Riemann Sum/Integral: \( \int_a^b f(x) \, dx = \lim_{\delta \to 0} \sum_i \delta f(a_i) \)

“Area is defined as rectangles and add up some thin ones.”

Derivative (Rate of change):
\[
F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}.
\]

“Rise over run of close together points.”

Fundamental Theorem: \( F(b) - F(a) = \int_a^b F'(x) \, dx \)

“Area (\( F(\cdot) \)) under \( f(x) \) grows at \( x, F'(x) \), by \( f(x) \)”

Thus \( F'(x) = f(x) \).
Continuous Probability 1

1. Examples
2. Events
3. Continuous Random Variables
Uniformly at Random in \([0, 1]\).

Choose a real number \(X\), uniformly at random in \([0, 1]\).

What is the probability that \(X\) is exactly equal to \(1/3\)? Well, ..., 0.

What is the probability that \(X\) is exactly equal to 0.6? Again, 0.

In fact, for any \(x \in [0, 1]\), one has \(Pr[X = x] = 0\).

How should we then describe ‘choosing uniformly at random in \([0, 1]\)’?

Here is the way to do it:

\[
Pr[X \in [a, b]] = b - a, \forall 0 \leq a \leq b \leq 1.
\]

Makes sense: \(b - a\) is the fraction of \([0, 1]\) that \([a, b]\) covers.
Poll

\[ F_X(x) = \Pr[X \leq x] \]

\[ f_X(x) = \lim_{\delta \to 0} \frac{\Pr[X \in (x, x + \delta)]}{\delta} \]

What is true?

(A) \( F_X(x) = \int_{-\infty}^{\infty} f_X(y)dy \)
(B) \( \int_{-\infty}^{\infty} f_X(x) = 1 \)
(C) \( F_X(x) = \int_{-\infty}^{x} f(y)dy \).
(D) \( f(x) = F_X'(x) \).
(E) \( \int_{-\infty}^{\infty} F_X(x)dx = 1 \).
(F) \( \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty}(1 - F(x))dx \).

(A) False. limits wrong. (B) cuz probability distribution.
(C) “sums up probability of rectangles”, e.g. calculus.
(D) calculus, fundamental theorem.

(F) is true since \( \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} F(x)dx = E[X] \).

Next lecture.
Uniformly at Random in $[0,1]$.

Let $[a, b]$ denote the event that the point $X$ is in the interval $[a, b]$.

\[
Pr[[a, b]] = \frac{\text{length of } [a, b]}{\text{length of } [0,1]} = \frac{b-a}{1} = b-a.
\]

Intervals like $[a, b] \subseteq \Omega = [0,1]$ are events. More generally, events in this space are unions of intervals. Example: the event $A$ - “within 0.2 of 0 or 1” is $A = [0, 0.2] \cup [0.8, 1]$. Thus,

\[
Pr[A] = Pr[[0, 0.2]] + Pr[[0.8, 1]] = 0.4.
\]

More generally, if $A_n$ are pairwise disjoint intervals in $[0,1]$, then

\[
Pr[\bigcup_n A_n] := \sum_n Pr[A_n].
\]

Many subsets of $[0,1]$ are of this form. Thus, the probability of those sets is well defined. We call such sets events.
Uniformly at Random in $[0, 1]$.

Note: A **radical** change in approach.

**Finite prob. space:** $\Omega = \{1, 2, \ldots, N\}$, with $Pr[\omega] = p_\omega$.

$\implies Pr[A] = \sum_{\omega \in A} p_\omega$ for $A \subset \Omega$.

**Continuous space:** e.g., $\Omega = [0, 1]$, $Pr[\omega]$ is typically 0.

Instead, start with $Pr[A]$ for some events $A$.
Event $A$ = interval, or union of intervals.
Uniformly at Random in $[0, 1]$.

$Pr[X \leq x] = x$ for $x \in [0, 1]$. Also, $Pr[X \leq x] = 0$ for $x < 0$. $Pr[X \leq x] = 1$ for $x > 1$.

Define $F(x) = Pr[X \leq x]$.

Then we have $Pr[X \in (a, b)] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a)$.

Thus, $F(\cdot)$ specifies the probability of all the events!
Uniformly at Random in $[0,1]$.

$$Pr[X \in (a,b)] = Pr[X \leq b] - Pr[X \leq a] = F(b) - F(a).$$

An alternative view is to define $f(x) = \frac{d}{dx} F(x) = 1\{x \in [0,1]\}$. Then

$$F(b) - F(a) = \int_a^b f(x)dx.$$

Thus, the probability of an event is the integral of $f(x)$ over the event:

$$Pr[X \in A] = \int_A f(x)dx.$$
Uniformly at Random in $[0, 1]$. 

Think of $f(x)$ as describing how one unit of probability is spread over $[0, 1]$: uniformly!

Then $Pr[X \in A]$ is the probability mass over $A$. 

Observe:

- This makes the probability automatically additive.
- We need $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x) \, dx = 1$. 

$$Pr[X \in A] = \int_{A} f(x) \, dx$$
Uniformly at Random in $[0, 1]$.

**Discrete Approximation:** Fix $N \gg 1$ and let $\varepsilon = \frac{1}{N}$.

Define $Y = n\varepsilon$ if $(n - 1)\varepsilon < X \leq n\varepsilon$ for $n = 1, \ldots, N$.

Then $|X - Y| \leq \varepsilon$ and $Y$ is discrete: $Y \in \{\varepsilon, 2\varepsilon, \ldots, N\varepsilon\}$.

Also, $Pr[Y = n\varepsilon] = \frac{1}{N}$ for $n = 1, \ldots, N$.

Thus, $X$ is ‘almost discrete.’

Calculus view: $Pr[Y = n\varepsilon]$ is area of rectangle in Riemann sum.
Nonuniformly at Random in $[0,1]$. 

This figure shows a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x)dx = 1$. It defines another way of choosing $X$ at random in $[0,1]$. Note that $X$ is more likely to be closer to 1 than to 0. One has $Pr[X \leq x] = \int_{-\infty}^{x} f(u)du = x^2$ for $x \in [0,1]$. Also, $Pr[X \in (x, x + \varepsilon)] = \int_{x}^{x+\varepsilon} f(u)du \approx f(x)\varepsilon$. 

\[ f(x) = 2x1\{0 \leq x \leq 1\} \] 

\[ \text{Pr}[X \in A] = \int_{A} f(x)dx \]
Another Nonuniform Choice at Random in $[0,1]$. 

This figure shows yet a different choice of $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x) dx = 1$. 

It defines another way of choosing $X$ at random in $[0,1]$. 

Note that $X$ is more likely to be closer to $1/2$ than to 0 or 1. 

For instance, $Pr[X \in [0, 1/3]] = \int_0^{1/3} 4x dx = 2 \left[ x^2 \right]_0^{1/3} = \frac{2}{9}$. 

Thus, $Pr[X \in [0, 1/3]] = Pr[X \in [2/3, 1]] = \frac{2}{9}$ and $Pr[X \in [1/3, 2/3]] = \frac{5}{9}$. 
General Random Choice in $\mathcal{R}$

Let $F(x)$ be a nondecreasing function with $F(-\infty) = 0$ and $F(+\infty) = 1$. Define $X$ by $\Pr[X \in (a, b)] = F(b) - F(a)$ for $a < b$. Also, for $a_1 < b_1 < a_2 < b_2 < \cdots < b_n$,

$$Pr[X \in (a_1, b_1] \cup (a_2, b_2] \cup (a_n, b_n)] = Pr[X \in (a_1, b_1)] + \cdots + Pr[X \in (a_n, b_n)] = F(b_1) - F(a_1) + \cdots + F(b_n) - F(a_n).$$

Let $f(x) = \frac{d}{dx} F(x)$. Then,

$$Pr[X \in (x, x + \varepsilon)] = F(x + \varepsilon) - F(x) \approx f(x) \varepsilon.$$

$F(x)$ is cumulative distribution function (cdf) of $X$

$f(x)$ is the probability density function (pdf) of $X$.

When $F$ and $f$ correspond RV $X$: $F_X(x)$ and $f_X(x)$. 
Pr[X ∈ (x, x + ε)]

An illustration of $Pr[X ∈ (x, x + ε)] ≈ f_X(x)ε$:

Thus, the pdf is the ‘local probability by unit length.’
It is the ‘probability density.’
Fix $\varepsilon \ll 1$ and let $Y = n\varepsilon$ if $X \in (n\varepsilon, (n+1)\varepsilon]$. 

Thus, $Pr[Y = n\varepsilon] = F_X((n+1)\varepsilon) - F_X(n\varepsilon)$.

Note that $|X - Y| \leq \varepsilon$ and $Y$ is a discrete random variable.

Also, if $f_X(x) = \frac{d}{dx} F_X(x)$, then $F_X(x + \varepsilon) - F_X(x) \approx f_X(x)\varepsilon$.

Hence, $Pr[Y = n\varepsilon] \approx f_X(n\varepsilon)\varepsilon$.

Thus, we can think of $X$ of being almost discrete with $Pr[X = n\varepsilon] \approx f_X(n\varepsilon)\varepsilon$. 
Example: hitting random location on gas tank. Random location on circle.

What is probability of being within $y$ of the center, for non-negative $y \leq 1$?

(A) 1.
(B) 0.
(C) $\int_0^y (2\pi y)\,dy$
(D) $y^2$.

(D) Next slide.
Example: CDF

Example: hitting random location on gas tank.
Random location on circle.

Random Variable: $Y$ distance from center.
Probability within $y$ of center:

$$Pr[Y \leq y] = \frac{\text{area of small circle}}{\text{area of dartboard}} = \frac{\pi y^2}{\pi} = y^2.$$ 

Hence,

$$F_Y(y) = Pr[Y \leq y] = \begin{cases} 
0 & \text{for } y < 0 \\
y^2 & \text{for } 0 \leq y \leq 1 \\
1 & \text{for } y > 1 
\end{cases}$$
Calculation of event with dartboard.

Probability between .5 and .6 of center?
Recall CDF.

\[ F_Y(y) = Pr[Y \leq y] = \begin{cases} 
0 & \text{for } y < 0 \\
y^2 & \text{for } 0 \leq y \leq 1 \\
1 & \text{for } y > 1 
\end{cases} \]

\[ Pr[0.5 < Y \leq 0.6] = Pr[Y \leq 0.6] - Pr[Y \leq 0.5] \]
\[ = F_Y(0.6) - F_Y(0.5) \]
\[ = .36 - .25 \]
\[ = .11 \]
Example: “Dart” board.
Recall that

\[ F_Y(y) = \Pr[Y \leq y] = \begin{cases} 
0 & \text{for } y < 0 \\
y^2 & \text{for } 0 \leq y \leq 1 \\
1 & \text{for } y > 1 
\end{cases} \]

\[ f_Y(y) = F'_Y(y) = \begin{cases} 
0 & \text{for } y < 0 \\
2y & \text{for } 0 \leq y \leq 1 \\
0 & \text{for } y > 1 
\end{cases} \]

The cumulative distribution function (cdf) and probability distribution function (pdf) give full information.

Use whichever is convenient.
Target

Random Variable
Event \{Y \leq y\}
Outcome

\( Y(\omega) \)

\( f_Y(y) \)

\( F_Y(y) \)

1 2

1 1

y y

\( y^2 \)
$U[a, b]$
Exponential derivation: Poll.

\[ Pr[X = i] = (1 - p)^{i-1}p. \]

Let \( p = \lambda / n \) and \( Y = X / n \).

What is true?

(A) \( X \sim G(p) \)
(B) \( Pr[X > i] = (1 - p)^i \).
(C) \( Pr[Y > i/n] = (1 - \lambda / n)^i \).
(D) \( Pr[Y > y] = (1 - \lambda / n)^{ny} \).
(E) \( \lim_{n \to \infty} (1 - \lambda / n)^{ny} = e^{-\lambda y} \).

(A) True by definition.
(B) \( Pr[X > i] = (1 - p)^i \) at least \( i \) coin flips fail.
(C) True, definition of \( Y \)
(D) True, \( y = i/n \) means \( i = ny \).
(E) \( (1 - \lambda / n)^{ny} = ((1 - \lambda / n)^{n/\lambda})^{\lambda y} \)
and \( \lim_{n \to \infty} (1 - \frac{\lambda}{n})^{n/\lambda} = e \).

The limit as \( n \to \infty \) of \( Y \) has \( Pr[Y > y] = e^{-\lambda y} \).

\( Pr[Y > y] \) is defined as “Survival function.”
**Expo(λ)**

“Limit of geometric.”

From last slide: $S(t) = Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

Note: $f_X(x) = F'(t) = (1 - S(t))' = -S'(t)$.

The exponential distribution with parameter $\lambda > 0$ is defined by

$$f_X(x) = \lambda e^{-\lambda x} 1\{x \geq 0\}$$

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$
Continuous Random Variables

Continuous random variable \( X \), specified by

1. \( F_X(x) = Pr[X \leq x] \) for all \( x \).

   **Cumulative Distribution Function (cdf).**
   \( Pr[a < X \leq b] = F_X(b) - F_X(a) \)
   1.1 \( 0 \leq F_X(x) \leq 1 \) for all \( x \in \mathbb{R} \).
   1.2 \( F_X(x) \leq F_X(y) \) if \( x \leq y \).

2. Or \( f_X(x) \), where \( F_X(x) = \int_{-\infty}^{x} f_X(u)du \) or \( f_X(x) = \frac{d(F_X(x))}{dx} \).

   **Probability Density Function (pdf).**
   \( Pr[a < X \leq b] = \int_{a}^{b} f_X(x)dx = F_X(b) - F_X(a) \)
   2.1 \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \).
   2.2 \( \int_{-\infty}^{\infty} f_X(x)dx = 1 \).

Recall that \( Pr[X \in (x, x + \delta)] \approx f_X(x)\delta \).
\( X \) “takes” value \( n\delta \), for \( n \in \mathbb{Z} \), with \( Pr[X = n\delta] = f_X(n\delta)\delta \).
The pdf $f_X(x)$ is a nonnegative function that integrates to 1.

The cdf $F_X(x)$ is the integral of $f_X$.

\[
Pr[x < X < x + \delta] \approx f_X(x) \delta
\]

\[
Pr[X \leq x] = F_X(x) = \int_{-\infty}^{x} f_X(u) du
\]
Multiple Continuous Random Variables

One defines a pair \((X, Y)\) of continuous RVs by specifying \(f_{X,Y}(x, y)\) for \(x, y \in \mathbb{R}\) where

\[
f_{X,Y}(x, y)\,dx\,dy = \Pr[X \in (x, x + dx), Y \in (y + dy)].
\]

The function \(f_{X,Y}(x, y)\) is called the joint pdf of \(X\) and \(Y\).

**Example:** Choose a point \((X, Y)\) uniformly in the set \(A \subset \mathbb{R}^2\). Then

\[
f_{X,Y}(x, y) = \frac{1}{|A|} 1\{(x, y) \in A\}
\]

where \(|A|\) is the area of \(A\).

**Interpretation.** Think of \((X, Y)\) as being discrete on a grid with mesh size \(\varepsilon\) and \(\Pr[X = m\varepsilon, Y = n\varepsilon] = f_{X,Y}(m\varepsilon, n\varepsilon)\varepsilon^2\).

Recall Marginal Distribution:

\[
\Pr[X = x] = \sum_y \Pr[X = x, Y = y].
\]

Similarly:

\[
f_X(x) = \int f_{X,Y}(x, y)\,dy.
\]

Sum “goes to” integral.
Example of Continuous \((X, Y)\)

Pick a point \((X, Y)\) uniformly in the unit circle.

Thus, \(f_{X,Y}(x, y) = \frac{1}{\pi} 1\{x^2 + y^2 \leq 1\}\).

Consequently,

\[
Pr[X > 0, Y > 0] = \frac{1}{4} \\
Pr[X < 0, Y > 0] = \frac{1}{4} \\
Pr[X^2 + Y^2 \leq r^2] = \frac{\pi r^2}{\pi} = r^2 \\
Pr[X > Y] = \frac{1}{2}.
\]
Definition: Continuous RVs $X$ and $Y$ independent if and only if

$$Pr[X \in A, Y \in B] = Pr[X \in A]Pr[Y \in B], \forall A, B.$$ 

Theorem: Continuous RVs $X$ and $Y$ independent if and only if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$ 

Note: $f_X(x)$ ($f_Y(y)$) is (marginal) distribution of $X$ ($Y$).


$$Pr[X \in A, Y \in B] = Pr[X \in A] \times Pr[Y \in B]$$
$$\approx f_X(x) \, dx \times f_Y(y) \, dy$$
$$= f_X(x)f_Y(y) \, dxdy.$$ 

Thus, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. 

□
Mutual Independence.

**Definition:** Continuous RVs $X_1, \ldots, X_n$ are mutually independent if

$$Pr[X_1 \in A_1, \ldots, X_n \in A_n] = Pr[X_1 \in A_1] \cdots Pr[X_n \in A_n], \forall A_1, \ldots, A_n.$$ 

**Theorem:** Continuous RVs $X_1, \ldots, X_n$ are mutually independent if and only if

$$f_X(x_1, \ldots, x_n) = f_{X_1}(x_1) \cdots f_{X_n}(x_n).$$

**Proof:** As in the discrete case.
Conditional density.

Conditional Density: \( f_{X|Y}(x, y) \).

Conditional Probability: \( Pr[X \in A | Y \in B] = \frac{Pr[X \in A, Y \in B]}{Pr[Y \in B]} \)

\[
Pr[X \in [x, x + dx] | Y \in [y, y + dy]] = \frac{f_{X,Y}(x,y) \, dx \, dy}{f_Y(dy)}
\]

\[
f_{X|Y}(x, y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{+\infty} f_{X,Y}(x,y) \, dx}
\]

Corollary: For independent random variables, \( f_{X|Y}(x, y) = f_X(x) \).
Independent Random Variables?

Uniform on a rectangle? Independent?

\[ f_{X|Y}(x, y) = f_X(x) \text{ for all } y \]

Also: \( \Pr[X \in A, Y \in B] \propto \text{Area of rectangle} \propto \Pr[X \in A] \times \Pr[Y \in B] \). Independent!

Uniform on a circle? Independent?

\[ f_{X|Y}(x, .5) \]

\[ f_{X|Y}(x, 0) \]

Not independent!
Continuous Probability 1

1. **pdf:** $Pr[X \in (x, x + \delta)] = f_X(x)\delta$.

2. **CDF:** $Pr[X \leq x] = F_X(x) = \int_{-\infty}^{x} f_X(y)dy$.

3. **$U[a, b]$:** $f_X(x) = \frac{1}{b-a}1\{a \leq x \leq b\}; F_X(x) = \frac{x-a}{b-a}$ for $a \leq x \leq b$.

4. **Expo($\lambda$):**
   
   $f_X(x) = \lambda \exp\{-\lambda x\}1\{x \geq 0\}; F_X(x) = 1 - \exp\{-\lambda x\}$ for $x \leq 0$.

5. **Target:** $f_X(x) = 2x1\{0 \leq x \leq 1\}; F_X(x) = x^2$ for $0 \leq x \leq 1$.

6. **Joint pdf:** $Pr[X \in (x, x + \delta), Y = (y, y + \delta)) = f_{X,Y}(x, y)\delta^2$.

   6.1 **Conditional Distribution:** $f_{X|Y}(x, y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$.

   6.2 **Independence:** $f_{X|Y}(x, y) = f_X(x)$
Continuous RVs are similar to discrete RVs (break into intervals.)

Think that $X \approx x$ with probability $f_X(x)\varepsilon$

Sums become integrals, ....