# CS70.

1. Random Variables: Brief Review
2. Joint Distributions.
3. Linearity of Expectation

# Random Variables: Definitions

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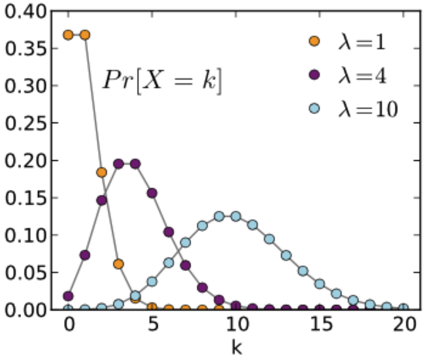

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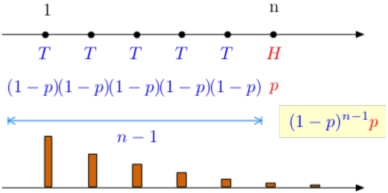

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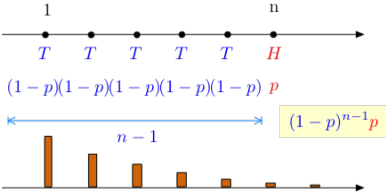


I could not find a picture of D. Binomial, sorry.

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Also,

$$\sum_{a} a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

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the Law of Large Numbers. (See later.)

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$$X_1(\omega) = \begin{cases} 1, & \text{if coin 1 is heads} \\ 0, & \text{otherwise} \end{cases} \qquad X_2(\omega) = \begin{cases} 1, & \text{if coin 2 is heads} \\ 0, & \text{otherwise} \end{cases}$$

# Multiple Random Variables: setup.

**Joint Distribution:** $\{(a, b, Pr[X = a, Y = b]) : a \in \mathscr{A}, b \in \mathscr{B}\}$, where $\mathscr{A}$ ($\mathscr{B}$) is possible values of $X$ ($Y$).

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| X/Y | 1 | 2 | 3 | X |
|---|---|---|---|---|
| 1 | .2 | .1 | .1 | .4 |
| 2 | 0 | 0 | .3 | .3 |
| 3 | .1 | 0 | .2 | .3 |
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Conditional Probability: $Pr[X=a|Y=b] = \frac{Pr[X=a,Y=b]}{Pr[Y=b]}$.

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Follows from $Pr[A \cap B] = Pr[A|B]Pr[B]$ (Product rule.)

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$$\begin{aligned}
E[X+Y] &= \sum_{\omega \in \Omega} (X(\omega) + Y(\omega)) Pr[\omega] \\
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&= \sum_{\omega \in \Omega} X(\omega) Pr[\omega] + \sum_{\omega \in \Omega} Y(\omega) Pr[\omega] \\
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This random variable $X(\omega)$ is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

# Indicators

**Definition**
Let $A$ be an event. The random variable $X$ defined by

$$X(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

is called the indicator of the event $A$.

Note that $Pr[X = 1] = Pr[A]$ and $Pr[X = 0] = 1 - Pr[A]$.

Hence,

$$E[X] = 1 \times Pr[X = 1] + 0 \times Pr[X = 0] = Pr[A].$$

This random variable $X(\omega)$ is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

Thus, we will write $X = 1_A$.

# Linearity of Expectation

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□

Note: If we had defined $Y = a_1 X_1 + \cdots + a_n X_n$ has had tried to compute $E[Y] = \sum_y y Pr[Y = y]$, we would have been in trouble!

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

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$\frac{n}{e} \approx 0.368n$ empty bins on average.

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# Review: Harmonic sum

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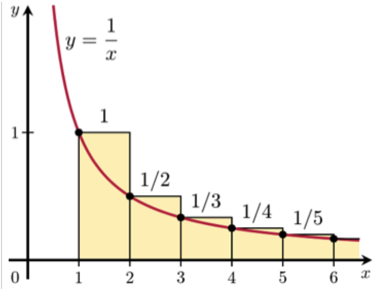

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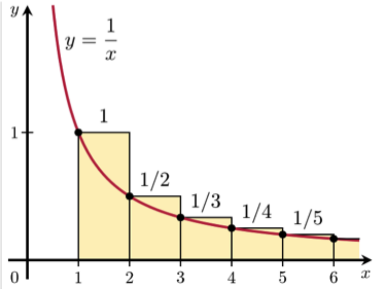


A good approximation is

$H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant).

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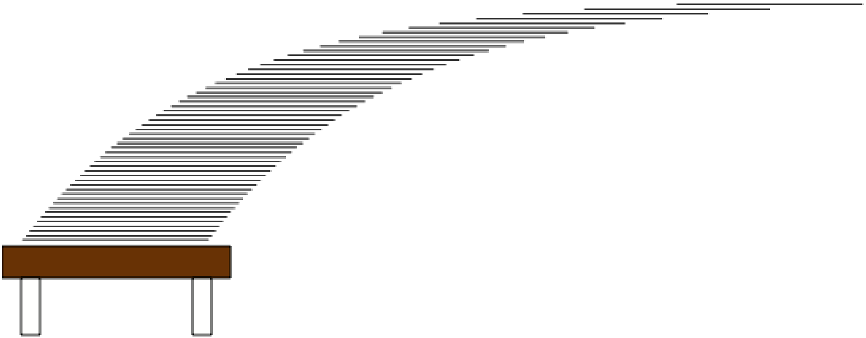

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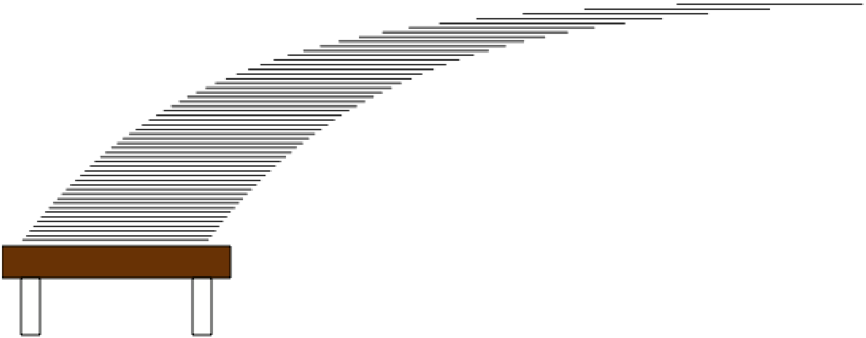

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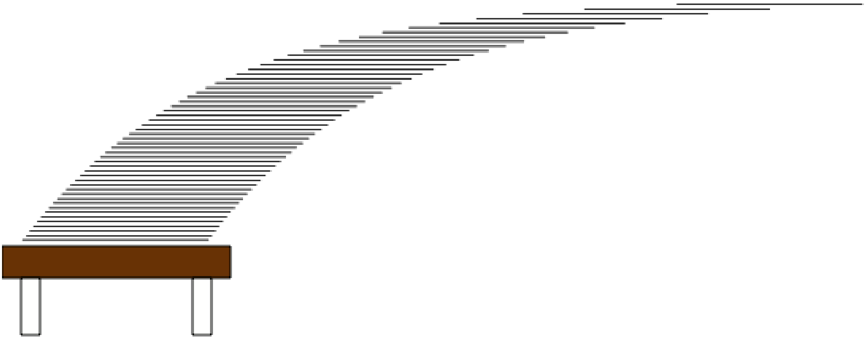


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As $n$ increases, you can go as far as you want!

# Paradox

## par·a·dox

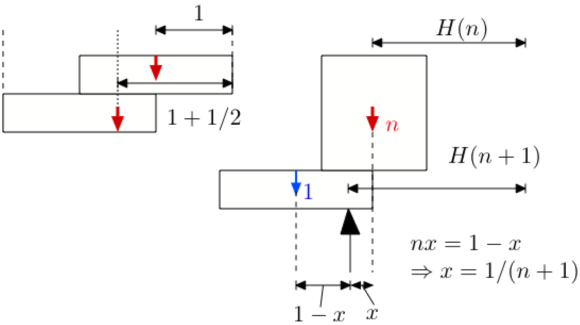
/ˈperəˌdäks/

*noun*

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
  *synonyms:* contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

# Stacking



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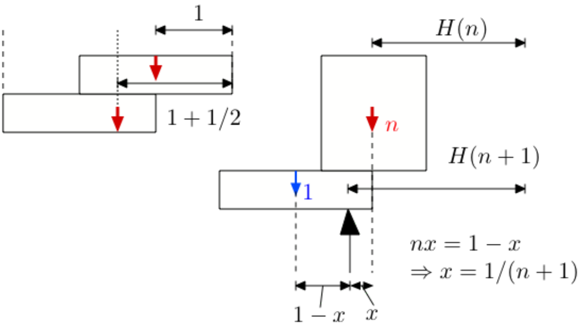


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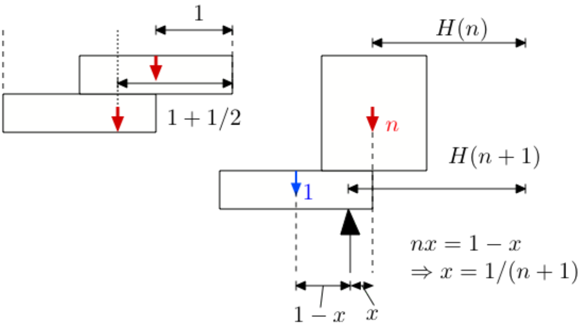


The cards have width 2. Induction shows that the center of gravity after $n$ cards is $H(n)$ away from the right-most edge.
Video.

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Thus,

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Distribution: Function $Pr[X = a] \geq 0$. $\sum_a Pr[X = a] = 1$.

Expectation: $E[X] = \sum_\omega Pr[\omega] = \sum_a Pr[X = a]$.

Many Random Variables: each one function on a sample space.

Joint Distributions: Function $Pr[X = a, Y = b] \geq 0$.
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