CS70: Markov Chains.
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Two-State Markov Chain

Here is a symmetric two-state Markov chain.
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Finite Markov Chain: Definition

- A finite set of states: \( X = \{1, 2, \ldots, K\} \)
- A probability distribution \( \pi_0 \) on \( X \):
  \[ \pi_0(i) \geq 0, \quad \sum_{i} \pi_0(i) = 1 \]
- Transition probabilities: \( P(i, j) \) for \( i, j \in X \):
  \[ P(i, j) \geq 0, \quad \forall i, j \in X; \quad \sum_{j} P(i, j) = 1, \quad \forall i \in X \]

- \( \{X_n, n \geq 0\} \) is defined so that
  \[ \Pr[X_0 = i] = \pi_0(i), \quad i \in X \] (initial distribution)
  \[ \Pr[X_{n+1} = j | X_0, \ldots, X_n = i] = P(i, j), \quad i, j \in X. \]
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Hitting Time - Example 1

Let's flip a coin with \( P[H] = p \) until we get \( H \). How many flips, on average?

Let's define a Markov chain:

- \( X_0 = S \) (start)
- \( X_n = S \) for \( n \geq 1 \), if last flip was \( T \) and no \( H \) yet
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First Passage Time - Example 1. Poll

Let’s flip a coin with $Pr[H] = p$ until we get $H$. How many flips, on average?

Let $\beta(S)$ be the average time until $E$, starting from $S$.

What is correct?

(A) $\beta(S)$ is at least 1.

(B) From $S$, in one step, go to $S$ with prob. $q = 1 - p$.

(C) From $S$, in one step, go to $E$ with prob. $p$.

(D) If you go back to $S$, you are back at $S$.

(D) $\beta(S) = 1 + q \beta(S) + p 0$.

All are correct. (D) is the “Markov property.” Only know where you are.
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Let's flip a coin with $Pr[H] = p$ until we get $H$. How many flips, on average?

Let $\beta(S)$ be the average time until $E$, starting from $S$. Then,

$$\beta(S) = 1 + q \beta(S) + p \beta_0.$$ 

(See next slide.)

Hence,

$$\beta(S) = 1 + (1 - p) \beta(S) \Rightarrow \beta(S) = 1,$$

so that $\beta(S) = 1/p$.

Note: Time until $E$ is $G(p)$. The mean of $G(p)$ is $1/p$!!!
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Justification:
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Let \( \beta(S) \) be the average time until \( E \). Then,

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**Justification:** \( N \) – number of steps until \( E \), starting from \( S \).
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$$N = 1 + (1 - Z) \times N' + Z \times 0.$$
First Passage Time - Example 1

Let’s flip a coin with \( Pr[H] = p \) until we get \( H \). How many flips, on average?

![Diagram of a coin flip with states S and E, and transition probabilities q = 1 - p and p.]

Let \( \beta(S) \) be the average time until \( E \).

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N = 1 + (1 - Z) \times N' + Z \times 0.
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\( Z \) and \( N' \) are “independent.”
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$Z$ and $N'$ are “independent.” $E[N'] = E[N] = \beta(S)$. 
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N = 1 + (1 - Z) \times N' + Z \times 0.
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\( Z \) and \( N' \) are “independent.” \( E[N'] = E[N] = \beta(S) \).
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\]
Let's flip a coin with $P[H] = p$ until we get two consecutive $H$s.

How many flips, on average?

Let's define a Markov chain:

- $X_0 = S$ (start)
- $X_n = E$, if we already got two consecutive $H$s (end)
- $X_n = T$, if last flip was $T$ and we are not done
- $X_n = H$, if last flip was $H$ and we are not done
Hitting Time - Example 2

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$H$
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$H T$
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$H \ T \ H$
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\[ H \ T \ H \ T \]
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$$H \ T \ H \ T \ T \ T \ H$$
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$HTHTTTHTH$
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\[
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Let’s define a Markov chain:

- $X_0 = S$
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\[
\begin{align*}
H & T \\
T & H \\
T & T \\
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\end{align*}
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Let’s define a Markov chain:

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Hitting Time - Example 2

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Which one is correct?
(A) $\beta(S) = 1 + p\beta(H) + q\beta(T)$
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Let $\beta(i)$ be the average time from state $i$ until the MC hits state $E$. 

![Diagram of a Markov Chain with states S, H, T, and E, and transitions labeled with p and q](image)

$S$: Start

$H$: Last flip = $H$

$T$: Last flip = $T$

$E$: Done
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(E.g., $\beta(S) = 6$ if $p = 1/2$.)
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Solving, we find $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$. 
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Hitting Time - Example 2

Let us justify the first step equation for $\beta(T)$. The others are similar.

$N(T)$ – number of steps, starting from $T$ until the MC hits $E$.

$N(H)$ – be defined similarly.

$N'(T)$ – number of steps after the second visit to $T$ until MC hits $E$.

$N(T) = 1 + Z \times N(H) + (1 - Z) \times N'(T)$

where $Z = 1\{\text{first flip in } T \text{ is } H\}$.

Since $Z$ and $N(H)$ are independent, and $Z$ and $N'(T)$ are independent, taking expectations, we get

$E[N(T)] = 1 + pE[N(H)] + qE[N'(T)]$,

i.e., $\beta(T) = 1 + p\beta(H) + q\beta(T)$.

Diagram:

- $S$: Start
- $H$: Last flip = $H$
- $T$: Last flip = $T$
- $E$: Done
- $X_0$
- $q \:= \: 1 - p$
- $p$
- $q$
- $1$
Hitting Time - Example 2

Let us justify the first step equation for $\beta(T)$.

Let $q := 1 - p$.

$S$: Start
$H$: Last flip = $H$
$T$: Last flip = $T$
$E$: Done
Hitting Time - Example 2

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The diagram shows the transitions between states:
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Hitting Time - Example 3

You roll a balanced six-sided die until the sum of the last two rolls is 8. How many times do you have to roll the die, on average?

$$\beta(S) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j); \beta(1) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j); \beta(i) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j), \forall i = 2, \ldots, 6.$$

Symmetry: $$\beta(2) = \cdots = \beta(6) =: \gamma.$$ Also, $$\beta(1) = \beta(S).$$ Thus, $$\beta(S) = 1 + \left(\frac{5}{6}\right) \gamma + \beta(S)/6; \gamma = 1 + \left(\frac{4}{6}\right) \gamma + \left(\frac{1}{6}\right) \beta(S).$$

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$S = \text{Start}; \ E = \text{Done}$
$i = \text{Last roll is } i, \text{ not done}$

$P(S, j) = 1/6, j = 1, \ldots, 6$
$P(1, j) = 1/6, j = 1, \ldots, 6$

$P(i, j) = 1/6, i = 2, \ldots, 6; 8 - i \neq j \in \{1, \ldots, 6\}$
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\[\Rightarrow \cdots \beta(S) = 8.4.\]
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. 

Let $\alpha(n)$ be the probability of reaching 100 before 0, starting from $n$, for $n = 0, 1, \ldots, 100$.

Which equations are correct?

(A) $\alpha(0) = 0$

(B) $\alpha(0) = 1$.

(C) $\alpha(100) = 1$.

(D) $\alpha(n) = 1 + p \alpha(n+1) + q \alpha(n-1)$, $0 < n < 100$.

(E) $\alpha(n) = p \alpha(n+1) + q \alpha(n-1)$, $0 < n < 100$.

(B) is incorrect, 0 is bad.

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Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. 

What is the probability that you reach $100$ before $0$?

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$q = 1 - p$
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$.
Start with $10$.
What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$. 
Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$.

Which equations are correct?

(A) $\alpha(0) = 0$

(B) $\alpha(0) = 1$.

(C) $\alpha(100) =$

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$.

Which equations are correct?

(A) $\alpha(0) = 0$

(B) $\alpha(0) = 1$.

(C) $\alpha(100) =$
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$.

Which equations are correct?

(A) $\alpha(0) = 0$
(B) $\alpha(0) = 1$.
(C) $\alpha(100) = 1$.
(D) $\alpha(n) = 1 + p\alpha(n+1) + q\alpha(n-1), 0 < n < 100$.
(E) $\alpha(n) =$
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10.
What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$.
Which equations are correct?

(A) $\alpha(0) = 0$
(B) $\alpha(0) = 1$.
(C) $\alpha(100) = 1$.
(D) $\alpha(n) = 1 + p\alpha(n + 1) + q\alpha(n - 1), 0 < n < 100$.
(E) $\alpha(n) = p\alpha(n + 1) + q\alpha(n - 1), 0 < n < 100$. 
Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $\$10$. Each step, flip yields ‘heads’, earn $\$1$. Otherwise, lose $\$1$. What is the probability that you reach $\$100$ before $\$0$?

Let $\alpha(n)$ be the probability of reaching 100 before 0, starting from $n$, for $n = 0, 1, \ldots, 100$.

Which equations are correct?

(A) $\alpha(0) = 0$
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(E) $\alpha(n) = p\alpha(n+1) + q\alpha(n-1), 0 < n < 100$.

(B) is incorrect, 0 is bad.
(D) is incorrect. Confuses expected hitting time with A before B.
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. 
Game of “heads or tails” using coin with ‘heads’ probability \( p < 0.5 \).
Start with $10.
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability \( p < 0.5 \).
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Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. 

Let $\alpha(n)$ be the probability of reaching 100 before 0, starting from $n$, for $n = 0, 1, \ldots, 100$.

$\alpha(0) = 0$; $\alpha(100) = 1$.

$\alpha(n) = p \alpha(n+1) + q \alpha(n-1)$, $0 < n < 100$.

$\Rightarrow \alpha(n) = 1 - \rho n^{100} \rho$ with $\rho = \frac{q}{p} - 1$. (See LN 22)
Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$. $\alpha(0) = 0$; $\alpha(100) = 1$. $\alpha(n) = p \alpha(n+1) + q \alpha(n-1)$, $0 < n < 100$. 

$\Rightarrow \alpha(n) = 1 - \rho n^{1 - \rho} 100$ with $\rho = qp - 1$. (See LN 22)
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10$.
What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$.

$\alpha(0) = 0$; $\alpha(100) = 1$.

$\alpha(n) = p \alpha(n + 1) + q \alpha(n - 1)$, $0 < n < 100$.

$\Rightarrow \alpha(n) = 1 - \rho n 1 - \rho 100$ with $\rho = qp - 1$.

(See LN 22)
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$. 

\begin{align*}
\alpha(0) &= 0; \\
\alpha(100) &= 1. \\
\alpha(n) &= p\alpha(n+1) + q\alpha(n-1), \\
&\quad 0 < n < 100. \\
\Rightarrow \alpha(n) &= \frac{1}{1 - \rho} - \rho n  \\
&\quad with \rho = q/p - 1.
\end{align*} 

(See LN 22)
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$.

$$\alpha(0) =$$
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability \( p < 0.5 \).
Start with $10.
What is the probability that you reach $100 before $0?

Let \( \alpha(n) \) be the probability of reaching 100 before 0, starting from \( n \), for \( n = 0, 1, \ldots, 100 \).

\[
\alpha(0) = 0;
\]
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. What is the probability that you reach $100$ before $0$?

Let $\alpha(n)$ be the probability of reaching $100$ before $0$, starting from $n$, for $n = 0, 1, \ldots, 100$.

\[ \alpha(0) = 0; \alpha(100) = \]

\[ q = 1 - p \]
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$. Start with $\$10$.
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$$\alpha(0) = 0; \alpha(100) = 1.$$
Here before There - A before B

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\[
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\]
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Here before There - A before B

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\]

\[
\Rightarrow \alpha(n) = \frac{1 - \rho^n}{1 - \rho^{100}} \text{ with } \rho = qp^{-1}. \text{ (See LN 22)}
\]
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p = .48$.
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p = .48$.
Start with $10$.

Moral of example: Money in Vegas stays in Vegas.
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p = 0.48$.
Start with $10$.

What is the probability that you reach $100$ before $0$?
Less than 1 in a 1000.

Moral of example: Money in Vegas stays in Vegas.
Game of “heads or tails” using coin with ‘heads’ probability \( p = .48 \).
Start with $10.
What is the probability that you reach $100 before $0?
Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p = .48$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. What is the probability that you reach $100$ before $0$?

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Game of “heads or tails” using coin with ‘heads’ probability $p = 0.48$. Start with $10$. Each step, flip yields ‘heads’, earn $1$. Otherwise, lose $1$. What is the probability that you reach $100$ before $0$?

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Less than 1 in a 1000. Moral of example:
Here before There - A before B

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Game of “heads or tails” using coin with ‘heads’ probability $p = .48$. Start with $10.
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Less than 1 in a 1000. Moral of example: Money in Vegas stays in Vegas.
First Step Equations

Let $X_n$ be a MC on $X$ and $A, B \subseteq X$ with $A \cap B \neq \emptyset$.

Define $T_A = \min \{ n \geq 0 | X_n \in A \}$ and $T_B = \min \{ n \geq 0 | X_n \in B \}$.

For $\beta(i) = \mathbb{E}[T_A | X_0 = i]$, first step equations are:

$\beta(i) = 0, i \in A$

$\beta(i) = 1 + \sum_j P(i, j) \beta(j), i / \in A$

For $\alpha(i) = \text{Pr}[T_A < T_B | X_0 = i]$, first step equations are:

$\alpha(i) = 1, i \in A$

$\alpha(i) = 0, i \in B$

$\alpha(i) = \sum_j P(i, j) \alpha(j), i \not\in A \cup B$.
First Step Equations

Let $X_n$ be a MC on $\mathcal{X}$ and $A, B \subset \mathcal{X}$ with $A \cap B = \emptyset$. 

Let $X_n$ be a MC on $\mathcal{X}$ and $A, B \subset \mathcal{X}$ with $A \cap B = \emptyset$. 

![Diagram showing a Markov chain with states $A$ and $B$ and transition $P(i, j)$ from $i$ to $j$.]
First Step Equations

Let $X_n$ be a MC on $\mathcal{X}$ and $A, B \subset \mathcal{X}$ with $A \cap B = \emptyset$. Define

$$T_A = \min \{ n \geq 0 \mid X_n \in A \}$$
First Step Equations

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For $\alpha(i) = Pr[T_A < T_B \mid X_0 = i], \; i \in \mathcal{X}$, first step equations are:
First Step Equations

Let $X_n$ be a MC on $\mathcal{X}$ and $A, B \subset \mathcal{X}$ with $A \cap B = \emptyset$. Define

$$T_A = \min\{n \geq 0 \mid X_n \in A\} \text{ and } T_B = \min\{n \geq 0 \mid X_n \in B\}.$$

For $\beta(i) = E[T_A \mid X_0 = i]$, first step equations are:

$$\beta(i) = \begin{cases} 0, & i \in A \\ 1 + \sum_j P(i,j)\beta(j), & i \notin A \end{cases}$$

For $\alpha(i) = \Pr[T_A < T_B \mid X_0 = i], i \in \mathcal{X}$, first step equations are:

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$$\beta(i) = 0, i \in A$$

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For $\alpha(i) = Pr[T_A < T_B \mid X_0 = i], i \in \mathcal{X}$, first step equations are:

$$\alpha(i) = 1, i \in A$$

$$\alpha(i) = 0, i \in B$$

$$\alpha(i) = \sum_j P(i,j)\alpha(j), i \notin A \cup B.$$
Let $X_n$ be a Markov chain on $X$ with $P$. Let also $g: X \rightarrow \mathbb{R}$ be some function. Define $\gamma(i) = E\left[\sum_{n=0}^{T_A} g(X_n) \mid X_0 = i\right]$, $i \in X$. Then $\gamma(i) = g(i)$, if $i \in A$ $g(i) + \sum_{j} P(i,j) \gamma(j)$, otherwise.
Let $X_n$ be a Markov chain on $\mathcal{X}$ with $P$. 

Let $A \subset \mathcal{X}$

Let also $g: \mathcal{X} \to \mathbb{R}$ be some function.

Define $\gamma(i) = \mathbb{E}\left[\sum_{n=0}^{\infty} g(X_n) \mid X_0 = i\right]$, $i \in \mathcal{X}$.

Then $\gamma(i) = g(i)$, if $i \in A$.

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Let $X_n$ be a Markov chain on $\mathcal{X}$ with $P$. Let $A \subset \mathcal{X}$
Accumulating Rewards

Let $X_n$ be a Markov chain on $\mathcal{X}$ with $P$. Let $A \subset \mathcal{X}$
Let also $g: \mathcal{X} \to \mathbb{R}$ be some function.
Let $X_n$ be a Markov chain on $\mathcal{X}$ with $P$. Let $A \subset \mathcal{X}$.

Let also $g : \mathcal{X} \rightarrow \mathbb{R}$ be some function.

Define

$$\gamma(i) = E\left[ \sum_{n=0}^{T_A} g(X_n) | X_0 = i \right], i \in \mathcal{X}.$$
Let $X_n$ be a Markov chain on $\mathcal{X}$ with $P$. Let $A \subset \mathcal{X}$

Let also $g : \mathcal{X} \to \mathbb{R}$ be some function.

Define

$$\gamma(i) = E\left[ \sum_{n=0}^{T_A} g(X_n) | X_0 = i \right], i \in \mathcal{X}.$$ 

Then

$$\gamma(i) = \begin{cases} g(i), & \text{if } i \in A \end{cases}$$
Let $X_n$ be a Markov chain on $\mathcal{X}$ with $P$. Let $A \subset \mathcal{X}$

Let also $g : \mathcal{X} \rightarrow \mathbb{R}$ be some function.

Define

$$\gamma(i) = E[\sum_{n=0}^{T_A} g(X_n) | X_0 = i], i \in \mathcal{X}.$$  

Then

$$\gamma(i) = \begin{cases} 
  g(i), & \text{if } i \in A \\
  g(i) + \sum_j P(i,j) \gamma(j), & \text{otherwise.}
\end{cases}$$
Let $X_n$ be a Markov chain on $\mathcal{X}$ with $P$. Let $A \subset \mathcal{X}$.
Let also $g : \mathcal{X} \rightarrow \mathbb{R}$ be some function.
Define
\[
\gamma(i) = E\left[ \sum_{n=0}^{T_A} g(X_n) | X_0 = i \right], i \in \mathcal{X}.
\]
Then
\[
\gamma(i) = \begin{cases} 
  g(i), & \text{if } i \in A \\
  g(i) + \sum_j P(i,j) \gamma(j), & \text{otherwise.}
\end{cases}
\]
Example
Example

Flip a fair coin until you get two consecutive $H$s.
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Flip a fair coin until you get two consecutive $H$s.
What is the expected number of $T$s that you see?
Example

Flip a fair coin until you get two consecutive $H$s. What is the expected number of $T$s that you see?

$$g(S) = g(H) = g(HH) = 0 \quad g(T) = 1$$
Example

Flip a fair coin until you get two consecutive $H$s.
What is the expected number of $T$s that you see?

FSE:

$$\gamma(S) = 0 + 0.5\gamma(H) + 0.5\gamma(T)$$
Example

Flip a fair coin until you get two consecutive Hs.
What is the expected number of Ts that you see?

FSE:

\[ \gamma(S) = 0 + 0.5\gamma(H) + 0.5\gamma(T) \]
\[ \gamma(H) = 0 + 0.5\gamma(HH) + 0.5\gamma(T) \]
Example

Flip a fair coin until you get two consecutive $H$s.
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$$g(S) = g(H) = g(HH) = 0$$
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FSE:

$$\gamma(S) = 0 + 0.5\gamma(H) + 0.5\gamma(T)$$
$$\gamma(H) = 0 + 0.5\gamma(HH) + 0.5\gamma(T)$$
$$\gamma(T) = 1 + 0.5\gamma(H) + 0.5\gamma(T)$$
Example

Flip a fair coin until you get two consecutive $H$s.
What is the expected number of $T$s that you see?

FSE:

\[
\begin{align*}
\gamma(S) &= 0 + 0.5\gamma(H) + 0.5\gamma(T) \\
\gamma(H) &= 0 + 0.5\gamma(HH) + 0.5\gamma(T) \\
\gamma(T) &= 1 + 0.5\gamma(H) + 0.5\gamma(T) \\
\gamma(HH) &= 0.
\end{align*}
\]
Example

Flip a fair coin until you get two consecutive Hs. What is the expected number of Ts that you see?

\[
\begin{align*}
\gamma(S) &= 0 + 0.5\gamma(H) + 0.5\gamma(T) \\
\gamma(H) &= 0 + 0.5\gamma(HH) + 0.5\gamma(T) \\
\gamma(T) &= 1 + 0.5\gamma(H) + 0.5\gamma(T) \\
\gamma(HH) &= 0.
\end{align*}
\]

Solving, we find \( \gamma(S) = 2.5 \).
Recap

Markov Chain:
- Finite set $X$;
- $\pi_0$;
- $P = \{P(i, j), i, j \in X\}$;
- $\Pr[X_0 = i] = \pi_0(i), i \in X$;
- $\Pr[X_n+1 = j | X_0, ..., X_n = i] = P(i, j), i, j \in X, n \geq 0$.

Note:
- $\Pr[X_0 = i_0, X_1 = i_1, ..., X_n = i_n] = \pi_0(i_0)P(i_0, i_1)\cdots P(i_{n-1}, i_n)$.

First Passage Time:
- $A \cap B = \emptyset$;
- $\beta(i) = \mathbb{E}[T_A | X_0 = i]$;
- $\alpha(i) = \Pr[T_A < T_B | X_0 = i]$.
- $\beta(i) = 1 + \sum_j P(i, j)\beta(j)$;
- $\alpha(i) = \sum_j P(i, j)\alpha(j)$.
- $\alpha(A) = 1, \alpha(B) = 0$. 
Recap

- Markov Chain:

- Finite set $X$; $\pi_0$; $P = \{P(i, j), i, j \in X\}$

- $\Pr[X_0 = i] = \pi_0(i), i \in X$

- $\Pr[X_{n+1} = j | X_0, ..., X_n = i] = P(i, j), i, j \in X, n \geq 0$

- Note: $\Pr[X_0 = i_0, X_1 = i_1, ..., X_n = i_n] = \pi_0(i_0)P(i_0, i_1)\cdots P(i_{n-1}, i_n)$

- First Passage Time:

- $A \cap B = \emptyset$; $\beta(i) = E[T_A | X_0 = i]$; $\alpha(i) = \Pr[T_A < T_B | X_0 = i]$

- $\beta(i) = 1 + \sum_j P(i, j)\beta(j)$

- $\alpha(i) = \sum_j P(i, j)\alpha(j)$

- $\alpha(A) = 1, \alpha(B) = 0$. 
Recap

Markov Chain:

- Finite set $\mathcal{X}$; $\pi_0$; $P = \{ P(i,j), i, j \in \mathcal{X} \}$;
- $Pr[X_0 = i] = \pi_0(i), i \in \mathcal{X}$
- $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i, j \in \mathcal{X}, n \geq 0$.

Note:
$Pr[X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] =$
Recap

- Markov Chain:
  - Finite set $\mathcal{X}$; $\pi_0$; $P = \{P(i,j), i,j \in \mathcal{X}\}$;
  - $Pr[X_0 = i] = \pi_0(i), i \in \mathcal{X}$
  - $Pr[X_{n+1} = j | X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathcal{X}, n \geq 0$.
  - Note:
    $Pr[X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = \pi_0(i_0)P(i_0, i_1) \cdots P(i_{n-1}, i_n)$.

- First Passage Time:
Recap

- **Markov Chain:**
  - Finite set $\mathcal{X}$; $\pi_0$; $P = \{P(i,j), i,j \in \mathcal{X}\}$;
  - $Pr[X_0 = i] = \pi_0(i), i \in \mathcal{X}$
  - $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathcal{X}, n \geq 0$.
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- **First Passage Time:**
  - $A \cap B = \emptyset$;
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  - $A \cap B = \emptyset$; $\beta(i) = E[T_A \mid X_0 = i]$;
Recap

► Markov Chain:

► Finite set $\mathcal{X}; \pi_0; P = \{P(i,j), i,j \in \mathcal{X}\}$;

► $Pr[X_0 = i] = \pi_0(i), i \in \mathcal{X}$

► $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathcal{X}, n \geq 0$.

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$Pr[X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] = \pi_0(i_0)P(i_0, i_1) \cdots P(i_{n-1}, i_n)$.

► First Passage Time:

► $A \cap B = \emptyset; \beta(i) = E[T_A|X_0 = i]; \alpha(i) = P[T_A < T_B|X_0 = i]$;

► $\beta(i) = 1 + \sum_j P(i,j)\beta(j)$;

► $\alpha(i) = \sum_j P(i,j)\alpha(j). \alpha(A) = 1, \alpha(B) = 0.$
Recall $\pi_n$ is a distribution over states for $X_n$.

Stationary distribution: $\pi = \pi P$.

Probability entering $i$: $\sum_j P(j, i) \pi(j)$.

Probability leaving $i$: $\pi_i$.

Are equal! Distribution same after one step.

Questions?

Does one exist? Is it unique?

If it exists and is unique.

Then what?

Sometimes the distribution as $n \to \infty$
Recall $\pi_n$ is a distribution over states for $X_n$.

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Stationary: Example

Example 1:

Balance Equations.

\[ P = \pi \]

\[ \pi = \pi \]

\[ \pi_1(1), \pi_2(2) \]

\[ 1 - a \]

\[ 1 - b \]

\[ = \pi_1(1), \pi_2(2) \]

\[ \pi_1(1)(1 - a) + \pi_2(2)b = \pi_2(2) \]

\[ \pi_1(1)a + \pi_2(2)(1 - b) = \pi_2(2) \]

These equations are redundant!

We have to add an equation:

\[ \pi_1 + \pi_2 = 1. \]

Then we find

\[ \pi = [b, a + b, a, a + b]. \]
Stationary: Example

Example 1:

\[ P = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix} \]
Stationary: Example

Example 1:

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\[ \pi P = \pi \iff [\pi(1), \pi(2)] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = [\pi(1), \pi(2)] \]
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Example 1:

![Graph Diagram]

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Stationary: Example

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\[
1 - a \quad 1 \quad 2 \quad 1 - b
\]

\[
\begin{bmatrix}
1 - a & a \\
\frac{a}{a+b} & \frac{b}{a+b}
\end{bmatrix}
\]

Balance Equations.

\[
\pi P = \pi \iff [\pi(1), \pi(2)] \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix} = [\pi(1), \pi(2)] \\
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Stationary distributions: Example 2

\[ P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Every distribution is invariant for this Markov chain. Since \( X_n = X_0 \) for all \( n \).

Hence, \( \Pr[X_n = i] = \Pr[X_0 = i], \forall (i, n) \).

Discussion.
We have seen a chain with one stationary, and a chain with many. When is there just one?
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\[
\begin{pmatrix}
1 & 1 \\
2 & 1
\end{pmatrix}
\]

\[
P = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
\pi P = \pi \Leftrightarrow [\pi(1), \pi(2)] \begin{bmatrix}
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\[
\begin{align*}
P &= \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\end{align*}
\]

\[
\pi P = \pi \iff \begin{bmatrix} \pi(1), \pi(2) \end{bmatrix} \begin{bmatrix}
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When is here just one?
Irreducibility.

**Definition** A Markov chain is irreducible if it can go from every state \(i\) to every state \(j\).
Irreducibility.

**Definition** A Markov chain is irreducible if it can go from every state $i$ to every state $j$ (possibly in multiple steps).

\[\begin{array}{ccc}
   1 & 0 & 0.8 \\
   0.8 & 1 & 0 \\
   0.7 & 0.3 & 1
\end{array}\]

[A] is not irreducible. It cannot go from (2) to (1).

[B] is not irreducible. It cannot go from (2) to (1).

[C] is irreducible. It can go from every $i$ to every $j$. If you consider the graph with arrows when $P(i,j) > 0$, irreducible means that there is a single connected component.
Irreducibility.

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**Examples:**
**Definition** A Markov chain is *irreducible* if it can go from every state $i$ to every state $j$ (possibly in multiple steps).

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If you consider the graph with arrows when $P(i,j) > 0$, irreducible means that there is a single connected component.
Existence and uniqueness of Invariant Distribution

A finite irreducible Markov chain has one and only one invariant distribution. That is, there is a unique positive vector \( \pi = [\pi(1), \ldots, \pi(K)] \) such that \( \pi P = \pi \) and \( \sum_k \pi(k) = 1 \).

Ok.

Now.

Only one stationary distribution if irreducible (or connected.)
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That is, there is a unique positive vector $\pi = [\pi(1), \ldots, \pi(K)]$ such that $\pi P = \pi$ and $\sum_k \pi(k) = 1$.

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Ok. Now.
Only one stationary distribution if irreducible (or connected.)
Theorem

Let $X_n$ be an irreducible Markov chain with invariant distribution $\pi$. Then, for all $i$,

$$\frac{1}{n} \sum_{m=0}^{n-1} 1\{X_m = i\} \rightarrow \pi(i),$$
as $n \rightarrow \infty$.

The left-hand side is the fraction of time that $X_m = i$ during steps $0, 1, \ldots, n-1$.

Thus, this fraction of time approaches $\pi(i)$.

Proof: Lecture note 21 gives a plausibility argument.
**Theorem** Let $X_n$ be an irreducible Markov chain with invariant distribution $\pi$. 
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Example 1: 

$P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ 

$\pi P = \pi$ 

$\pi = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ 

The fraction of time in state 1 converges to $\frac{1}{2}$, which is $\pi(1)$. 
Long Term Fraction of Time in States

**Theorem** Let $X_n$ be an irreducible Markov chain with invariant distribution $\pi$. Then, for all $i$, $\frac{1}{n} \sum_{m=0}^{n-1} 1\{X_m = i\} \to \pi(i)$, as $n \to \infty$.

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Long Term Fraction of Time in States

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**Example 2:**

\[ \pi = [0.4, 0.6] \]
Convergence to Invariant Distribution

Question:
Assume that the MC is irreducible. Does $\pi_n$ approach the unique invariant distribution $\pi$?

Answer:
Not necessarily. Here is an example:

$$
\begin{align*}
P &= \begin{pmatrix}
0 & 1 \\
1 & 0 \\
\end{pmatrix} \\
\pi_0 &= \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \\
\pi_1 &= \begin{pmatrix} 0/2 \\ 1/2 \end{pmatrix} \\
\pi_2 &= \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \\
\pi_3 &= \begin{pmatrix} 0/2 \\ 1/2 \end{pmatrix} \\
\end{align*}
$$

Thus, if $\pi_0 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$, $\pi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\pi_2 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, $\pi_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, etc.

Hence, $\pi_n$ does not converge to $\pi = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.

Notice, all cycles or closed walks have even length.
Convergence to Invariant Distribution

**Question:** Assume that the MC is irreducible.
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See the image for a mathematical example involving transition probabilities $P$ and distributions $\pi_n$.
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![Diagram of the Markov chain with states 1 and 2.](image)

Assume $X_0 = 1$. Then $X_1 = 2$, $X_2 = 1$, $X_3 = 2$, ...

Hence, $\pi_n$ does not converge to $\pi = [1/2, 1/2]$. Notice, all cycles or closed walks have even length.
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$$n$$
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![Diagram showing transition probabilities and sample paths]

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Notice, all cycles or closed walks have even length.
Convergence to stationary distribution.

**Theorem** Let $X_n$ be an irreducible Markov chain with invariant distribution $\pi$. Then, for all $i$, \[
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$\pi = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$

As $n$ gets large the probability of being in either state approaches $1/2$. (The stationary distribution.)

Notice cycles of length 1 and 2.
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Periodicity

Definition:
Periodicity is gcd of the lengths of all closed walks in irreducible chain.

Previous example: 2.

Definition
If periodicity is 1, Markov chain is said to be aperiodic. Otherwise, it is periodic.

Example
\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array}
\]
Which one is converges to stationary?

(A) \[A\]
(B) \[B\]
(C) both
(D) neither.

(A). 

\[A\]:
Closed walks of length 3 and length 4 ⇒ periodicity = 1.

\[B\]:
All closed walks multiple of 3 ⇒ periodicity = 2.
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1 & 2 & 3 \\
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\end{array}
\]

\[
\begin{array}{ccc}
A & & \\
& B & \\
& & \\
\end{array}
\]

Which one is converges to stationary?

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(B) \([B]\) 

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Which one is converges to stationary?

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[A]:

\[
\begin{align*}
    1 & \rightarrow 2 & \rightarrow 3 \\
    4 & \rightarrow 5 & \rightarrow 6
\end{align*}
\]

[B]:

\[
\begin{align*}
    1 & \rightarrow 2 & \rightarrow 3 \\
    4 & \rightarrow 5 & \rightarrow 6
\end{align*}
\]
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Convergence of $\pi_n$

Let $X_n$ be an irreducible and aperiodic Markov chain with invariant distribution $\pi$. Then, for all $i \in X$, $\pi_n(i) \to \pi(i)$, as $n \to \infty$.

Example 1.

<table>
<thead>
<tr>
<th>m</th>
<th>1</th>
<th>0.8</th>
<th>0.7</th>
<th>0.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td></td>
<td>0.3</td>
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</tr>
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<td>0.3</td>
<td></td>
<td></td>
<td></td>
<td>0.1</td>
</tr>
</tbody>
</table>

$\pi_0 = [1, 0, 0]$ $\pi_0 = [0, 1, 0]$
Convergence of $\pi_n$

**Theorem** Let $X_n$ be an irreducible and aperiodic Markov chain with invariant distribution $\pi$. 

Example 1

\[
\begin{array}{c}
p(1) = [1, 0, 0] \\
p(2) = [0, 1, 0] \\
p(3) = m
\end{array}
\]
Convergence of $\pi_n$

**Theorem** Let $X_n$ be an irreducible and aperiodic Markov chain with invariant distribution $\pi$. Then, for all $i \in \mathcal{X}$,

$$
\pi_n(i) \to \pi(i), \text{ as } n \to \infty.
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Convergence of $\pi_n$

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$$\pi_n(i) \to \pi(i), \text{ as } n \to \infty.$$ 

**Example**

\[
\begin{align*}
\pi_0 &= [0, 1, 0] \\
\pi_m(1) &\approx [0.2, 0.8, 0.0] \\
\pi_m(2) &\approx [0.4, 0.6, 0.0] \\
\pi_m(3) &\approx [0.7, 0.3, 0.0] \\
\end{align*}
\]
Convergence of $\pi_n$

**Theorem** Let $X_n$ be an irreducible and aperiodic Markov chain with invariant distribution $\pi$. 

Example

$\pi = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}$
Convergence of $\pi_n$

**Theorem** Let $X_n$ be an irreducible and aperiodic Markov chain with invariant distribution $\pi$. Then, for all $i \in \mathcal{X}$,

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**Example**

\[ \pi = [0.5, 0.5] \]
Summary

Markov Chains

- \[ P[X_{n+1} = j | X_0, \ldots, X_n = i] = P(i, j) \]

- \( FSE: \beta(i) = 1 + \sum_j P(i, j) \beta(j); \alpha(i) = \sum_j P(i, j) \alpha(j) \)

- \( \pi_n = \pi_0 P^n \)

- \( \pi \) is invariant iff \( \pi P = \pi \)

- Irreducible \( \Rightarrow \) one and only one invariant distribution \( \pi \)

- Irreducible \( \Rightarrow \) fraction of time in state \( i \) approaches \( \pi(i) \)

- Irreducible + Aperiodic \( \Rightarrow \pi_n \to \pi \)

- Calculating \( \pi \): One finds \( \pi = [0, 0, \ldots, 1] \) where \( Q = \cdots \).
Summary

Markov Chains

Markov Chain: $P_{X_{n+1} \mid X_0, \ldots, X_n} = P(i, j)$

FSE: $
\begin{align*}
\beta(i) &= 1 + \sum_j P(i, j) \beta(j); \\
\alpha(i) &= \sum_j P(i, j) \alpha(j).
\end{align*}$

$\pi_n = \pi_0 P^n$

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Irreducible $\Rightarrow$ one and only one invariant distribution $\pi$

Irreducible $\Rightarrow$ fraction of time in state $i$ approaches $\pi(i)$

Irreducible + Aperiodic $\Rightarrow \pi_n \rightarrow \pi$.

Calculating $\pi$: One finds $\pi = [0, 0, \ldots, 1] Q^{-1}$ where $Q = \cdots$.
Summary

- **Markov Chain:**

  - $P[X_{n+1} = j | X_0, ..., X_n = i] = P(i, j)$

- **FSE:**
  - $\beta(i) = 1 + \sum_j P(i, j) \beta(j)$
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- **Irreducible** $\Rightarrow$ one and only one invariant distribution $\pi$

- **Irreducible** $\Rightarrow$ fraction of time in state $i$ approaches $\pi(i)$

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- **Calculating $\pi$:**
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