Summary of Last Lecture

• **Random variable** = function $X: \Omega \rightarrow \mathbb{R}$

Examples:
- $\Omega = \text{seq. 8 coin tosses}$
  $X(\omega) = \# \text{Heads in } \omega$
- $\Omega = \text{two dice rolls}$
  $X(\omega) = \text{sum of numbers on the dice}$

• **Distribution of a r.v. $X$**:
  $\Pr[X = a]$ for each possible value $a$ of $X$

  Can think of this as a histogram:
  $\sum_{a} \Pr[X = a] = 1$
Summary (continued)

• **Expectation** (= mean)
  
  \[ E[X] = \sum_a a \cdot \Pr[X = a] \]

  Measures the “center of mass” of the distribution

• **Linearity of expectation**:
  For any r.v.’s \( X, Y \) and constants \( a, b \)
  
  \[ E[aX + bY] = aE[X] + bE[Y] \]

• Use with indicator r.v.’s to do counting
  
  E.g. \( X = \) no. of fixed points in a random permutation
  
  \[ X = \sum_{i=1}^{\infty} X_i \]
  where \( X_i = \begin{cases} 1 & \text{if } i \text{ a fixed point} \\ 0 & \text{otherwise} \end{cases} \]
Summary (continued)

• **Binomial Distribution**  \( \text{Bin}(n, p) \)
  
  \( X = \# \) Heads in \( n \) tosses of a biased coin (Heads \( p \))

\[
\Pr [X=k] = \binom{n}{k} p^k (1-p)^{n-k} \quad k = 0, 1, \ldots, n
\]

• **Hypergeometric Distribution**  \( \text{HyperGeom}(N, n, B) \)
  
  \( X = \# \) black balls in a sample of size \( n \) drawn from a box containing \( N \) balls, \( B \) of which are black

\[
\Pr [X=k] = \frac{\binom{B}{k} \binom{N-B}{n-k}}{\binom{N}{n}}
\]
Today

- Joint distributions & independence of random variables

- Two more important distributions:
  - Geometric distribution
  - Poisson distribution
Joint Distributions

Defn: The joint distribution of two r.v.'s $X, Y$ on the same prob. space is the set
\[
\{(a, b, \Pr[X=a, Y=b] : a \in A, b \in B)\}
\]
where $A, B$ are the possible values of $X, Y$ resp.

The marginal distribution of $X$ is given by
\[
\Pr[X=a] = \sum_{b \in B} \Pr[X=a, Y=b]
\]

$X, Y$ are independent if
\[
\Pr[X=a, Y=b] = \Pr[X=a] \times \Pr[Y=b] \quad \forall a, b
\]

\[
\Pr[E \cap F] = \Pr[E] \times \Pr[F]
\]
Joint Distributions

**Example**: Throw two fair dice

Random variables:

- \( X = \) score on first die
- \( Y = \) score on second die
- \( Z = \) sum of scores

\[
\begin{align*}
\Pr[X=3, Y=5] &= \frac{1}{36} \\
\Pr[X=3, Z=9] &= \frac{1}{36}
\end{align*}
\]

\( X, Y \) independent? \( \Pr[X=a, Y=b] = \frac{1}{36} \)

\( X, Z \) independent?

\[
\begin{align*}
\Pr[X=a] &= \frac{1}{6} \\
\Pr[Y=b] &= \frac{1}{6}
\end{align*}
\]

\[
\Pr[X=3, Z=12] = 0
\]

\[
\Pr[X=3] = \frac{1}{6} \\
\Pr[Z=12] = \frac{1}{36}
\]
**Geometric distribution**

Toss a biased coin (Heads prob. $p$) until you see the first Head

Random variable $X :=$ number of tosses

What is the distribution of $X$?

**Note:** $X$ takes values in $\{1, 2, 3, \ldots \}$

\[
\begin{align*}
    \Pr[X=1] &= p \\
    \Pr[X=2] &= (1-p)p \\
    \Pr[X=3] &= (1-p)^2p \\
    \vdots \\
    \Pr[X=k] &= (1-p)^{k-1}p
\end{align*}
\]

We say $X$ has the \underline{Geometric distribution} with parameter $p$

$X \sim \text{Geom}(p)$
\[ \Pr [X = k] = (1-p)^{k-1} p \quad k = 1, 2, 3, \ldots \]

Check that \( \sum_{k=1}^{\infty} \Pr [X = k] = 1 \) !

\[
\sum_{k=1}^{\infty} \Pr [X = k] = \sum_{k=1}^{\infty} (1-p)^{k-1} p \\
= p \sum_{k=0}^{\infty} (1-p)^k \\
= p \times \frac{1}{1-(1-p)} \\
= 1 \quad \text{[sum of geometric series]} \\
\]
What does the Geometric distribution look like?

Note: Always decreases geometrically (for any p)
Expectation of $\text{Geom}(p)$

Compute $E[X]$ two ways:

(i) **Calculus**

\[
E[X] = \sum_{k=1}^{\infty} k \times \Pr[X = k]
\]

\[
= \sum_{k=1}^{\infty} k \times p (1-p)^{k-1}
\]

\[
= p \sum_{k=1}^{\infty} k (1-p)^{k-1} = -\frac{d}{dp} \left( \sum_{k=0}^{\infty} (1-p)^k \right)
\]

\[
= -\frac{d}{dp} \left( \frac{1}{p} \right) = \frac{1}{p^2}
\]

\[
= \frac{1}{p}
\]
**Expectation of Geom(p)**

Compute $E[X]$ two ways:

**(ii) Tail Sum Formula**

Fact: For any r.v. that takes values in $\{0,1,2,\ldots\}$ we have

$$E[X] = \sum_{i=1}^{\infty} \Pr [X \geq i]$$

Proof: Write $P_i = \Pr [X = i]$ for $i = 0, 1, 2, \ldots$

Then $E[X] = (0 \times p_0) + (1 \times p_1) + (2 \times p_2) + (3 \times p_3) + \cdots$

$$= P_1 + (P_2 + P_2) + (P_3 + P_3 + P_3) + \cdots$$

$$= (P_1 + P_2 + P_3 + \cdots) + (P_2 + P_3 + P_4 + \cdots) + (P_3 + P_4 + \cdots)$$

$$= \Pr [X \geq 1] + \Pr [X \geq 2] + \Pr [X \geq 3] + \cdots$$
Fact: For any r.v. that takes values in \( \{0, 1, 2, \ldots \} \) we have

\[
E[X] = \sum_{i=1}^{\infty} \Pr [X \geq i]
\]

Apply to \( X \sim \text{Geom}(p) \)

Note that \( \Pr[X \geq i] = \Pr[\text{first } (i-1) \text{ losses are Tails}] = (1-p)^{i-1} \)

Hence

\[
E[X] = \sum_{i=1}^{\infty} (1-p)^{i-1} = \sum_{i=0}^{\infty} (1-p)^i = \frac{1}{p}
\]

Bottom line: Expected no. of trials (tosses) until we see first Head is \( \frac{1}{p} \)

(\( = 2 \) for fair coin)
**Geometric distribution is Memoryless**

**Claim:** Time until next Head is independent of how long we’ve been waiting — i.e.

\[ Pr[X > m + k \mid X > m] = Pr[X > k] \]

**Proof:** \( \forall k, \ Pr(X > k) = (1-p)^k \)

Therefore:

\[ Pr[X > m + k \mid X > m] = \frac{Pr[X > m + k]}{Pr[X > m]} = \frac{(1-p)^{m+k}}{(1-p)^m} = (1-p)^k = Pr(X > k) \]
Coupon collecting revisited

Recall:
- n different coupons
  - sequence of uniform random samples
  - $X = \#\text{samples until we get at least one of each}$

Write $X = X_1 + X_2 + \ldots + X_n$

where $X_i = \#\text{of samples until we get the ith new coupon, starting after we got the (i-1)th}$

Claim: $X_i \sim \text{Geom} \left( \frac{n-i+1}{n} \right)$

Hence $E[X_i] = \frac{n}{n-i+1}$

Linearity: $E[X] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right)$

$\sim n \ln n + g$
Poisson Distribution

Suppose some event (e.g., a radioactive emission, a disconnected phone call etc.) occurs randomly at a certain average density \( \lambda \) per unit time, and occurrences are independent. Then the no. of occurrences in a unit of time is modeled by a Poisson r.v.

\[
\Pr [X = k] = e^{-\lambda} \frac{\lambda^k}{k!} \quad k = 0, 1, 2, \ldots
\]

Check: \[
\sum_{k=0}^{\infty} \Pr [X = k] = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda} = 1
\]
E.g., # goals in a World Cup soccer match
\[ \lambda = 2.5 \]

\[
\Pr[X=0] = e^{-2.5} \frac{(2.5)^0}{0!} = e^{-2.5} \approx 0.082
\]
\[
\Pr[X=1] = e^{-2.5} \frac{2.5}{1!} \approx 0.205
\]
\[
\Pr[X=2] = e^{-2.5} \frac{(2.5)^2}{2!} \approx 0.257
\]
\[
\Pr[X=3] = e^{-2.5} \frac{(2.5)^3}{3!} \approx 0.214
\]
\[
\Pr[X > 3] \approx 0.242
\]
Histograms of $\text{Pois}(\lambda)$

$\lambda = 1$

$\lambda = 5$

$\lambda = 20$

Note: The distribution is unimodal, peaks at $\lfloor \lambda \rfloor$
Expectation of Pois(\lambda)

\[ \Pr [X = k] = e^{-\lambda} \frac{\lambda^k}{k!} \]

\[ E[X] = \sum_{k=0}^{\infty} k \times \Pr [X = k] \]

\[ = \sum_{k=1}^{\infty} k \times e^{-\lambda} \frac{\lambda^k}{k!} \]

\[ = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \]

\[ = \lambda e^{-\lambda} e^\lambda \]

\[ = \lambda \]

\[ = e^\lambda \]
Sum of Independent Poisson R.V.'s

Thus: Suppose $X \sim \text{Pois} (\lambda)$ and $Y \sim \text{Pois} (\mu)$ are independent. Then $X+Y \sim \text{Pois} (\lambda+\mu)$

Proof: \[ Pr[X+Y=k] = \sum_{j=0}^{k} Pr[X=j, Y=k-j] \]
\[ = \sum_{j=0}^{k} Pr[X=j] \cdot Pr[Y=k-j] \] (indep)
\[ = \sum_{j=0}^{k} e^{-\lambda} \frac{\lambda^j}{j!} \cdot e^{-\mu} \frac{\mu^{k-j}}{(k-j)!} \]
\[ = e^{-(\lambda+\mu)} \cdot \frac{1}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \lambda^j \mu^{k-j} \]
\[ = e^{-(\lambda+\mu)} \cdot \frac{1}{k!} (\lambda+\mu)^k \] (binomial theorem)
**Poisson vs. Binomial**

**Example**: Balls & bins with \(n\) balls, \(n\) bins

R.v. \(X = \# \text{ balls in bin 1}\)

Then \(X \sim \text{Bin}(n, \frac{1}{n})\)

So \(E[X] = n \times \frac{1}{n} = 1\)

So: \(\Pr[X = k] = \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1-\frac{1}{n}\right)^{n-k} \quad k = 0, 1, 2, \ldots\)

Now fix \(k\) and let \(n \to \infty\)

\(\Pr[X = k] = \binom{n}{k} \frac{1}{n^k} \left(1-\frac{1}{n}\right)^{n-k}\)

\[
\frac{1}{n^k} \left(\frac{n}{k}\right) = \frac{1}{k!} \frac{n(n-1)\ldots(n-k+1)}{n^k} \to \frac{1}{k!} \quad \text{as } n \to \infty
\]

\[
\left(1-\frac{1}{n}\right)^{n-k} \to e^{-\frac{k}{n}} \to e^{-1}
\]

So as \(n \to \infty\), \(X \sim \text{Pois}(1)\)

E.g. \(\Pr[X = 0] \to e^{-1}\) \(\Pr[X = 1] \to e^{-1}\)

\[
(l-\frac{1}{n})^n \to e^{-1}
\]
More generally, for any constant \( \lambda \),

\[
\text{Bin}(n, \frac{\lambda}{n}) \xrightarrow{n \to \infty} \text{Pois}(\lambda)
\]

**Connection with “rare events”**

Assume

- expect \( \lambda \) events per unit interval
- events are “independent”

Divide interval into \( n \) equal-sized pieces

\[
\text{Pr}[\text{event happens in one piece}] = \frac{\lambda}{n}
\]

Events in different pieces mutually independent

\[
X = \# \text{events in interval} : X \sim \text{Bin}(n, \frac{\lambda}{n}) \to \text{Pois}(\lambda)
\]