Today

Probability:

Keep building it formally..

And our intuition.

Poll: blows my mind.

Flip 300 million coins.

Which is more likely?
(A) 300 million heads.
(B) 300 million tails.
(C) Alternating heads and tails.
(D) A tail every third spot.

Given the history of the universe up to right now.

What is the likelihood of our universe?
(A) The likelihood is 1. Cuz here it is.
(B) As likely as any other. Cuz of probability.
(C) Well. Quantum. IDK- TBH.

Perhaps a philosophical (“wastebasket”) question.

Also, “cuz” $==$ “because”

Probability Basics.

Probability Space.

1. **Sample Space:** Set of outcomes, Ω.

2. **Probability:** $Pr[\omega]$ for all $\omega \in \Omega$.
   2.1 $0 \leq Pr[\omega] \leq 1$.
   2.2 $\sum_{\omega \in \Omega} Pr[\omega] = 1$.

Example: Two coins.

1. $\Omega = \{HH, HT, TH, TT\}$
   (Note: Not $\Omega = \{H, T\}$ with two picks!)
2. $Pr[HH] = \cdots = Pr[TT] = 1/4$

Consequences of Additivity

Theorem

(a) Inclusion/Exclusion: $Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$;

(b) Union Bound: $Pr[A_1 \cup \cdots \cup A_n] \leq Pr[A_1] + \cdots + Pr[A_n]$;

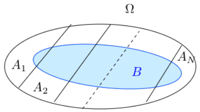
(c) Law of Total Probability:

If $A_1, \ldots A_N$ are a partition of Ω, i.e.,

pairwise disjoint and $\cup_{m=1}^{N} A_m = \Omega$, then

$$Pr[B] = Pr[B \cap A_1] + \cdots + Pr[B \cap A_N].$$

Proof Idea: Total probability.



Add it up!

Add it up. Poll.

What does Rao mean by “Add it up.”

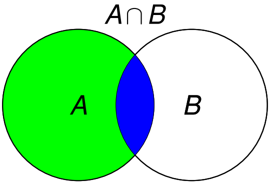
(A) Organize intuitions/proofs around $Pr[\omega]$.
(B) Organize intuition/proofs around $Pr[A]$.
(C) Some weird song whose refrain he heard in his youth.

(A), (B), and (C)

Conditional Probability.

Definition: The **conditional probability** of B given A is

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$$



In A!

In B?

Must be in $A \cap B$.

$$Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}.$$

Note also:

$$Pr[A \cap B] = Pr[B|A] Pr[A]$$

Product Rule

Def: $Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$.

Also: $Pr[A \cap B] = Pr[B|A]Pr[A]$

Theorem Product Rule
Let $A_1, A_2, \ldots, A_n$ be events. Then

$$Pr[A_1 \cap \cdots \cap A_n] = Pr[A_1]Pr[A_2|A_1] \cdots Pr[A_n|A_1 \cap \cdots \cap A_{n-1}].$$

Simple Bayes Rule.

$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$, $Pr[B|A] = \frac{Pr[A \cap B]}{Pr[A]}$.

$Pr[A \cap B] = Pr[A|B]Pr[B] = Pr[B|A]Pr[A]$.

Bayes Rule: $Pr[A|B] = \frac{Pr[B|A]Pr[A]}{Pr[B]}$.

Is you coin loaded?

Your coin is fair w.p. $1/2$ or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

Analysis:

$$A = \text{ 'coin is fair'}, B = \text{ 'outcome is heads'}$$

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2, P[B|\bar{A}] = 0.6, Pr[A] = 1/2 = Pr[\bar{A}]$

Now,

$$\begin{aligned} Pr[B] &= Pr[A \cap B] + Pr[\bar{A} \cap B] = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}] \\ &= (1/2)(1/2) + (1/2)0.6 = 0.55. \end{aligned}$$

Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

Independence

Definition: Two events A and B are **independent** if

$$Pr[A \cap B] = Pr[A]Pr[B].$$

Examples:

- When rolling two dice, $A =$ sum is 7 and $B =$ red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$.
- When rolling two dice, $A =$ sum is 3 and $B =$ red die is 1 are not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right)\left(\frac{1}{6}\right)$.
- When flipping coins, $A =$ coin 1 yields heads and $B =$ coin 2 yields tails are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$.
- When throwing 3 balls into 3 bins, $A =$ bin 1 is empty and $B =$ bin 2 is empty are not independent; $Pr[A \cap B] = \frac{1}{27}$, $Pr[A]Pr[B] = \left(\frac{8}{27}\right)\left(\frac{8}{27}\right)$.

Independence and conditional probability

Fact: Two events A and B are **independent** if and only if

$$Pr[A|B] = Pr[A].$$

Indeed: $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$, so that

$$Pr[A|B] = Pr[A] \Leftrightarrow \frac{Pr[A \cap B]}{Pr[B]} = Pr[A] \Leftrightarrow Pr[A \cap B] = Pr[A]Pr[B].$$

Conditional Probability: Review

Recall:

- $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$.
- Hence, $Pr[A \cap B] = Pr[B]Pr[A|B] = Pr[A]Pr[B|A]$.
- A and B are *positively correlated* if $Pr[A|B] > Pr[A]$,

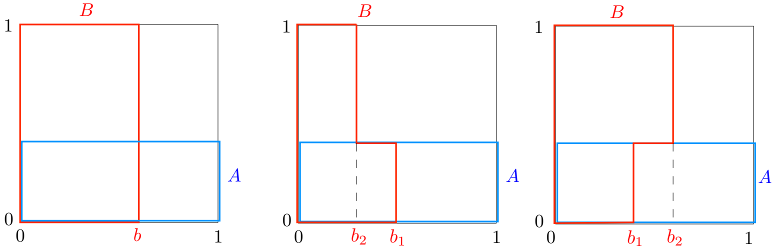
  i.e., if $Pr[A \cap B] > Pr[A]Pr[B]$.
- A and B are *negatively correlated* if $Pr[A|B] < Pr[A]$,

  i.e., if $Pr[A \cap B] < Pr[A]Pr[B]$.
- A and B are *independent* if $Pr[A|B] = Pr[A]$,

  i.e., if $Pr[A \cap B] = Pr[A]Pr[B]$.
- Note: $B \subset A \Rightarrow A$ and B are positively correlated.
  ($Pr[A|B] = 1 > Pr[A]$)
- Note: $A \cap B = \emptyset \Rightarrow A$ and B are negatively correlated.
  ($Pr[A|B] = 0 < Pr[A]$)

Conditional Probability: Pictures/Poll.

Illustrations: Pick a point uniformly in the unit square



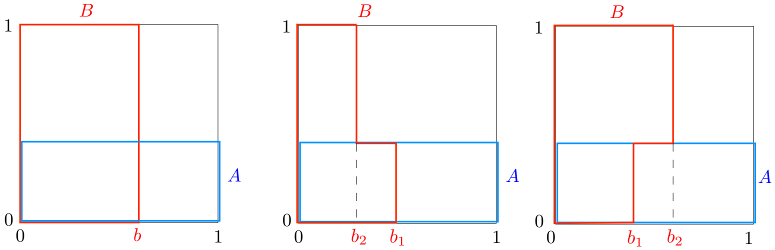
Which *A* and *B* are independent?

(A) Left.
(B) Middle.
(B) Right.

See next slide.

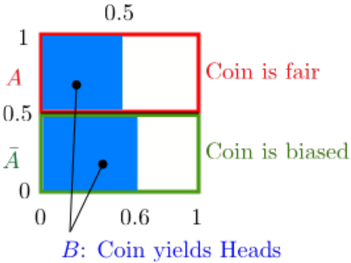
Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square



- Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.
- Middle: A and B are positively correlated.
  $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.
- Right: A and B are negatively correlated.
  $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_1, b_2)$.

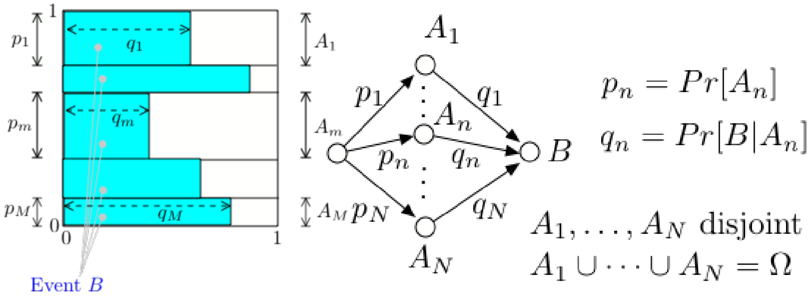
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$
\begin{aligned}
&Pr[A] = 0.5; Pr[\bar{A}] = 0.5\\
&Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] = 0.5 \times 0.5\\
&Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]\\
&Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}\\
&\qquad\quad \approx 0.46 = \text{fraction of } B \text{ that is inside } A
\end{aligned}
$$

Bayes: General Case



Pick a point uniformly at random in the unit square. Then

$$Pr[A_n] = p_n, n = 1, \ldots, N$$

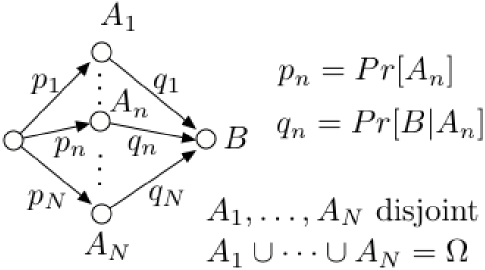
$$Pr[B|A_n] = q_n, n = 1, \ldots, N; Pr[A_n \cap B] = p_n q_n$$

$$Pr[B] = p_1 q_1 + \cdots p_N q_N$$

$$Pr[A_n|B] = \frac{p_n q_n}{p_1 q_1 + \cdots p_N q_N} = \text{ fraction of } B \text{ inside } A_n.$$

Bayes Rule

A general picture: We imagine that there are N possible causes $A_1, \ldots, A_N$.



100 situations: $100 p_n q_n$ where A_n and B occur, for $n = 1, \ldots, N$.
In $100 \sum_m p_m q_m$ occurences of B, $100 p_n q_n$ occurrences of A_n.

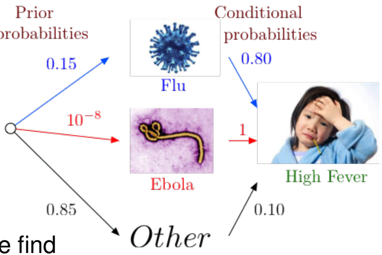
Hence,

$$Pr[A_n|B] = \frac{p_n q_n}{\sum_m p_m q_m}.$$

But, $p_n = Pr[A_n], q_n = Pr[B|A_n], \sum_m p_m q - m = Pr[B]$, hence,

$$Pr[A_n|B] = \frac{Pr[B|A_n] Pr[A_n]}{Pr[B]}.$$

Why do you have a fever?



Using Bayes' rule, we find

$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

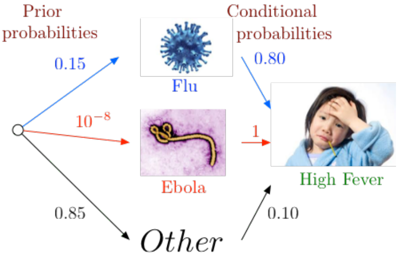
$$Pr[\text{Ebola}|\text{High Fever}] = \frac{10^{-8} \times 1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 5 \times 10^{-8}$$

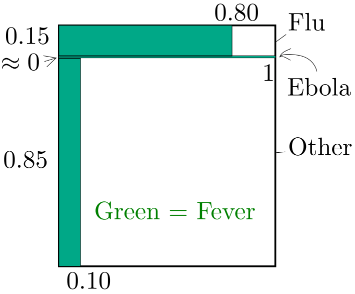
$$Pr[\text{Other}|\text{High Fever}] = \frac{0.85 \times 0.1}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.42$$

The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

Why do you have a fever?

Our “Bayes’ Square” picture:





$$58\% \text{ of Fever} = \text{Flu}$$
$$\approx 0\% \text{ of Fever} = \text{Ebola}$$
$$42\% \text{ of Fever} = \text{Other}$$

Note that even though $Pr[\text{Fever}|\text{Ebola}] = 1$, one has

$$Pr[\text{Ebola}|\text{Fever}] \approx 0.$$

This example shows the importance of the prior probabilities.

Why do you have a fever?

We found

$$\begin{aligned} Pr[\text{Flu}|\text{High Fever}] &\approx 0.58, \\ Pr[\text{Ebola}|\text{High Fever}] &\approx 5 \times 10^{-8}, \\ Pr[\text{Other}|\text{High Fever}] &\approx 0.42 \end{aligned}$$

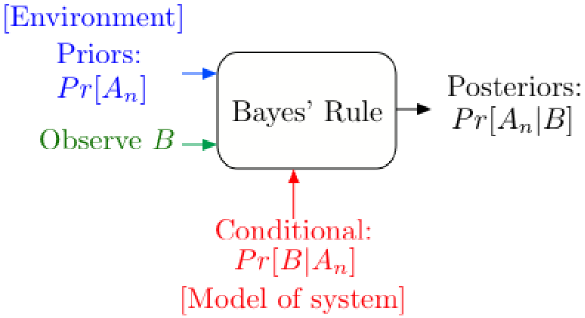
'Flu' is Most Likely a Posteriori (MAP) cause of high fever.
'Ebola' is Maximum Likelihood Estimate (MLE) of cause:
causes fever with largest probability.
Recall that

$$p_m = Pr[A_m], q_m = Pr[B|A_m], Pr[A_m|B] = \frac{p_m q_m}{p_1 q_1 + \cdots + p_M q_M}.$$

Thus,

- MAP = value of m that maximizes $p_m q_m$.
- MLE = value of m that maximizes q_m.

Bayes' Rule Operations



Bayes' Rule: canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A $\qquad\qquad\qquad\qquad$ B

FIG. 3. *Joshua Bayes (1671–1746).*

$$\xrightarrow{Pr[B|A]}$$

Max over B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Random Experiment: Pick a random male.
Outcomes: $(test, disease)$
A - prostate cancer.
B - positive PSA test.

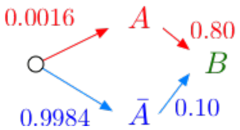
- $Pr[A] = 0.0016$, (.16 % of the male population is affected.)
- $Pr[B|A] = 0.80$ (80% chance of positive test with disease.)
- $Pr[B|\overline{A}] = 0.10$ (10% chance of positive test without disease.)

From http://www.cpcn.org/01_psa_tests.htm and http://seer.cancer.gov/statfacts/html/prost.html (10/12/2011.)

Positive PSA test (B). Do I have disease?

$$Pr[A|B]???$$

Bayes Rule.



Using Bayes' rule, we find

$$P[A|B] = \frac{0.0016 \times 0.80}{0.0016 \times 0.80 + 0.9984 \times 0.10} = .013.$$

A 1.3% chance of prostate cancer with a positive PSA test.

Surgery anyone?

Impotence...

Incontinence..

Death.

Quick Review

$\boxed{\text{Events, Conditional Probability, Independence, Bayes' Rule}}$

Key Ideas:

- Conditional Probability:

  $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$

- Independence: $Pr[A \cap B] = Pr[A]Pr[B]$.
- Bayes' Rule:

$$Pr[A_n|B] = \frac{Pr[A_n]Pr[B|A_n]}{\sum_m Pr[A_m]Pr[B|A_m]}.$$

$Pr[A_n|B] =$ posterior probability; $Pr[A_n] =$ prior probability .

- All these are possible:

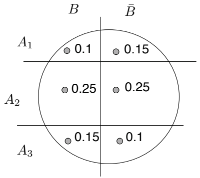
  $Pr[A|B] < Pr[A]; Pr[A|B] > Pr[A]; Pr[A|B] = Pr[A].$

Independence

Recall :

$$\begin{aligned} & A \text{ and } B \text{ are independent} \\ & \Leftrightarrow Pr[A \cap B] = Pr[A]Pr[B] \\ & \Leftrightarrow Pr[A|B] = Pr[A]. \end{aligned}$$

Consider the example below:

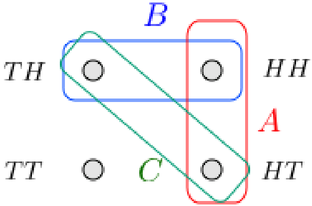


Which are independent? (A) (A_2, B) (B) $(A_2, \bar{B})$ (C) (A_1, B).
(A_2, B) are independent: $Pr[A_2|B] = 0.5 = Pr[A_2]$.
$(A_2, \bar{B})$ are independent: $Pr[A_2|\bar{B}] = 0.5 = Pr[A_2]$.
(A_1, B) are not independent: $Pr[A_1|B] = \frac{0.1}{0.5} = 0.2 \neq Pr[A_1] = 0.25$.

Pairwise Independence

Flip two fair coins. Let

- $A =$ 'first coin is H' $= \{HT, HH\}$;
- $B =$ 'second coin is H' $= \{TH, HH\}$;
- $C =$ 'the two coins are different' $= \{TH, HT\}$.



A, C are independent; B, C are independent;

$A \cap B, C$ are not independent. ($Pr[A \cap B \cap C] = 0 \neq Pr[A \cap B] Pr[C]$.)

False: If A did not say anything about C and B did not say anything about C, then $A \cap B$ would not say anything about C.

Example 2

Flip a fair coin 5 times. Let A_n = 'coin n is H', for $n = 1, \ldots, 5$.

Then,

$$A_m, A_n \text{ are independent for all } m \neq n.$$

Also,

$$A_1 \text{ and } A_3 \cap A_5 \text{ are independent.}$$

Indeed,

$$Pr[A_1 \cap (A_3 \cap A_5)] = \frac{1}{8} = Pr[A_1]Pr[A_3 \cap A_5].$$

Similarly,

$$A_1 \cap A_2 \text{ and } A_3 \cap A_4 \cap A_5 \text{ are independent.}$$

This leads to a definition ....

Mutual Independence

Definition Mutual Independence

(a) The events $A_1, \ldots, A_5$ are mutually independent if

$$Pr[\cap_{k \in K} A_k] = \prod_{k \in K} Pr[A_k], \text{ for all } K \subseteq \{1, \ldots, 5\}.$$

(b) More generally, the events $\{A_j, j \in J\}$ are mutually independent if

$$Pr[\cap_{k \in K} A_k] = \prod_{k \in K} Pr[A_k], \text{ for all finite} K \subseteq J.$$

Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

Mutual Independence

Theorem

(a) If the events $\{A_j, j \in J\}$ are mutually independent and if K_1 and K_2 are disjoint finite subsets of J, then

$$\cap_{k \in K_1} A_k \text{ and } \cap_{k \in K_2} A_k \text{ are independent.}$$

(b) More generally, if the K_n are pairwise disjoint finite subsets of J, then the events

$$\cap_{k \in K_n} A_k \text{ are mutually independent.}$$

(c) Also, the same is true if we replace some of the A_k by $\bar{A}_k$.

Conditional Probability: Review

Recall:

- $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$.
- Hence, $Pr[A \cap B] = Pr[B]Pr[A|B] = Pr[A]Pr[B|A]$.
- A and B are *positively correlated* if $Pr[A|B] > Pr[A]$,

  i.e., if $Pr[A \cap B] > Pr[A]Pr[B]$.
- A and B are *negatively correlated* if $Pr[A|B] < Pr[A]$,

  i.e., if $Pr[A \cap B] < Pr[A]Pr[B]$.
- A and B are *independent* if $Pr[A|B] = Pr[A]$,

  i.e., if $Pr[A \cap B] = Pr[A]Pr[B]$.
- Note: $B \subset A \Rightarrow A$ and B are positively correlated.
  $(Pr[A|B] = 1 > Pr[A])$
- Note: $A \cap B = \emptyset \Rightarrow A$ and B are negatively correlated.
  $(Pr[A|B] = 0 < Pr[A])$

Quick Review.

$\boxed{\text{Bayes' Rule, Mutual Independence, Collisions and Collecting}}$

Main results:

- Bayes' Rule: $Pr[A_m|B] = p_m q_m/(p_1 q_1 + \cdots + p_M q_M)$.
- Product Rule:
  $Pr[A_1 \cap \cdots \cap A_n] = Pr[A_1]Pr[A_2|A_1]\cdots Pr[A_n|A_1 \cap \cdots \cap A_{n-1}]$.