Coupon Collector

Suppose there's a coupon inside every cereal box.

Suppose there are $n$ types of coupons and each box has a random type.

How many boxes do you need to purchase until you get all coupons?
**Coupon Collector**

Experiment: Get random coupons from \( S_1, \ldots, S_3 \) until you have all types of coupons.

\( X \) - # of boxes until you get all \( n \) types.

**Example:** \( n = 3 \)

\[ 2 \ 2 \ 2 \ 1 \ 1 \ 2 \ 1 \ 3 \]

**Idea:** Write \( X \) as the sum \( X_1 + X_2 + X_3 \)
**Coupon Collector**

**Experiment:** Get random coupons from $i_1, \ldots, i_3$ until you have all types of coupons.

$x$ - # of boxes until you get all $n$ types.

**Example:** $n=3$

<table>
<thead>
<tr>
<th>2</th>
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<tbody>
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<td>$x_1$</td>
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**Idea:** Write $X$ as the sum $X_1 + X_2 + X_3$

- $X_1$ - # boxes until we get first type
- $X_2$ - # boxes until we get second type after we got first type.
- $X_3$ - # boxes until we get third type after we got second type.
Coupon Collector

Experiment: Get random coupons from \(i_1, \ldots, i_3\) until you have all types of coupons.

\(X\) - # of boxes until you get all n types.

Example: \(n=3\)

\[
\begin{array}{c|ccc}
2 & 2 & 2 & 1 & 1 & 2 & 1 & 3 \\
X_1 & X_2 & X_3
\end{array}
\]

\([E[X]]\) = ?

Idea: Write \(X\) as the sum \(X_1 + X_2 + X_3\)

1 = \(X_1\) - # boxes until we get first type

\(\text{Geo}(\frac{2}{3})\) \(\leftarrow\) \(X_2\) - # boxes until we get second type after we got first type.

\(\text{Geo}(\frac{1}{3})\) \(\leftarrow\) \(X_3\) - # boxes until we get third type after we got second type
**Coupon Collector**

**Experiment:** Get random coupons from \( S_1, \ldots, S_3 \) until you have all types of coupons.

**X:** \# of boxes until you get all \( n \) types.

**Example:** \( n=3 \)

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</tr>
</tbody>
</table>

\[
\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3]
\]

\[
= 1 + \frac{3}{2} + 3 = 5.5
\]

**Idea:** Write \( X \) as the sum \( X_1 + X_2 + X_3 \)

1 = \( X_1 \) - \# boxes until we get first type

\( \text{Geo}(\frac{2}{3}) \) \( \leftarrow \) \( X_2 \) - \# boxes until we get second type after we got first type.

\( \text{Geo}(\frac{1}{3}) \) \( \leftarrow \) \( X_3 \) - \# boxes until we get third type after we got second type.
General Case:

Write \( X = X_1 + X_2 + \ldots + X_n \)

\( X_i \) is the number of boxes until we get \( i \) unique types after we got \( i-1 \) unique types.
General Case:

Write \( X = X_1 + X_2 + \ldots + X_n \)

\( X_i \) — #boxes until we get \( i \) unique types after we got \((i - 1)\) unique types.

- \( X_1 = 1 \)
- \( X_2 = \begin{cases} 1, & \text{w.p.} \frac{1}{2} \\ 2, & \text{w.p.} \frac{1}{2} \\ \vdots \end{cases} \)
- \( X_3 \sim \)
- \( \vdots \)
- \( X_i \sim \)
General Case:

Write $X = X_1 + X_2 + \ldots + X_n$

$X_i$ - #boxes until we get $i$ unique types after we got $(i-1)$ unique types.

- $X_1 = 1$
- $X_2 = \sum_{i=1}^n 1 \quad \text{w.p.} \quad \frac{n-1}{n}$
  \[ 2 \quad \text{w.p.} \quad \frac{i-1}{n} \]
- $X_3 \sim \text{Geo} \left( \frac{n-2}{n} \right)$
- $X_i \sim \text{Geo} \left( \frac{n-i+1}{n} \right)$

$E[X] = E[X_1] + \ldots + E[X_n] =$
Balls in Bins

Suppose you throw balls in $n$ bins. How many balls do you need to throw until all bins are non-empty?
Functions of Random Variables

Suppose $X$ is a r.v. and $g: \mathbb{R} \rightarrow \mathbb{R}$

Then $Y = g(X)$ is also a r.v.

$Y(w) = g(X(w))$. **Q:** Calculate $\mathbb{E}[Y]$
Functions of Random Variables

Suppose $X$ is a r.v. and $g: \mathbb{R} \rightarrow \mathbb{R}$.

Then $Y = g(X)$ is also a r.v.

$Y(w) = g(X(w))$.

**Q:** Calculate $E[Y]$.

**Method 1:** Compute the dist of $Y$ from the dist of $X$.

**Method 2:**

$E[Y] = \sum_{\omega \in \Omega} g(X(\omega)) \cdot Pr[\omega]$

$= \sum_{x} \sum_{\omega: X(\omega) = x} g(x) \cdot Pr[\omega]$

$= \sum_{x} g(x) \cdot \sum_{\omega: X(\omega) = x} Pr[\omega] = \sum_{x} g(x) \cdot Pr[X = x]$
Variance

Suppose we have some r.v. $X$

$E(X)$ - the average value $X$ would get if we run the experiment many times.

But if we run the experiment only once, we still would like a guarantee on how close $X$ is to its mean.
Variance

The variance measures the deviation of $X$ from its mean.
**Variance**

The variance measures the deviation of $X$ from its mean.

**Defn:** The variance of a r.v. $X$ is defined as

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$$

Measures the expected deviation squared.

$$\sigma(X) = \text{stdev}(X) = \sqrt{\text{Var}(X)}$$
Variance

**Defn:** The variance of a r.v. $X$ is defined as

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$$

**Fact:** $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. 
**Variance**

**Defn:** The variance of a r.v. $X$ is defined as

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

**Fact:** $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

**Proof:** Let $\mu = \mathbb{E}[X]$.

$$\text{Var}(X) = \mathbb{E}[(X - \mu)^2]$$

$$= \mathbb{E}[X^2 - 2\mu X + \mu^2]$$

$$= \mathbb{E}[X^2] - 2\mu \mathbb{E}[X] + \mu^2$$

$$= \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$
Simple Example

\[ X = \begin{cases} 
  m + \sigma & \text{w.p. } \frac{1}{2} \\
  m - \sigma & \text{w.p. } \frac{1}{2}
\end{cases} \]

\[ \mathbb{E}[X] = ? \]

\[ \text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = ? \]
**Simple Example**

\[ X = \begin{cases} 
\mu + \sigma & \text{w.p. } \frac{1}{2} \\
\mu - \sigma & \text{w.p. } \frac{1}{2}
\end{cases} \]

\[ \mathbb{E} X = \frac{(\mu + \sigma) + (\mu - \sigma)}{2} = \mu \]

\[ \text{Var}(X) = \mathbb{E}[(X - \mathbb{E}X)^2] = \frac{1}{2} (\mu + \sigma - \mu)^2 + \frac{1}{2} (\mu - \sigma - \mu)^2 \]

\[ = \frac{1}{2} \sigma^2 + \frac{1}{2} \sigma^2 = \sigma^2 \]
Facts: For any \( c \in \mathbb{R} \), r.v. \( X \)

1. \( \text{Var} (c \cdot X) = c^2 \cdot \text{Var} (X) \). \( \text{Scales by } c^2 \)

2. \( \text{Var} (c + X) = \text{Var} (X) \). \( \text{shifts center} \)
Facts: For any $c \in \mathbb{R}$, r.v. $X$

1. $\text{Var}(c \cdot X) = c^2 \cdot \text{Var}(X)$. Scales by $c^2$

2. $\text{Var}(c + X) = \text{Var}(X)$. Shifts center

Proof:

1. $\text{Var}(c \cdot X) = \mathbb{E}[(cX)^2] - \mathbb{E}[cX]^2$
   
   $= c^2 \cdot \mathbb{E}[X^2] - c^2 \cdot \mathbb{E}[X]^2$
   
   $= c^2 \cdot \text{Var}(X)$.

2. $\text{Var}(c+X) = \mathbb{E}[(c+X - (\mathbb{E}(c+X))^2)]$
   
   $= \mathbb{E}[(c+X - c - \mathbb{E}X)^2]$
   
   $= \mathbb{E}[(X - \mathbb{E}X)^2] = \text{Var}(X)$. 
Facts: For any $c \in \mathbb{R}$, r.v. $X$

1. $\text{Var}(c \cdot X) = c^2 \cdot \text{Var}(X)$. Scales by $c^2$
2. $\text{Var}(c + X) = \text{Var}(X)$. Shifts center

If $X$ and $Y$ are r.v.s.
what's $\text{Var}(X + Y) =$?
**Independent Random Variables**

**Defn:** We say that two r.v.s $X$ and $Y$ are independent if

$$\Pr[X = a, Y = b] = \Pr[X = a] \cdot \Pr[Y = b]$$

for all $a \in \text{range}(X)$ and $b \in \text{range}(Y)$.

**Fact:** If $X$ & $Y$ are independent,

$$\Pr[X = a \mid Y = b] = \Pr[X = a]$$
Theorem: If X and Y are independent r.v.s, then
\[ E[X \cdot Y] = E[X] \cdot E[Y] \]
Theorem: If $X$ and $Y$ are independent r.v.s, then $$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

Proof: $$\mathbb{E}[X \cdot Y] = \sum_{a, b} a \cdot b \cdot \Pr[X = a, Y = b]$$

$$= \sum_{a, b} a \cdot b \cdot \Pr[X = a] \cdot \Pr[Y = b]$$

$$= \left( \sum_{a} a \cdot \Pr[X = a] \right) \cdot \left( \sum_{b} b \cdot \Pr[Y = b] \right)$$

$$= \mathbb{E}[X] \cdot \mathbb{E}[Y].$$
Theorem: If $X$ and $Y$ are independent r.v.s, then

$$\text{Var} (X+Y) = \text{Var} (X) + \text{Var} (Y).$$
Theorem: If $X$ and $Y$ are independent r.v.s, then

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Proof:

Theorem: If $X$ and $Y$ are independent r.v.s, then

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y).$$

Proof:


$$\left(E[X+Y]\right)^2 = (E[X] + E[Y])^2$$

$$= (E[X])^2 + 2(E[X])(E[Y]) + (E[Y])^2$$
**Theorem:** If $X$ and $Y$ are independent r.v.s, then

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y).$$

**Proof:**

$$\left\{ \begin{align*}
\mathbb{E}[(X+Y)^2] &= \mathbb{E}[x^2] + 2\mathbb{E}[xy] + \mathbb{E}[y^2] \\
\left(\mathbb{E}[X+Y]\right)^2 &= \left(\mathbb{E}[X] + \mathbb{E}[Y]\right)^2 \\
&= (\mathbb{E}[X])^2 + 2(\mathbb{E}[X])(\mathbb{E}[Y]) + (\mathbb{E}[Y])^2
\end{align*} \right.$$ 

$$\text{Var}(X+Y) = \mathbb{E}[x^2] - (\mathbb{E}[x])^2 + 2\mathbb{E}[xy] - 2(\mathbb{E}[x])(\mathbb{E}[y]) + \mathbb{E}[y^2] - (\mathbb{E}[y])^2$$
**Theorem:** If X and Y are independent r.v.s, then

\[
\text{Var} (X+Y) = \text{Var} (X) + \text{Var} (Y).
\]

**Proof:**

\[
\begin{align*}
\left[ \mathbb{E}[(X+Y)^2] \right] &= \mathbb{E}[x^2] + 2\mathbb{E}[XY] + \mathbb{E}[y^2] \\
- \left[ (\mathbb{E}[X+Y])^2 \right] &= (\mathbb{E}X + \mathbb{E}Y)^2 \\
&= (\mathbb{E}X)^2 + 2(\mathbb{E}X)(\mathbb{E}Y) + (\mathbb{E}Y)^2
\end{align*}
\]

\[
\text{Var} (X+Y) = \mathbb{E}[x^2] - (\mathbb{E}X)^2 + 2 \mathbb{E}[XY] - 2(\mathbb{E}X)(\mathbb{E}Y) + \mathbb{E}[y^2] - (\mathbb{E}Y)^2
\]

\[
= \text{Var}(X) + 0 + \text{Var}(Y)
\]

\[
\text{since } X, Y \text{ are indep.}
\]
Theorem: If \( X_1, X_2, \ldots, X_n \) are pairwise independent r.v.'s then

\[
\text{Var}(X_1 + \ldots + X_n) = \text{Var}(X_1) + \ldots + \text{Var}(X_n)
\]
Theorem: If $X_1, X_2, ..., X_n$ are pairwise independent r.v.s then

$$\text{Var}(X_1 + ... + X_n) = \text{Var}(X_1) + ... + \text{Var}(X_n)$$

Proof: (Similar)

\[
\begin{align*}
\mathbb{E}[(X_1 + ... + X_n)^2] &= \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j] \\
\mathbb{E}[X_1 + ... + X_n]^2 &= \sum_{i=1}^{n} \mathbb{E}[X_i]^2 + \sum_{i \neq j} \mathbb{E}[X_i] \mathbb{E}[X_j]
\end{align*}
\]
**Theorem:** If \( X_1, X_2, \ldots, X_n \) are pairwise independent r.v.s then

\[
\text{Var}(X_1 + \ldots + X_n) = \text{Var}(X_1) + \ldots + \text{Var}(X_n)
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**Proof:** (Similar)

\[
\begin{align*}
\mathbb{E}[(X_1 + \ldots + X_n)^2] &= \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j] \\
\mathbb{E}[(X_1 + \ldots + X_n)^2] &= \sum_{i=1}^{n} \mathbb{E}[X_i]^2 + \sum_{i \neq j} \mathbb{E}[X_i] \mathbb{E}[X_j] \\
\end{align*}
\]

\[
\text{Var}(X_1 + \ldots + X_n) = \sum_{i=1}^{n} \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 + \sum_{i \neq j} \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j] = \sum_{i=1}^{n} \text{Var}(X_i) + 0.
\]
Facts: For any $c \in \mathbb{R}$, r.v. $X$
1. $\text{Var}(c \cdot X) = c^2 \cdot \text{Var}(X)$. Scales by $c^2$
2. $\text{Var}(c + X) = \text{Var}(X)$. Shifts center

Q: If $X$ and $Y$ are r.v.s. What's $\text{Var}(X+Y)$ = ?

A: It depends.
Facts: For any $c \in \mathbb{R}$, r.v. $X$

1. $\text{Var}(c \cdot X) = c^2 \cdot \text{Var}(X)$.  
Scales by $c^2$

2. $\text{Var}(c + X) = \text{Var}(X)$.  
shifts center

Q: If $X$ and $Y$ are r.v.s.

What's $\text{Var}(X+Y) =$ ?

A: It depends.

Example 1: $X, Y$ ind.

Example 2: $X, Y$ r.v.s $Y = X$

Example 3: $X, Y$ r.v.s $Y = -X$
Variance of a Binomial Distribution

\( X \sim \text{Bin}(n, p) \) models \# of heads in \( n \) coin tosses w. heads prob. \( p \).

\[
\Pr[X = i] = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}
\]

\( \text{Var}(X) = \) ?
Variance of a Binomial Distribution

$X \sim \text{Bin}(n, p)$ models # of heads in $n$ coin tosses w. heads prob. $p$.

$$\Pr[X = i] = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$

$$X = X_1 + \ldots + X_n$$

$X_i = \text{indicator that } i^{th} \text{ flip is heads.}$

$$\mathbb{E}X = \mathbb{E}X_1 + \ldots + \mathbb{E}X_n = n \cdot p.$$ 

$$\text{Var}(X) = ?$$
Variance of a Binomial Distribution

\[ X \sim \text{Bin}(n, p) \] models \# of heads in \( n \) coin tosses \( \omega \) heads prob. \( p \).

\[ \Pr[X = i] = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i} \]

\[ X = X_1 + \ldots + X_n \quad \text{where } X_i = \text{indicator that } i\text{th flip is heads}. \]

\[ \mathbb{E}X = \mathbb{E}X_1 + \ldots + \mathbb{E}X_n = n \cdot p. \]

\[ \text{Var}(X) = \text{Var}(X_1 + \ldots + X_n) = \text{Var}(X_1) + \ldots + \text{Var}(X_n) \]

\[ \text{Var}(X) \uparrow \text{ as } X_1, \ldots, X_n \text{ are pairwise indep}. \]
**Variance of a Binomial Distribution**

\( X \sim \text{Bin}(n, p) \) models the number of heads in \( n \) coin tosses with heads probability \( p \).

\[
\Pr[X = i] = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}
\]

\( X = X_1 + \ldots + X_n \)  \( X_i \) = indicator that \( i \)th flip is heads.

\[
\mathbb{E}X = \mathbb{E}X_1 + \ldots + \mathbb{E}X_n = n \cdot p.
\]

\[
\text{Var}(X) = \text{Var}(X_1 + \ldots + X_n) = \text{Var}(X_1) + \ldots + \text{Var}(X_n)
\]

\[
\text{Var}(X_i) = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = p - p^2 = p \cdot (1-p)
\]
# Variance of a Binomial Distribution

\( X \sim \text{Bin}(n, p) \) models the number of heads in \( n \) coin tosses with heads probability \( p \).

\[
\Pr[X = i] = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}
\]

\( \mathbb{E} X = np \)

\( \text{Var}(X) = np(1-p) \leq \frac{n}{4} \)

\( \text{std}(X) = \sqrt{np(1-p)} \leq \frac{\sqrt{n}}{2} \)

The variance scales linearly with \( n \), while the standard deviation scales like \( \sqrt{n} \).
Number of Fixed Points - Variance

Handout assignment at random to \( n \) students

\( X = \) number of students who got their own assignment.

\( \Omega = \{ \pi : \pi \text{ is a permutation of } \{1, \ldots, n\} \} \)
Handout assignment at random to n students

\( X = \text{number of students who got their own assignment.} \)

\( \Omega = \{ \pi : \pi \text{ is a permutation of } \{1,...,n\} \} \)

\( X = X_1 + X_2 + ... + X_n \quad \text{where } X_i = \text{indicator that } \pi(i) = i \)

\( \text{Var}(X) = \text{Var}(x) - \text{Var}(x)^2 \)
Number of Fixed Points - Variance

Handout assignment at random to n students

$X = \text{number of students who got their own assignment}$

$\Omega = \{ \pi : \pi \text{ is a permutation of } \{1, \ldots, n\} \}$

$X = X_1 + X_2 + \ldots + X_n$  

$X_i = \text{indicator that } \pi(i) = i$

$\text{Var}(X) = E[X^2] - (E[X])^2$

$E[X^2] = E\left[ (X_1 + \ldots + X_n)^2 \right] = E\left[ \sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j \right]$

$= \sum_{i=1}^{n} E[X_i^2] + \sum_{i \neq j} E[X_i X_j]$
Number of Fixed Points - Variance

Handout assignment at random to \( n \) students

\( X = \text{number of students who got their own assignment} \)

\( \Omega = \{ \pi : \pi \text{ is a permutation of } \{1, \ldots, n\} \} \)

\( X = X_1 + X_2 + \ldots + X_n \quad X_i = \text{indicator that } \pi(i) = i \)

\( \text{Var}(X) = E[X^2] - E[X]^2 \)

\( E[X^2] = E[(X_1 + \ldots + X_n)^2] = E\left[ \sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j \right] \)

\( = \sum_{i=1}^{n} E[X_i^2] + \sum_{i \neq j} E[X_i X_j] \)

\( E[X_i X_j] = Pr[\pi(i) = i, \pi(j) = j] = \frac{1}{n} \cdot \frac{1}{n-1} \quad \text{are } X_i, X_j \text{ indep?} \)

\( E[X^2] = \sum_{i=1}^{n} E[X_i^2] + \sum_{i \neq j} \frac{1}{n} \cdot \frac{1}{n-1} = n \cdot \frac{1}{n} + n \cdot (n-1) \cdot \frac{1}{n} \cdot \frac{1}{n-1} = 2 \).
Poll: What's true

1. $X_i$ & $X_j$ are ind.

2. $E[X_i X_j] = Pr[X_i X_j = 1]$

3. $E[X_i X_j] = Pr[X_i = 1 \cap X_j = 1]$

4. $X_i^2 = X_i$
Covariance

If \( x \) and \( y \) are r.v.s their co-variance is

\[
\text{Cov}(x, y) = \mathbb{E}[(x - \mathbb{E}x)(y - \mathbb{E}y)]
\]
Covariance

If \( x \) and \( y \) are r.v.s their co-variance is

\[
\text{Cov}(x, y) = \mathbb{E}[(x - \mathbb{E}x)(y - \mathbb{E}y)]
\]

Fact: \( \text{Cov}(x, y) = \mathbb{E}[xy] - (\mathbb{E}x) \cdot (\mathbb{E}y) \)
Covariance

If \( x \) and \( y \) are r.v.s, their co-variance is

\[
\text{cov}(x,y) = \mathbb{E}[(x - \mathbb{E}x)(y - \mathbb{E}y)]
\]

**Fact:** \( \text{cov}(x,y) = \mathbb{E}[xy] - (\mathbb{E}x) \cdot (\mathbb{E}y) \)

**Proof:** Let \( \mu_x = \mathbb{E}x \quad \mu_y = \mathbb{E}y \)

\[
\text{cov}(x,y) = \mathbb{E}[(x - \mu_x)(y - \mu_y)]
\]

\[
= \mathbb{E}[xy] - \mu_x \mathbb{E}[y] - \mu_y \mathbb{E}[x] + \mu_x \mu_y
\]

\[
= \mathbb{E}[xy] - \mu_x \mu_y = \mathbb{E}[xy] - (\mathbb{E}x)(\mathbb{E}y).
\]
Covariance

If \( x \) and \( y \) are r.v.s their co-variance is

\[
\text{Cov}(x, y) = \mathbb{E}[(x - \mathbb{E}x)(y - \mathbb{E}y)]
\]

Fact: \( \text{Cov}(x, y) = \mathbb{E}[xy] - (\mathbb{E}x)(\mathbb{E}y) \)

Corollary: If \( x \) and \( y \) are independent then \( \text{Cov}(x, y) = 0 \).
Recap

**Theorem:** If $X$ and $Y$ are independent r.v.s, then

$$\text{Var} \ (X+Y) = \text{Var} \ (X) + \text{Var} \ (Y).$$

**Proof:**

$$\begin{align*}
\left\{ \mathbb{E}[(X+Y)^2] \right\} &= \left( \mathbb{E}[X^2] + 2 \mathbb{E}[XY] + \mathbb{E}[Y^2] \right) \\
- \left\{ (\mathbb{E}[X+Y])^2 \right\} &= (\mathbb{E}X + \mathbb{E}Y)^2 \\
&= (\mathbb{E}X)^2 + 2(\mathbb{E}X)(\mathbb{E}Y) + (\mathbb{E}Y)^2
\end{align*}$$

$$\begin{align*}
\text{Var} \ (X+Y) &= \mathbb{E}[X^2] - (\mathbb{E}X)^2 + 2 \mathbb{E}[XY] - 2(\mathbb{E}X)(\mathbb{E}Y) \\
&\quad + \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 \\
&= \text{Var} \ (X) + 0 + \text{Var} \ (Y)
\end{align*}$$

The only point where we used that $X$ & $Y$ are independent.
Theorem: If $X$ and $Y$ are r.v.s

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X, Y)$$

Proof:

$$\begin{align*}
\sum \left( \mathbb{E}[(X+Y)^2] \right) &= \mathbb{E}(X^2) + 2 \mathbb{E}(XY) + \mathbb{E}(Y^2) \\
\left( \mathbb{E}[X+Y] \right)^2 &= (\mathbb{E}X + \mathbb{E}Y)^2 \\
&= (\mathbb{E}X)^2 + 2(\mathbb{E}X)(\mathbb{E}Y) + (\mathbb{E}Y)^2
\end{align*}$$

$$\begin{align*}
\text{Var}(X+Y) &= \mathbb{E}[X^2] - (\mathbb{E}X)^2 + 2 \mathbb{E}[XY] - 2(\mathbb{E}X)(\mathbb{E}Y) \\
&\quad + \mathbb{E}[Y^2] - (\mathbb{E}Y)^2 \\
&= \text{Var}(X) + 2 \cdot \text{Cov}(X,Y) + \text{Var}(Y)
\end{align*}$$