Today

Estimation.

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MMSE: Best Function that predicts X from Y.

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Finish Linear Regression:

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Applications to random processes.

Estimation: cs70 style

Given distribution for Y.

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Estimation: Expectation and Mean Squared Error.

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Definitions Let X and Y be RVs on Ω.

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Today: we view as a predicted value for Y.

Properties of CE

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(a) Obvious

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Properties of CE

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(e) Let $h(X) = 1$ in (d).

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It gives that $E[Y|X]$ is best estimator for Y given X.

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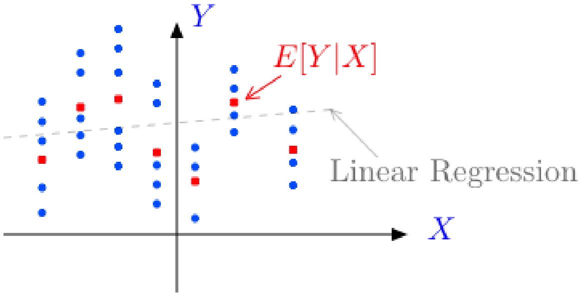

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Thus, $E[(Y - h(X))^2] \geq E[(Y - g(X))^2]$. $\square$

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Consider a social network (e.g., Twitter).

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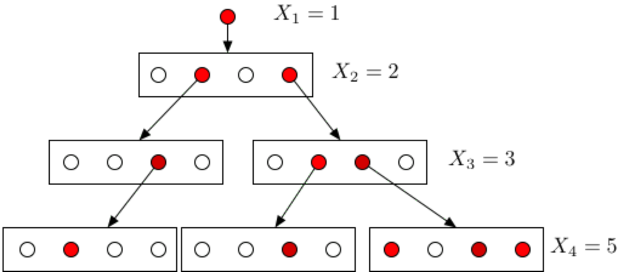

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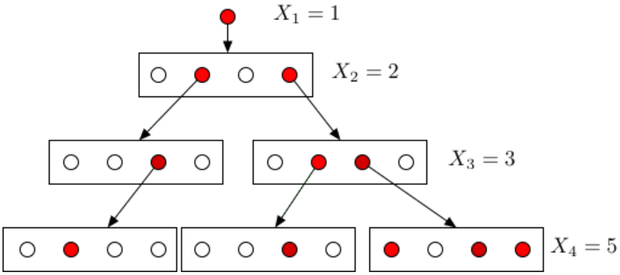

In this example, $d = 4$.

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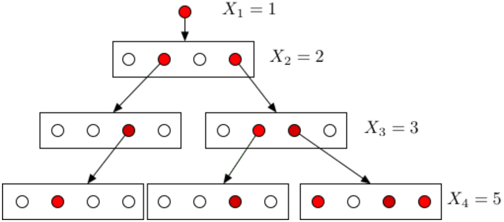

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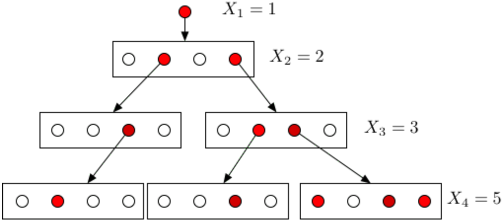
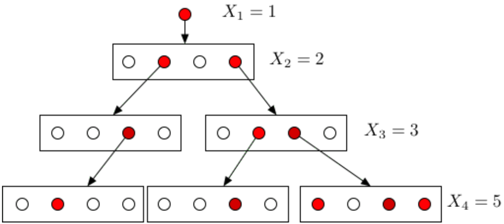


Fact:

Application: Going Viral



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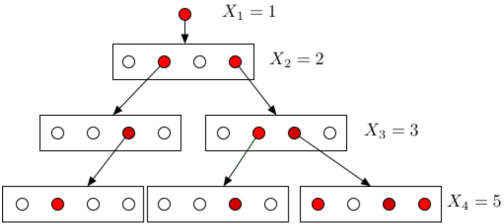

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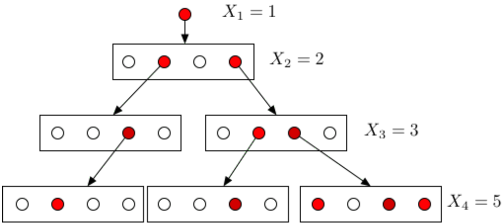

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Given $X_n = k$, $X_{n+1} = B(kd, p)$.

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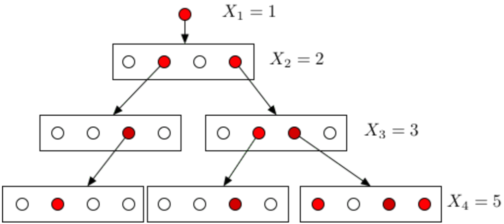

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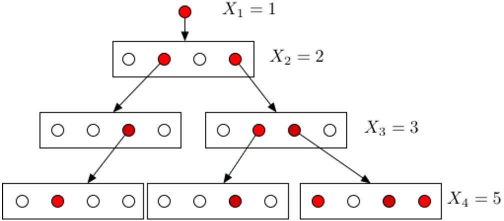

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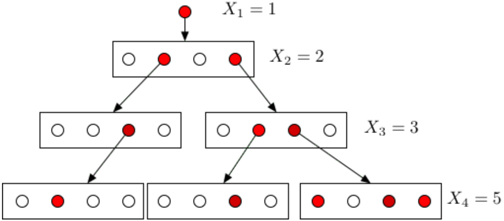

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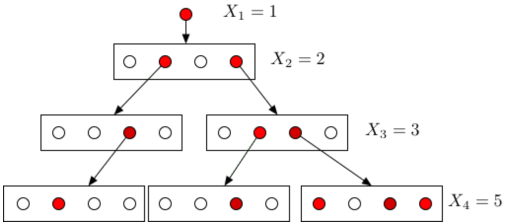

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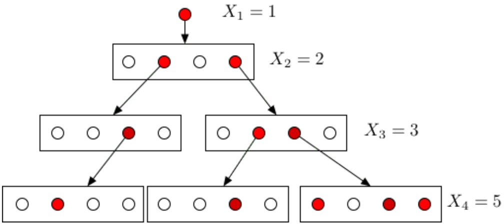

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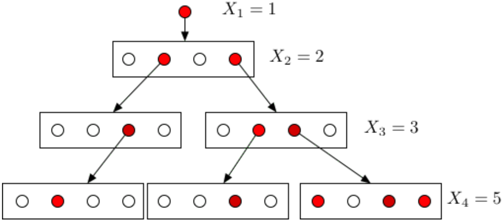

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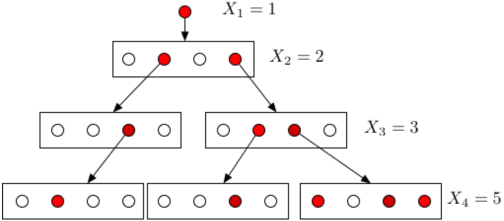

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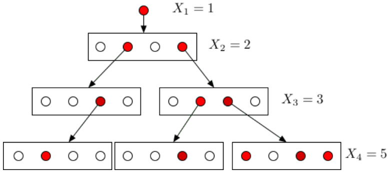

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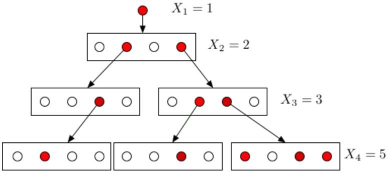

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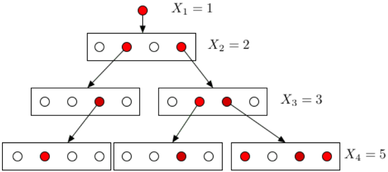

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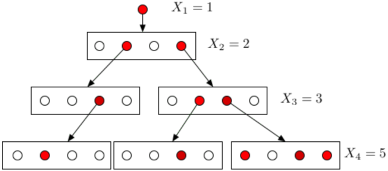

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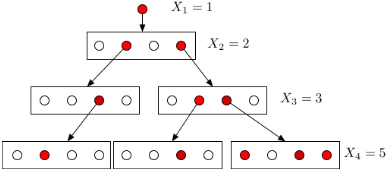

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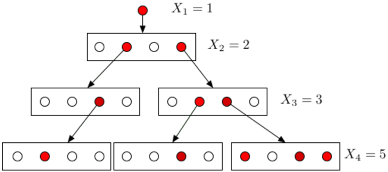

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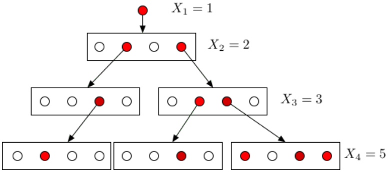

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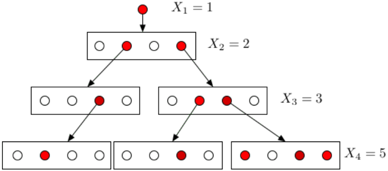

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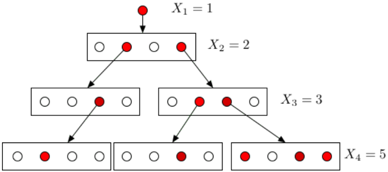

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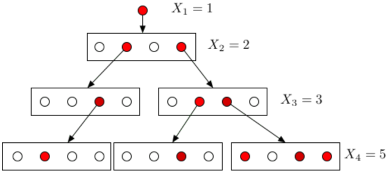

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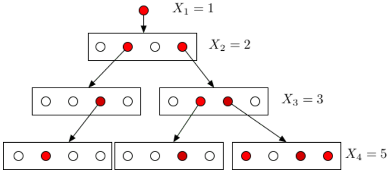

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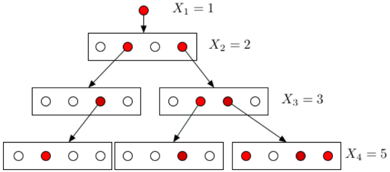

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We conclude as before.

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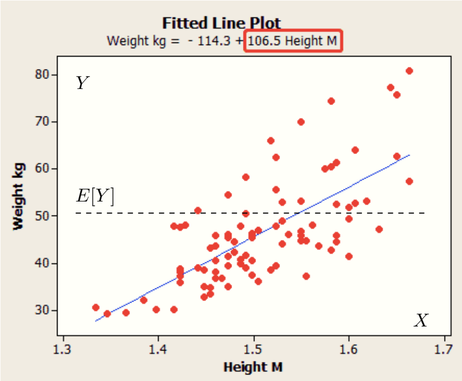

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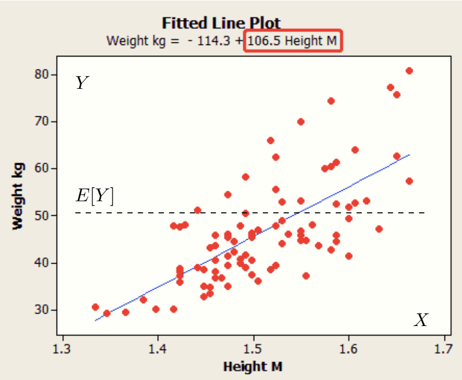

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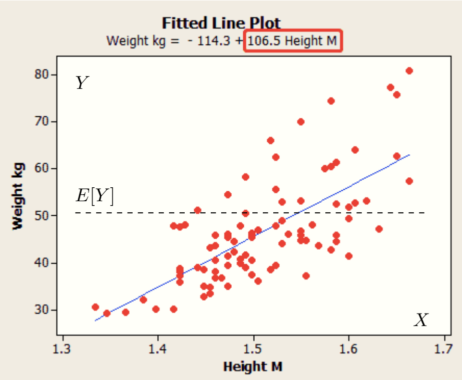

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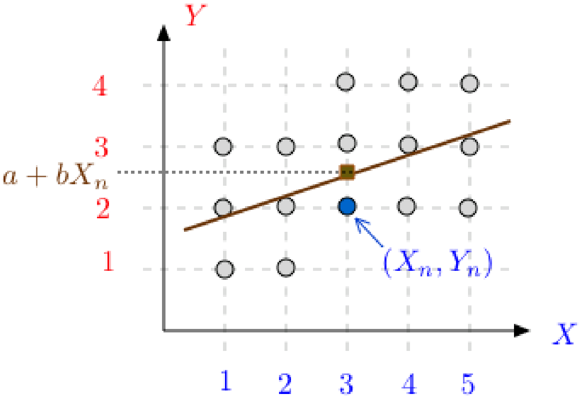

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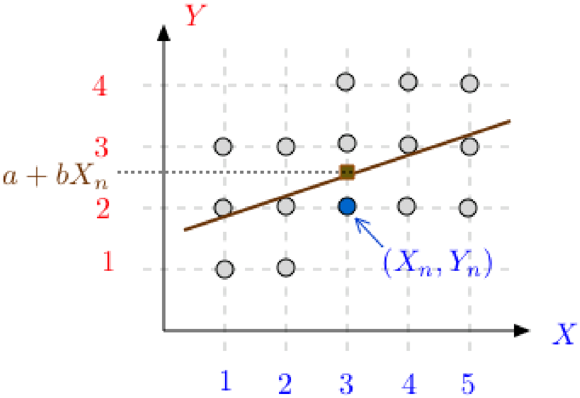

The line $Y = a + bX$ is the linear regression.

LLSE

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$^{(*)}$ Recall that $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$ and $var[X] = E[(X - E[X])^2]$.

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Without observations, the estimate is $E[Y]$. The error is $var(Y)$. Observing X reduces the error.

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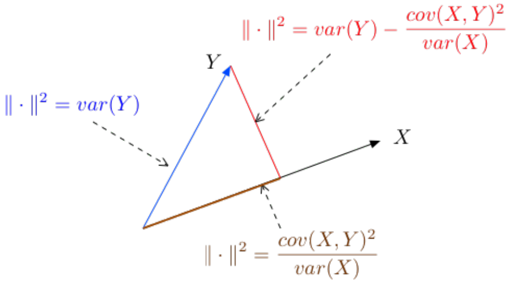

Vector Y at dimension ω is $\frac{1}{\sqrt{\Omega}}Y(\omega)$

Linear Regression Examples

Example 1:

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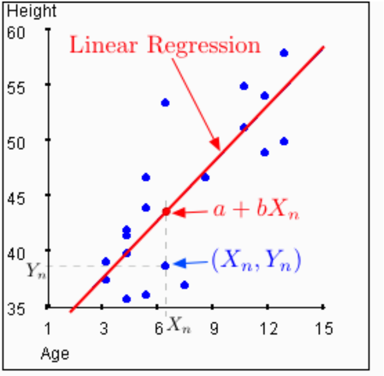

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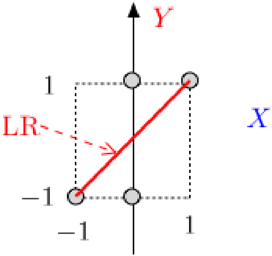

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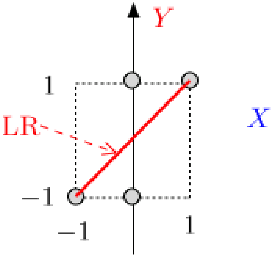

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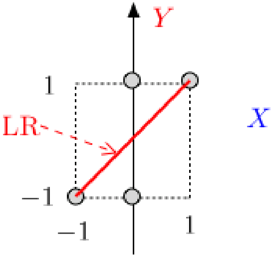

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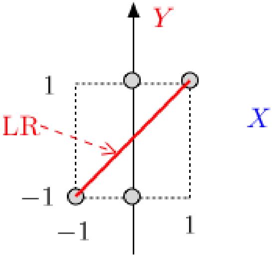

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Linear Regression Examples

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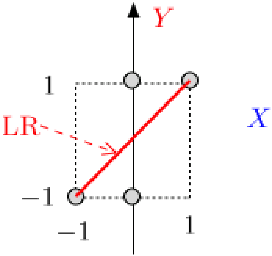

We find:

$$E[X] = 0; E[Y] = 0;$$

Linear Regression Examples

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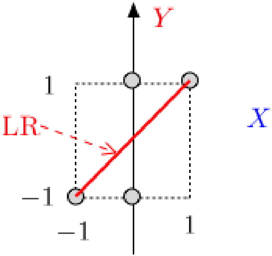

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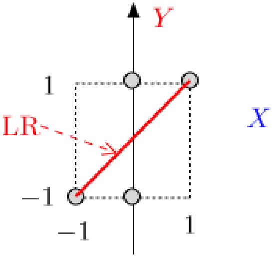

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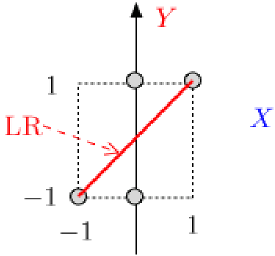

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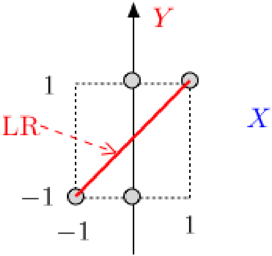

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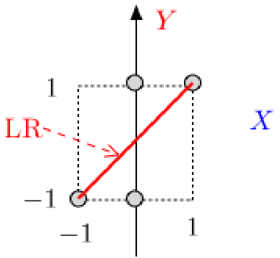

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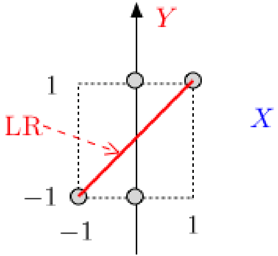

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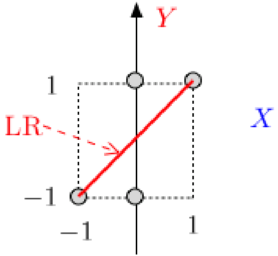

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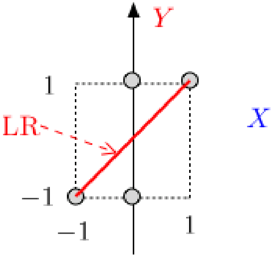

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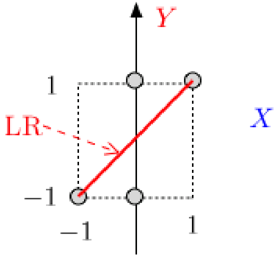

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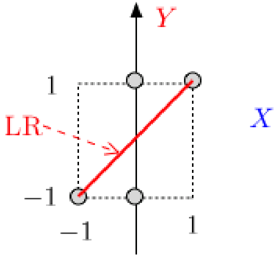

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Linear Regression Examples

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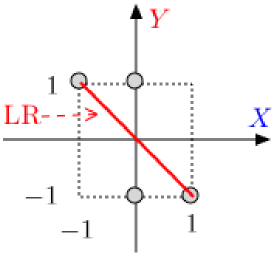

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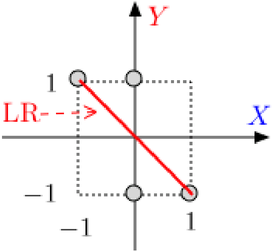

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$E[X] =$

Linear Regression Examples

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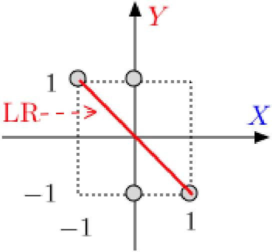

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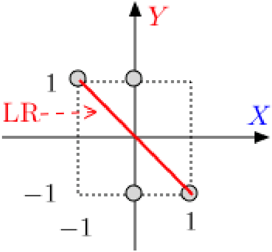

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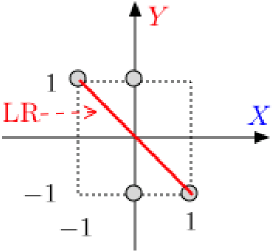

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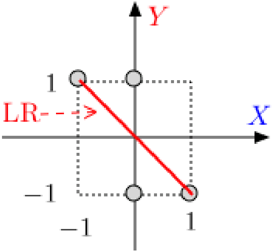

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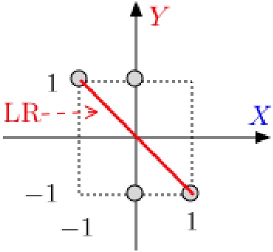

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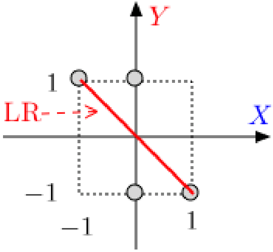

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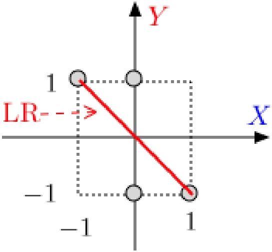

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Linear Regression Examples

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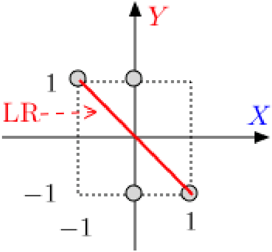

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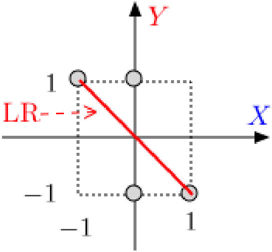

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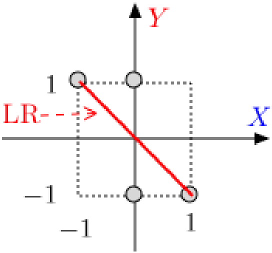

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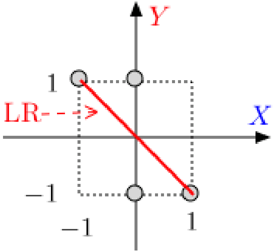

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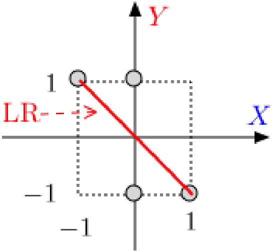

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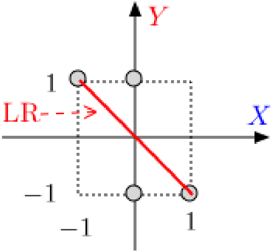

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Linear Regression Examples

Example 4:

Linear Regression Examples

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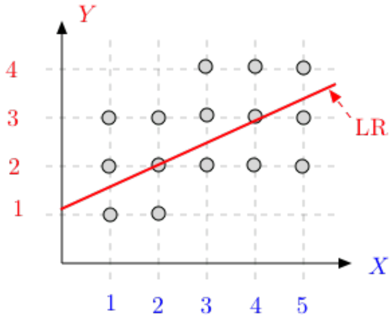

Linear Regression Examples

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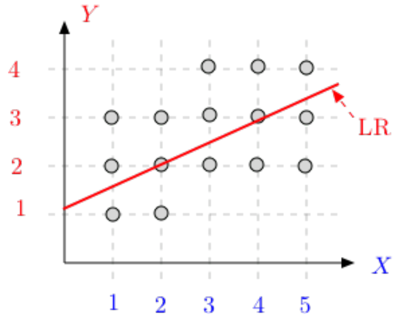

We find:

$$E[X] =$$

Linear Regression Examples

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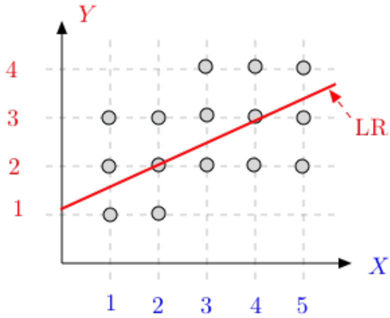

We find:

$E[X] = 3;$

Linear Regression Examples

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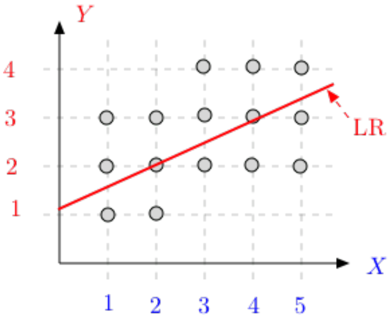

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Linear Regression Examples

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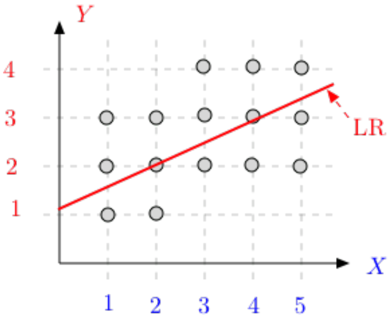

We find:

$$E[X] = 3; E[Y] = 2.5;$$

Linear Regression Examples

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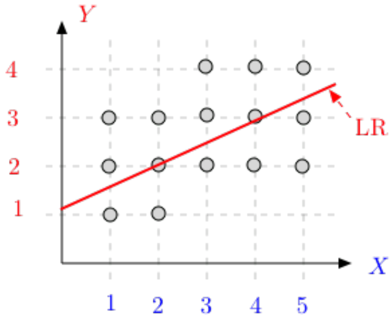

We find:

$$E[X] = 3; E[Y] = 2.5; E[X^2] = (3/15)(1 + 2^2 + 3^2 + 4^2 + 5^2) = 11;$$

Linear Regression Examples

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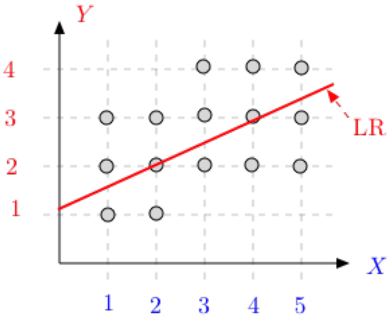

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Linear Regression Examples

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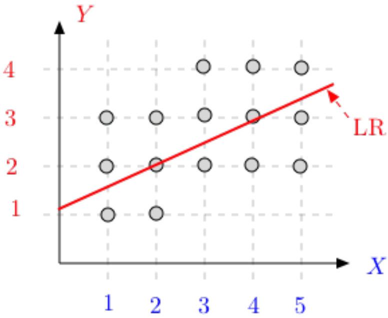

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Linear Regression Examples

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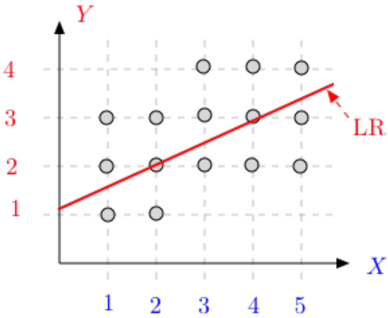

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Linear Regression Examples

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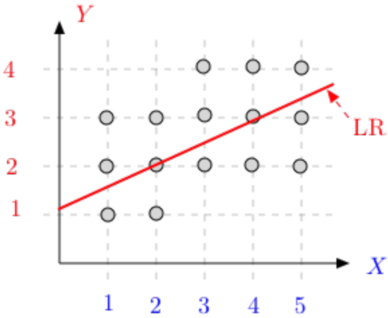

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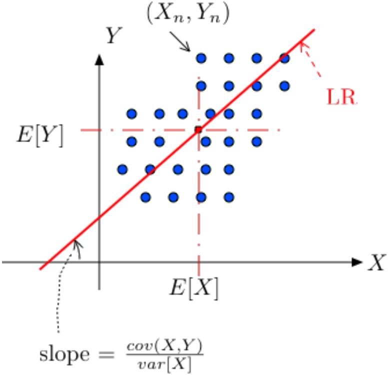
$$var[X] = 11 - 9 = 2; cov(X, Y) = 8.4 - 3 \times 2.5 = 0.9;$$

$$\text{LR: } \hat{Y} = 2.5 + \frac{0.9}{2}(X - 3) = 1.15 + 0.45X.$$

LR: Another Figure



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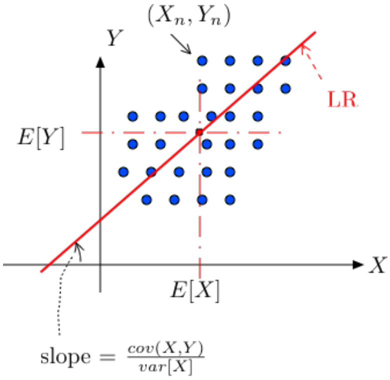

Note that

- the LR line goes through $(E[X], E[Y])$

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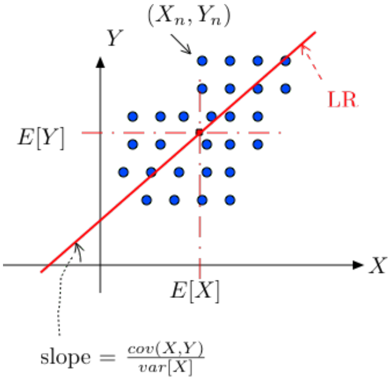

Note that

- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

Quadratic Regression

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where a, b, c are chosen to minimize $E[(Y - a - bX - cX^2)^2]$.

Derivation: We set to zero the derivatives w.r.t. a, b, c. We get

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We solve these three equations in the three unknowns (a, b, c).

Quadratic Regression

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Statistics: Fix the assumption above.