number:
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .7
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .500000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

... 

Construct “diagonal” number: \( .77 \)
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \( .500000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

Construct “diagonal” number: \( .776 \)
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .5000000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

Construct “diagonal” number: \( .7767 \)
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \[.5000000000\ldots\]
1: \[.785398162\ldots\]
2: \[.367879441\ldots\]
3: \[.632120558\ldots\]
4: \[.345212312\ldots\]

Construct “diagonal” number: \[.77677\]
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677…
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .\overline{500000000} \)
1: \( .\overline{785398162} \)
2: \( .\overline{367879441} \)
3: \( .\overline{632120558} \)
4: \( .\overline{345212312} \)

...

Construct “diagonal” number: \( .\overline{77677} \)...

Diagonal Number:
Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .500000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

...

Construct “diagonal” number: \( .77677... \)

Diagonal Number: Digit \( i \) is 7 if number \( i \)’s \( i \)th digit is not 7
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: \[.500000000\ldots\]
1: \[.785398162\ldots\]
2: \[.367879441\ldots\]
3: \[.632120558\ldots\]
4: \[.345212312\ldots\]

::

Construct “diagonal” number: \[.77677\ldots\]

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Diagonalization.

If countable, there a listing, \( L \) contains all reals. For example

0: \( .500000000... \)
1: \( .785398162... \)
2: \( .367879441... \)
3: \( .632120558... \)
4: \( .345212312... \)

: 

Construct “diagonal” number: \( .77677... \)

Diagonal Number: Digit \( i \) is 7 if number \( i \)'s \( i \)th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list!
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

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3: \( .632120558 \ldots \)
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\( \vdots \)

Construct “diagonal” number: \( .77677 \ldots \)

Diagonal Number: Digit \( i \) is 7 if number \( i \)’s \( i \)th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list! Diagonal number not in list.
Diagonalization.

If countable, there a listing, $L$ contains all reals. For example

0: .500000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

Construct “diagonal” number: .77677...

Diagonal Number: Digit $i$ is 7 if number $i$’s $i$th digit is not 7 and 6 otherwise.

Diagonal number for a list differs from every number in list! Diagonal number not in list.
Diagonal number is real.
Diagonalization.

If countable, there a listing, *L contains all reals*. For example

0: .5000000000...
1: .785398162...
2: .367879441...
3: .632120558...
4: .345212312...

::

Construct “diagonal” number: .77677...

Diagonal Number: Digit \(i\) is 7 if number \(i\)’s \(i\)th digit is not 7
and 6 otherwise.

Diagonal number for a list differs from every number in list!
**Diagonal number not in list.**

Diagonal number is real.

Contradiction!
Diagonalization.

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0: \( .500000000 \ldots \)
1: \( .785398162 \ldots \)
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Diagonal number is real.
Contradiction!
Subset \([0, 1]\) is not countable!!
All reals?

Subset $[0, 1]$ is not countable!!
All reals?

Subset [0, 1] is not countable!!

What about all reals?
All reals?

Subset \([0, 1]\) is not countable!!

What about all reals?
No.
All reals?

Subset [0, 1] is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.
All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.
If reals are countable then so is $[0, 1]$. 
1. Assume that a set $S$ can be enumerated.
Diagonalization.

1. Assume that a set $S$ can be enumerated.
2. Consider an arbitrary list of all the elements of $S$. 

   Show that the constructed element $t$ is different from all elements in the list $\Rightarrow t$ is not in the list.

   Show that $t$ is in $S$.

   Contradiction.
Diagonalization.

1. Assume that a set \( S \) can be enumerated.
2. Consider an arbitrary list of all the elements of \( S \).
3. Use the diagonal from the list to construct a new element \( t \).
Diagonalization.

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1. Assume that a set $S$ can be enumerated.
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4. Show that $t$ is different from all elements in the list $\iff t$ is not in the list.
5. Show that $t$ is in $S$.
6. Contradiction.
Diagonalization.

1. Assume that a set $S$ can be enumerated.
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1. Assume that a set $S$ can be enumerated.
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5. Show that $t$ is in $S$.
6. Contradiction.
Another diagonalization.

The set of all subsets of $N$. 

Example subsets of $N$: 

* $\{0\}$, 
* $\{0, \ldots, 7\}$, 
* evens, 
* odds, 
* primes,

Assume is countable.

There is a listing, $L$, that contains all subsets of $N$.

Define a diagonal set, $D$:

If $i$th set in $L$ does not contain $i$, $i \in D$. Otherwise $i \not\in D$.

$D$ is different from $i$th set in $L$ for every $i$.

$= \Rightarrow D$ is not in the listing.

$D$ is a subset of $N$.

$L$ does not contain all subsets of $N$.

Contradiction.

Theorem: The set of all subsets of $N$ is not countable.

(The set of all subsets of $S$, is the powerset of $N$.)
Another diagonalization.

The set of all subsets of \( N \).

Example subsets of \( N \): \( \{0\} \),
Another diagonalization.

The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: $\{0\}$, $\{0, \ldots, 7\}$,
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: \{0\}, \{0, \ldots, 7\},
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0, \ldots, 7\}$, evens,
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: $\{0\}$, $\{0,\ldots,7\}$, evens, odds,
Another diagonalization.

The set of all subsets of $N$.

Example subsets of $N$: \{0\}, \{0,\ldots,7\},
evens, odds, primes,
Another diagonalization.

The set of all subsets of $\mathbb{N}$.

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Another diagonalization.

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Define a diagonal set, $D$:
If $i$th set in $L$ does not contain $i$, $i \in D$.
\[\text{otherwise } i \notin D.\]

$D$ is different from $i$th set in $L$ for every $i$.
\[\implies D \text{ is not in the listing.}\]
Another diagonalization.

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$D$ is a subset of $N$.

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Contradiction.

**Theorem:** The set of all subsets of $N$ is not countable.
Another diagonalization.

The set of all subsets of $\mathbb{N}$.

Example subsets of $\mathbb{N}$: $\{0\}$, $\{0, \ldots, 7\}$, evens, odds, primes,

Assume is countable.

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$D$ is a subset of $\mathbb{N}$.

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Contradiction.

**Theorem:** The set of all subsets of $\mathbb{N}$ is not countable.
(The set of all subsets of $S$, is the **powerset** of $\mathbb{N}$.)
Poll: diagonalization Proof.

Mark parts of proof.
Poll: diagonalization Proof.

Mark parts of proof.

(A) Integers are larger than naturals cuz obviously.
(B) Integers are countable cuz, interleaving bijection.
(C) Reals are uncountable cuz obviously!
(D) Reals can’t be in a list: diagonal number not on list.
(E) Powerset in list: diagonal set not in list.
Poll: diagonalization Proof.

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(E) Powerset in list: diagonal set not in list.

(B), (C)?, (D), (E)
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals.
The Continuum hypothesis.

There is no set with cardinality between the naturals and the reals. First of Hilbert’s problems!
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : \mathbb{R}^+ \to [0, 1]$.

$f(x) = x + 1/2$ if $0 \leq x \leq 1/2$.

One to one.

If both in $[0, 1/2]$, a shift $= \Rightarrow f(x) \neq f(y)$.

If neither in $[0, 1/2]$, a division $= \Rightarrow f(x) \neq f(y)$.

If one is in $[0, 1/2]$ and one isn't, different ranges $= \Rightarrow f(x) \neq f(y)$.

Bijection!

$[0, 1]$ is same cardinality as nonnegative reals!
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Cardinalities of uncountable sets?

Cardinality of $[0,1]$ smaller than all the reals?

$f : \mathbb{R}^+ \rightarrow [0,1]$.

$$f(x) = \begin{cases} 
    x + \frac{1}{2} & 0 \leq x \leq 1/2 \\
    \frac{1}{4x} & x > 1/2
\end{cases}$$
Cardinalities of uncountable sets?

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One to one.  $x \neq y$
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If both in $[0, 1/2]$, a shift $\implies f(x) \not= f(y)$. 
Cardinalities of uncountable sets?

Cardinality of [0, 1] smaller than all the reals?

\[ f : \mathbb{R}^+ \rightarrow [0, 1]. \]

\[ f(x) = \begin{cases} 
  x + \frac{1}{2} & 0 \leq x \leq \frac{1}{2} \\
  \frac{1}{4x} & x > \frac{1}{2}
\end{cases} \]

One to one. \( x \neq y \)

If both in \([0, 1/2]\), a shift \( f(x) \neq f(y) \).

If neither in \([0, 1/2]\)
Cardinalities of uncountable sets?

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If one is in $[0, 1/2]$ and one isn’t,
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$$f(x) = \begin{cases} 
    x + \frac{1}{2} & 0 \leq x \leq 1/2 \\
    \frac{1}{4x} & x > 1/2 
\end{cases}$$

One to one. $x \neq y$

If both in $[0,1/2]$, a shift $\implies f(x) \neq f(y)$.

If neither in $[0,1/2]$ a division $\implies f(x) \neq f(y)$.

If one is in $[0,1/2]$ and one isn’t, different ranges $\implies f(x) \neq f(y)$. 
Cardinalities of uncountable sets?

Cardinality of $[0, 1]$ smaller than all the reals?

$f : \mathbb{R}^+ \rightarrow [0, 1]$.

$$f(x) = \begin{cases} \frac{1}{2} + \frac{1}{4x} & 0 \leq x \leq 1/2 \\ \frac{1}{4x} & x > 1/2 \end{cases}$$

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- If neither in $[0, 1/2]$ a division $\Rightarrow f(x) \neq f(y)$.
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Bijection!
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If one is in $[0, 1/2]$ and one isn’t, different ranges $\implies f(x) \neq f(y)$.

Bijection!

$[0, 1]$ is same cardinality as nonnegative reals!
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.
Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.
Resolution of hypothesis?

Gödel. 1940.

Can't use math!

If math doesn't contain a contradiction.

This statement is a lie.

Is the statement above true?

The barber shaves every person who does not shave themselves.

Who shaves the barber?

Self reference.

Can a program refer to a program?

Can a program refer to itself?

Uh oh....
Resolution of hypothesis?

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Uh oh....
The barber shaves every person who does not shave themselves.
The Barber!

The barber shaves every person who does not shave themselves.

(A) Barber not Mark. Barber shaves Mark.
(B) Mark shaves the Barber.
(C) Barber doesn’t shave themself.
(D) Barber shaves themself.
The barber shaves every person who does not shave themselves.

(A) Barber not Mark. Barber shaves Mark.
(B) Mark shaves the Barber.
(C) Barber doesn’t shave themself.
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Its all true.
The barber shaves every person who does not shave themselves.

(A) Barber not Mark. Barber shaves Mark.
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(C) Barber doesn’t shave themself.
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It’s all true. It’s all a problem.
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Generalized Continuum hypothesis.

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Recall: powerset of the naturals is not countable.
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Changing Axioms?

Goedel:
Any set of axioms is either
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Changing Axioms?

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Any set of axioms is either inconsistent (can prove false statements) or incomplete (true statements cannot be proven.)

Concrete example:

BTW:
Cantor..bipolar disorder..
Goedel..starved himself out of fear of being poisoned..
Russell..was fine...
..but for...
two schizophrenic children..

Dangerous work?
See Logicomix by Doxiaidis, Papadimitriou (was professor here), Papadatos, Di Donna.
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Write me a program checker!
Is it actually useful?

Write me a program checker!
Check that the compiler works!
Is it actually useful?

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How about.. Check that the compiler terminates on a certain input.
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$HALT(P, I)$
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Program is a text string.
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Program can be an input to a program.
Implementing HALT.

HALT (P, I)  

P - program  
I - input.  

Determines if P(I) (P run on I) halts or loops forever.

Run P on I and check!

How long do you wait?

Something about infinity here, maybe?
Implementing HALT.

\[ \text{HALT}(P, I) \]
Implementing HALT.

\[ \text{HALT}(P, I) \]
\[ P - \text{program} \]
Implementing HALT.

$HALT(P, I)$

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How long do you wait?

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Halt does not exist.
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**Theorem:** There is no program $HALT$. 
Halt does not exist.

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Determines if $P(I)$ ($P$ run on $I$) halts or loops forever.

**Theorem:** There is no program $HALT$.

**Proof:** Yes!
Halt does not exist.

\[ \text{HALT}(P, I) \]
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**Proof:** Yes! No!
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Yes! No!...

What is he talking about?
Yes! No!...

What is he talking about?

(A) He is confused.
(B) Diagonalization.
(C) Welch-Berlekamp
(D) Professor is just strange.
Yes! No!...

What is he talking about?

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What is he talking about?
(A) He is confused.  
(B) Diagonalization.  
(C) Welch-Berlekamp  
(D) Professor is just strange.

(B) and (D)
Yes! No!...

What is he talking about?

(A) He is confused.
(B) Diagonalization.
(C) Welch-Berlekamp
(D) Professor is just strange.

(B) and (D) maybe?
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What is he talking about?

(A) He is confused.
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(D) Professor is just strange.

(B) and (D) maybe? and maybe (A).

Professor does me some love Welch-Berlekamp though!
Halt and Turing.

Proof:

Assume there is a program HALT(·, ·).

Turing(P)
1. If HALT(P, P) = "halts", then go into an infinite loop.
2. Otherwise, halt immediately.

Assumption: there is a program HALT.

There is text that "is" the program HALT.

There is text that is the program Turing.

Can run Turing on Turing!

Does Turing(Turing) halt?

Turing(Turing) halts = ⇒ HALTS(Turing, Turing) = halts = ⇒ Turing(Turing) loops forever.

Turing(Turing) loops forever = ⇒ HALTS(Turing, Turing)̸= halts = ⇒ Turing(Turing) halts.

Contradiction.

Program HALT does not exist!

Questions?
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot, \cdot)$. 

1. If $HALT(P, P) = \text{halts}$, then go into an infinite loop.
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Can run $Turing$ on $Turing$!

Does $Turing(Turing)$ halt?

$Turing(Turing)$ halts $\Rightarrow$ then $HALTS(Turing, Turing) = \text{halts} = \Rightarrow$ $Turing(Turing)$ loops forever.

$Turing(Turing)$ loops forever $\Rightarrow$ then $HALTS(Turing, Turing) \neq \text{halts} = \Rightarrow$ $Turing(Turing)$ halts.

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Program $HALT$ does not exist!
Halt and Turing.

**Proof:** Assume there is a program $HALT(\cdot,\cdot)$.

Turing($P$)
Halt and Turing.

**Proof:** Assume there is a program \( HALT(P,P) \).

**Turing(P)**

1. If \( HALT(P,P) = \)“halts”, then go into an infinite loop.
Halt and Turing.

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Does $Turing(Turing)$ halt?

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Does $\text{Turing}(\text{Turing})$ halt?

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Questions?
Another view of proof: diagonalization.

Any program is a fixed length string.
Another view of proof: diagonalization.

Any program is a fixed length string. Fixed length strings are enumerable.
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Any program is a fixed length string. Fixed length strings are enumerable. Program halts or not on any input, which is a string.
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Halt - diagonal. Turing - is not Halt. and is different from every $P_i$ on the diagonal. Turing is not on list. Turing is not a program. Halt does not exist!
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\[
\begin{array}{c|cccc}
\text{ } & P_1 & P_2 & P_3 & \ldots \\
\hline
P_1 & H & H & L & \ldots \\
P_2 & L & L & H & \ldots \\
P_3 & L & H & H & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

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Halt does not exist!
What are programs?
What are programs?
(A) Instructions.
(B) Text.
(C) Binary String.
(D) They run on computers.
Programs?

What are programs?

(A) Instructions.
(B) Text.
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All are correct.
Proof play by play.

Assumed HALT($P, I$) existed.
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

What is $P$?
Assumed $\text{HALT}(P,I)$ existed.

What is $P$? Text.
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

What is $P$? Text.
What is $I$?
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Assumed $\text{HALT}(P, I)$ existed.

What is $P$? Text.
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What does it mean to have a program $\text{HALT}(P, I)$.
Proof play by play.

Assumed HALT($P, I$) existed.

What is $P$? Text.
What is $I$? Text.

What does it mean to have a program HALT($P, I$).
You have Text that is the program HALT($P, I$).
Proof play by play.

Assumed HALT\((P, I)\) existed.

What is \(P\)? Text.
What is \(I\)? Text.

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Proof play by play.

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What does it mean to have a program $\text{HALT}(P, I)$.
   You have $Text$ that is the program $\text{HALT}(P, I)$.

Have ___ that is the program $\text{TURING}$.
Proof play by play.

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Have Text that is the program TURING.
Here it is!!
Proof play by play.

Assumed \( \text{HALT}(P, I) \) existed.

What is \( P \)? Text.
What is \( I \)? Text.

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  You have Text that is the program \( \text{HALT}(P, I) \).

Have Text that is the program Turing.
Here it is!!

\( \text{Turing}(P) \)
Proof play by play.

Assumed $\text{HALT}(P, I)$ existed.

What is $P$? Text.

What is $I$? Text.

What does it mean to have a program $\text{HALT}(P, I)$.

You have Text that is the program $\text{HALT}(P, I)$.

Have Text that is the program TURING.
Here it is!!

$\text{Turing}(P)$

1. If $\text{HALT}(P, P) = \text{"halts"}$, then go into an infinite loop.
Proof play by play.

Assumed \( \text{HALT}(P, I) \) existed.

What is \( P \)? Text.
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\( \text{Turing}(P) \)
   1. If \( \text{HALT}(P, P) = \text{"halts"} \), then go into an infinite loop.
   2. Otherwise, halt immediately.
Proof play by play.

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Turing “diagonalizes” on list of program.
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It is not a program!!!!
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\text{Turing}(P)

1. If \( \text{HALT}(P, P) = \text{"halts"} \), then go into an infinite loop.
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Turing “diagonalizes” on list of program.
It is not a program!!!!

\( \implies \) HALT is not a program.
Proof play by play.

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Turing “diagonalizes” on list of program.
It is not a program!!!!

$\implies$ $\text{HALT}$ is not a program.

Questions?
We are so smart!

Wow, that was easy!
We are so smart!

Wow, that was easy!
We should be famous!
No computers for Turing!

In Turing’s time.
No computers for Turing!

In Turing’s time.
No computers.
No computers for Turing!

In Turing’s time.
No computers.
Adding machines.
No computers for Turing!

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e.g., Babbage (from table of logarithms) 1812.
No computers for Turing!

In Turing’s time.
No computers.
Adding machines.
e.g., Babbage (from table of logarithms) 1812.
Concept of program as data wasn’t really there.
Turing machine.
A Turing machine.
– an (infinite) tape with characters
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– be in a state, and read a character
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Universal Turing machine
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Universal Turing machine
– an interpreter program for a Turing machine
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Now that’s a computer!
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Turing: AI,
A Turing machine.  
– an (infinite) tape with characters  
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Universal Turing machine  
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Turing: AI, self modifying code,
Turing machine.

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– an (infinite) tape with characters
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Turing: AI, self modifying code, learning...
Turing and computing.

Just a mathematician?
Turing and computing.

Just a mathematician?

“Wrote” a chess program.
Turing and computing.

Just a mathematician?

“Wrote” a chess program.

Simulated the program by hand to play chess.
Turing and computing.

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The polish machine...the *bomba*. 
Computing on top of computing...

Computer, assembly code, programming language, browser, html, javascript..
Computing on top of computing...

Computer, assembly code, programming language, browser, html, javascript..
We can’t get enough of building more Turing machines.
Undecidable problems.

Does a program, $P$, print “Hello World”?
Undecidable problems.

Does a program, $P$, print “Hello World”? How?
Undecidable problems.

Does a program, $P$, print “Hello World”? How? What is $P$?

Can a set of notched tiles tile the infinite plane? Proof: simulate a computer. Halts if finite.

Does a set of integer equations have a solution? Example: $x^n + y^n = 1$?

Problem is undecidable.

Be careful!

Is there an integer solution to $x^n + y^n = 1$? (Diophantine equation.)

The answer is yes or no. This "problem" is not undecidable.

Undecidability for Diophantine set of equations $\Rightarrow$ no program can take any set of integer equations and always correctly output whether it has an integer solution.
Undecidable problems.

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Person:

- embryo is blob.
- Legs, arms, head....
- How?
- Fly: blob.
- Torso becomes striped.
  - Developed chemical reaction-diffusion networks that break symmetry.
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Turing: personal.

Tragic ending...
Turing: personal.

Tragic ending...

- Arrested as a homosexual, (not particularly closeted)

(A bite from the apple....) accident?

Tragic ending...

▶ Arrested as a homosexual, (not particularly closeted)
▶ given choice of prison or (quackish) injections to eliminate sex drive;
▶ took injections.
▶ lost security clearance...
▶ suffered from depression;
▶ (possibly) suicided with cyanide at age 42 in 1954.
Turing: personal.

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(A bite from the apple...)
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This statement is a lie.
This statement is a lie. *Neither true nor false!*
This statement is a lie. Neither true nor false!

Every person who doesn’t shave themselves is shaved by the barber.

Back to technical..
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Who shaves the barber?
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  *Who shaves the barber?*

def Turing(P):

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*Who shaves the barber?*

```python
def Turing(P):
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        return
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**Who shaves the barber?**

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Halt Progam $\implies$ Turing Program.
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Halt Program $\implies$ Turing Program. $(P \implies Q)$
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Turing(“Turing”)? Neither halts nor loops!
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    else:
        return

...Text of Halt...
```

Halt Program $\implies$ Turing Program. ($P \implies Q$)

Turing(“Turing”)? Neither halts nor loops! $\implies$ No Turing program.

No Turing Program
This statement is a lie. Neither true nor false!

Every person who doesn’t shave themselves is shaved by the barber.

Who shaves the barber?

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Program is text, so we can pass it to itself,
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Program is text, so we can pass it to itself, or refer to self.
Summary: decidability.

Computer Programs are an interesting thing.
Summary: decidability.

Computer Programs are an interesting thing. Like Math.
Summary: decidability.

Computer Programs are an interesting thing. Like Math. Formal Systems.
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Computer Programs are an interesting thing. Like Math. Formal Systems.
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Computer Programs cannot completely “understand” computer programs.
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Computer Programs are an interesting thing.
   Like Math.
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Computer Programs cannot completely “understand” computer programs.
Computer Programs are an interesting thing. Like Math. Formal Systems.

Computer Programs cannot completely “understand” computer programs.

Computation is a lens for other action in the world.
Kolmogorov Complexity, Google, and CS70

Of strings, $s$. 

What Kolmogorov complexity of a string of 1,000,000, one's?

What is Kolmogorov complexity of a string of $n$ one's?

for $i = 1$ to $n$: print '1'.
Of strings, $s$.

Minimum sized program that prints string $s$. 

Kolmogorov Complexity, Google, and CS70

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Of strings, \( s \).

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What is Kolmogorov complexity of a string of \( n \) one’s?

for \( i = 1 \) to \( n \): print ’1’.
What is the minimum I need to know (remember) to know stuff.
Kolmogorov Complexity, Google, and CS70

What is the minimum I need to know (remember) to know stuff.

Radius of the earth?
What is the minimum I need to know (remember) to know stuff.

Radius of the earth? Distance to the sun?
Kolmogorov Complexity, Google, and CS70

What is the minimum I need to know (remember) to know stuff.

Radius of the earth? Distance to the sun? Population of the US?
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Radius of the earth? Distance to the sun? Population of the US? Acceleration due to gravity on earth?
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Syntax of pandas?
What is the minimum I need to know (remember) to know stuff.

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Syntax of pandas? Google + Stackoverflow.
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Syntax of pandas? Google + Stackoverflow.
Plus “how to program”
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Kolmogorov Complexity View:
What is the minimum I need to know (remember) to know stuff.

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  Kolmogorov Complexity View:
  perimeter of a circle/diameter.
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Calculus:
Kolmogorov Complexity, Google, and CS70

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Depends on your skills!
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Conceptualization.
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Calculus: what is minimum you need to know?
  Depends on your skills!
  Conceptualization.
  Reason and understand an argument and you can generate a lot.
What is the first half of calculus about?
What is the first half of calculus about?

The slope of a tangent line to a function at a point.
What is the first half of calculus about?

The slope of a tangent line to a function at a point.

Slope is rise/run.
What is the first half of calculus about?

The slope of a tangent line to a function at a point.

Slope is rise/run. Oh, yes: \( \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \).
Calculus

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Chain rule? Derivative of a function composition.
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Intuition: composition of two linear functions?
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Chain rule? Derivative of a function composition. Intuition: composition of two linear functions?

\[ f(x) = ax, \ g(x) = bx. \]
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\( f(x) = ax, \ g(x) = bx. \ f(g(x)) = ab \ x. \) Slope is \( ab \). Multiply slopes!
Calculus

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But...but...
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For function slopes of tangent differ at different places.
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So, where?
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So, where? \( f(g(x)) \)

slope of \( f \) at \( g(x) \) times slope of \( g \) at \( x \).
What is the first half of calculus about?

The slope of a tangent line to a function at a point.

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Product Rule.

Idea: use rise in function value!
Product Rule.

Idea: use rise in function value!

\[ d(uv) = (u + du)(v + dv) - uv = udv + vdu + dudv \rightarrow udv + vdu. \]
Product Rule.

Idea: use rise in function value!
\[ d(uv) = (u + du)(v + dv) - uv = udv + vdu + dudv \rightarrow udv + vdu. \]

Any concept:
Product Rule.

Idea: use rise in function value!

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Any concept:

A quick argument from basic concept of slope of a tangent line.
Product Rule.

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Any concept:
A quick argument from basic concept of slope of a tangent line.
Perhaps.
Derivative of sine?

\[ \sin(x). \]
Derivative of sine?

\[ \sin(x) . \]

What is \( x \)? An angle in radians.
Derivative of sine?

\[ \sin(x). \]

What is \( x \)? An angle in radians.

Let's call it \( \theta \) and do derivative of \( \sin \theta \).
Derivative of sine?

\( \sin(x) \).

What is \( x \)? An angle in radians.

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\( \theta \) - Length of arc of unit circle
Derivative of sine?

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Rise. Similar triangle!!!
Fundamental Theorem of Calculus.

Conceptual: Height times Width = Area.
Fundamental Theorem of Calculus.

Conceptual: Height times Width = Area.
Useful?
Fundamental Theorem of Calculus.

Conceptual: Height times Width = Area.
Useful?
   Speed times Time is Distance.
Fundamental Theorem of Calculus.

Conceptual: Height times Width = Area.

Useful?
    Speed times Time is Distance.

Conceptual: Area is proportional to height.
Fundamental Theorem of Calculus.

Conceptual: Height times Width = Area.

Useful?
  Speed times Time is Distance.

Conceptual: Area is proportional to height.

If you change width, change in area is proportional to height.
Fundamental Theorem of Calculus.

Conceptual: Height times Width = Area.

Useful?
   Speed times Time is Distance.

Conceptual: Area is proportional to height.

If you change width, change in area is proportional to height.

Derivative (rate of change) of Area (Integral) under curve, is height of curve.
Riemann Sum/Integral:
\[ R_{b}^{a} f(x) \, dx = \lim_{\delta \to 0} \sum_{i} \delta f(a_i) \]

“Area is defined as rectangles and add up some thin ones.”

Derivative (Rate of change):
\[ F'(x) = \lim_{h \to 0} \frac{F(x+h)-F(x)}{h} \]

“Rise over run of close together points.”

Fundamental Theorem:
\[ F(b) - F(a) = \int_{a}^{b} F'(x) \, dx \]

“Area (\( F(\cdot) \)) under \( f(x) \) grows at \( x \), \( F'(x) \), by \( f(x) \)”

Thus \( F'(x) = f(x) \).
Calculus

Riemann Sum/Integral: \( \int_a^b f(x) \, dx = \lim_{\delta \to 0} \sum_i \delta f(a_i) \)
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“Area is defined as rectangles and add up some thin ones.”

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Fundamental Theorem:  
\( F(b) - F(a) = \int_{a}^{b} F'(x) \, dx \).  
“Area (\( F(\cdot) \)) under \( f(x) \) grows at \( x, F'(x) \), by \( f(x) \)”
Riemann Sum/Integral: $\int_{a}^{b} f(x)dx = \lim_{\delta \to 0} \sum_{i} \delta f(a_i)$

“Area is defined as rectangles and add up some thin ones.”

Derivative (Rate of change):
\[ F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}. \]

“Rise over run of close together points.”

Fundamental Theorem: $F(b) - F(a) = \int_{a}^{b} F'(x)dx$.

“Area ($F(\cdot)$) under $f(x)$ grows at $x$, $F'(x)$, by $f(x)$”

Thus $F'(x) = f(x)$. 
Arguments, reasoning.

What you know: slope, limit.
Arguments, reasoning.

What you know: slope, limit.  
Plus: definition.
Arguments, reasoning.

What you know: slope, limit.
   Plus: definition.
yields calculus.
Arguments, reasoning.

What you know: slope, limit.
   Plus: definition.
yields calculus.
   Minimization, optimization, ….
Arguments, reasoning.

What you know: slope, limit.
   Plus: definition.
yields calculus.
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Knowing how to program
Arguments, reasoning.

What you know: slope, limit.
  Plus: definition.
yields calculus.
  Minimization, optimization, ….

Knowing how to program plus some syntax (google) gives the ability to program.
Arguments, reasoning.

What you know: slope, limit.
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yields calculus.
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Knowing how to reason
Arguments, reasoning.

What you know: slope, limit.
   Plus: definition.
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Knowing how to reason plus some definition
Arguments, reasoning.

What you know: slope, limit.
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Knowing how to reason plus some definition gives calculus.
Arguments, reasoning.

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Knowing how to program plus some syntax (google) gives the ability to program.

Knowing how to reason plus some definition gives calculus.

Discrete Math: basics are counting, how many, when are two sets the same size?
Arguments, reasoning.

What you know: slope, limit.
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Probability:
Arguments, reasoning.

What you know: slope, limit.
   Plus: definition.
yields calculus.
   Minimization, optimization, ….

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...plus reasoning.
CS 70 : ideas.

Induction
Induction $\equiv$ every integer has a next one.
Induction \( \equiv \) every integer has a next one. Graph theory. 
Number of edges is sum of degrees. 
\( \Delta + 1 \) coloring. Neighbors only take up \( \Delta \). 
Connectivity plus connected components. 
Eulerian paths: if you enter you can leave. 
Euler’s formula: tree has \( v - 1 \) edges and 1 face plus 
each extra edge makes additional face. 
\( v - 1 + (f - 1) = e \)
Number theory.
A divisor of $x$ and $y$ divides $x - y$.
The remainder is always smaller than the divisor.
$\implies$ Euclid’s GCD algorithm.
Multiplicative Inverse.
Fermat’s theorem from function with inverse is a bijection.
Gives RSA.
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Error Correction.
(Any) Two points determine a line.
(well, and $d$ points determine a degree $d+1$-polynomials.
Cuz, factoring.
Find line by linear equations.
If a couple are wrong, then multiply them by zero, i.e., Error polynomial.
CS70 and your future?

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....and you will pursue probability in this course.