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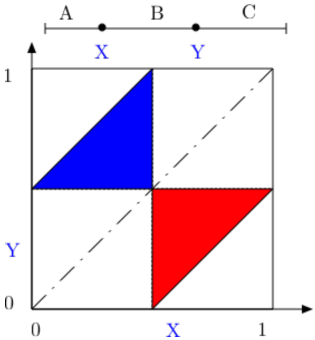

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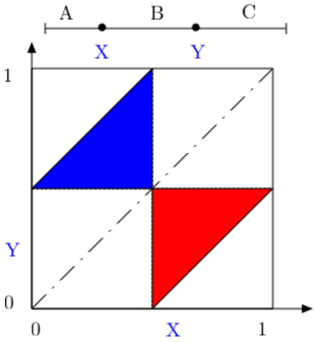

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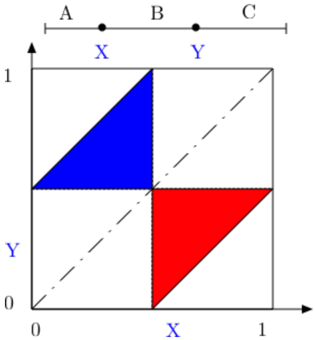

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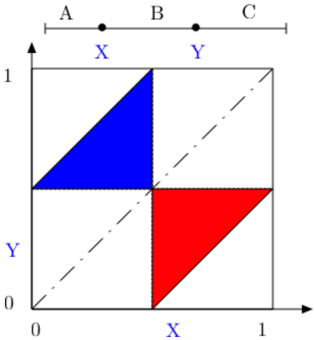

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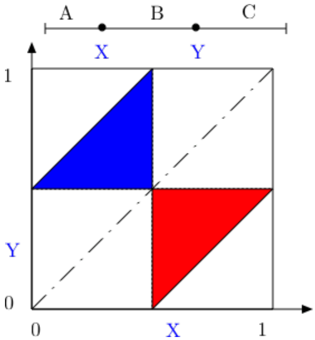

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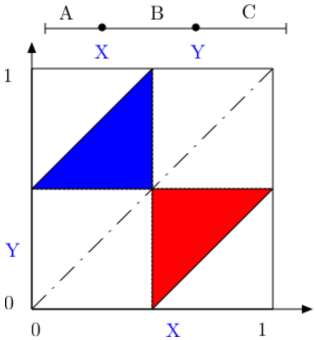

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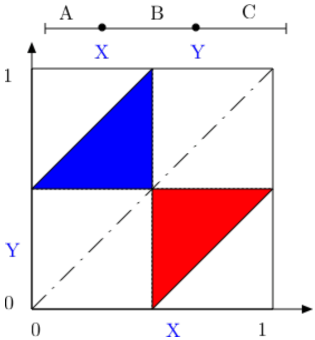

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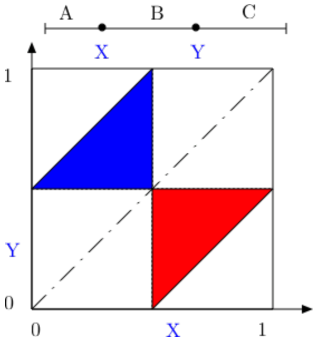

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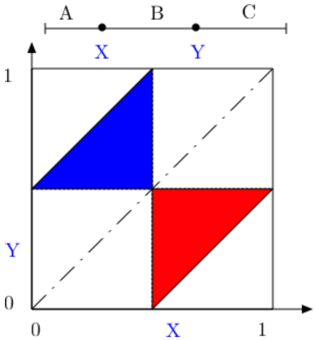

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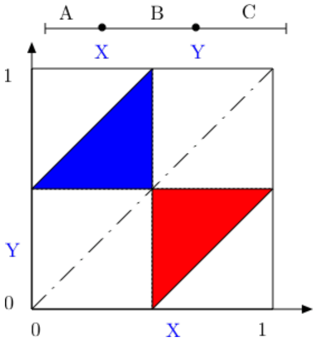

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Thus, $Pr[\text{make triangle}] = 1/4$.

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(C) is an argument for (A), (B) and (D).

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$$E[Z] = A_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = H(n).$$

CS70: Markov Chains.

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Markov Chains

1. Examples
2. Definition
3. Stationary Distribution
4. Peridoicity.
5. Hitting Time.
6. Here before there.

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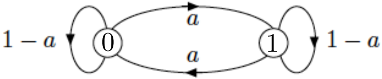

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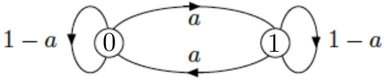

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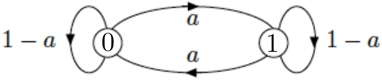

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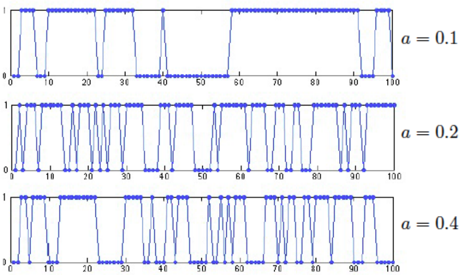

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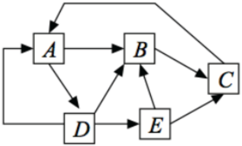
At each step, the MC follows one of the outgoing arrows of the current state, with equal probabilities.



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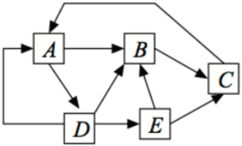

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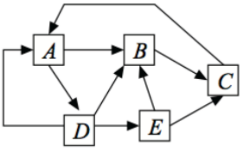

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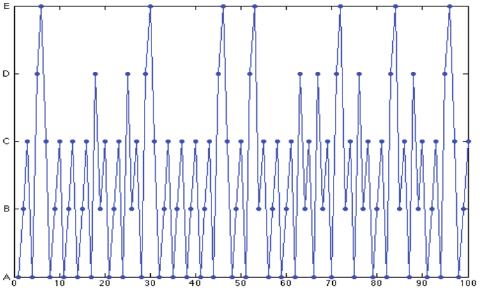

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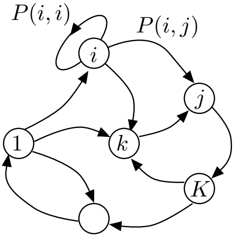

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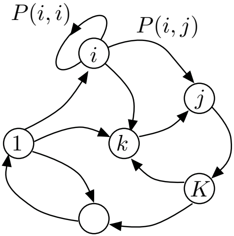

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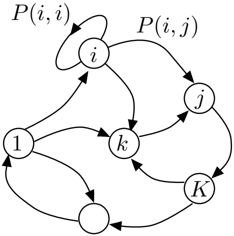

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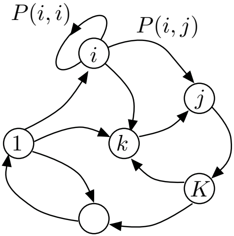

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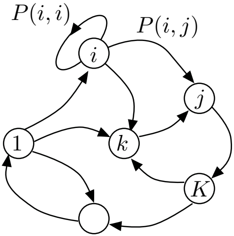

- A finite set of states: $\mathscr{X} = \{1, 2, \ldots, K\}$
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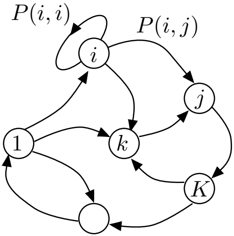

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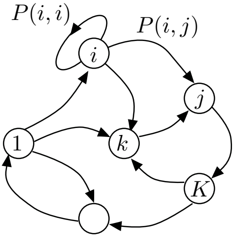

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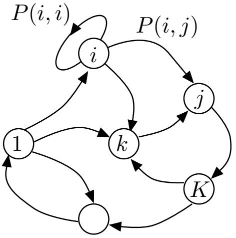

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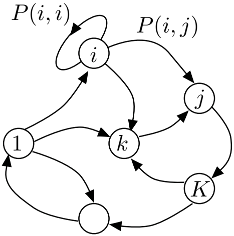

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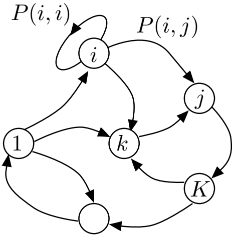

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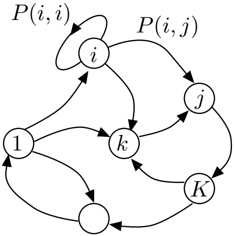

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  $Pr[X_{n+1} = j \mid X_0, \ldots, X_n = i] = P(i,j), i,j \in \mathscr{X}$.

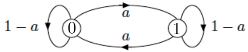
Two-State Markov Chain

Symmetric two-state Markov chain for a random motion on $\{0, 1\}$.

Two-State Markov Chain

Symmetric two-state Markov chain for a random motion on $\{0, 1\}$.

Recall *a* is the probability of a state change in a step.



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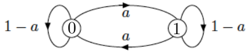

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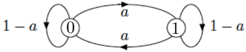

$$P = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{pmatrix} 1-a & a \\ a & 1-a \end{pmatrix} \end{array}$$

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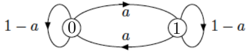

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Sum of row entries?

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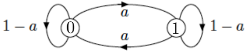

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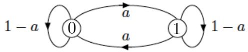

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Sum of row entries? 1. Always.

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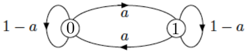

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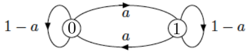

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If $\pi_0 = [1, 0]$

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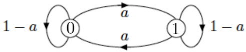

$$P = \begin{matrix} & \begin{matrix} 0 & & 1 \end{matrix} \\ \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} 1-a & a \\ a & 1-a \end{pmatrix} \end{matrix}$$

Sum of row entries? 1.   Always.

Evolving distribution.

If $\pi_0 = [1,0]$ what is π_1?

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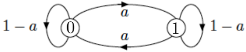

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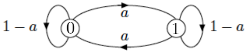

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Sum of row entries? 1. Always.

Evolving distribution.
If $\pi_0 = [1, 0]$ what is π_1? $\pi_1 P \quad = [1-a, a]$.
What is π_2?

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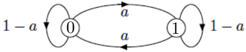

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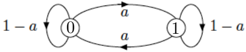

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Evolving distribution.

If $\pi_0 = [1,0]$ what is π_1? $\pi_1 P \quad = [1-a, a]$.

What is π_2? $\pi_1 P$ $[(1-a)(1-a)+a^2, (1-a)a+a(1-a)]$

What is π_{100}?

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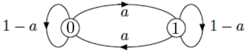

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What is π_{100}? Just guessing, but close to $[.5, .5]$.

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Symmetric two-state Markov chain for a random motion on $\{0, 1\}$.

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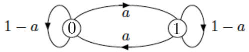

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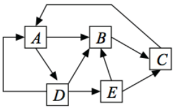
If $\pi_0 = [1, 0]$ what is π_1? $\pi_1 P \quad = [1-a, a]$.
What is π_2? $\pi_1 P$ $[(1-a)(1-a)+a^2, (1-a)a+a(1-a)]$
What is π_{100}? Just guessing, but close to $[.5, .5]$. Later.

Five-State Markov Chain

MC follows each outgoing arrows of current state with equal probabilities.



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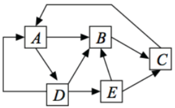

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & .5 & 0 & 0 & .5 \\ 0 & .5 & .5 & 0 & 0 \end{pmatrix} \end{array}$$

Evolving distribution from $\pi_0 = [1, 0, 0, 0, 0]$?

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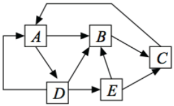

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What is π_1?

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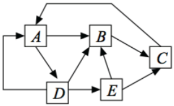

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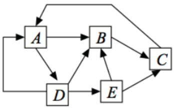

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Evolving distribution from $\pi_0 = [1,0,0,0,0]$?

What is π_1? $\pi_1 P = [0,.5,0,.5,0]$.

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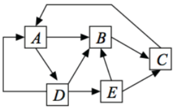

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Evolving distribution from $\pi_0 = [1,0,0,0,0]$?

What is π_1? $\pi_1 P = [0,.5,0,.5,0]$.

If $\pi_t[.2,.2,.2.,.2,.2]$, what is π_{t+1}?

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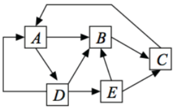
MC follows each outgoing arrows of current state with equal probabilities.



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Evolving distribution from $\pi_0 = [1, 0, 0, 0, 0]$?
What is π_1? $\pi_1 P = [0, .5, 0, .5, 0]$.

If $\pi_t[.2, .2, .2., .2, .2]$, what is π_{t+1}? $\pi_t P$

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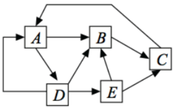

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Evolving distribution from $\pi_0 = [1, 0, 0, 0, 0]$?

What is π_1? $\pi_1 P = [0, .5, 0, .5, 0]$.

If $\pi_t[.2, .2, .2., .2, .2]$, what is π_{t+1}? $\pi_t P$ $[.2, .3, .3, .1, .1]$.

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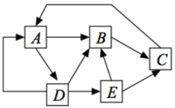

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Evolving distribution from $\pi_0 = [1, 0, 0, 0, 0]$?

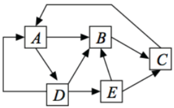
What is π_1? $\pi_1 P = [0, .5, 0, .5, 0]$.

If $\pi_t[.2, .2, .2., .2, .2]$, what is π_{t+1}? $\pi_t P$ $[.2, .3, .3, .1, .1]$.

This is just taking scaled (by .2) in-degree.

Five-State Markov Chain

MC follows each outgoing arrows of current state with equal probabilities.



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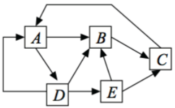

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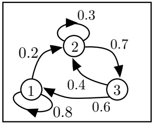
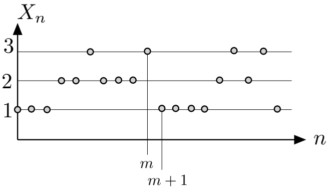
What is it at π_{10000}?

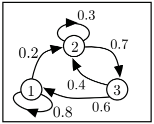
Distribution of X_n

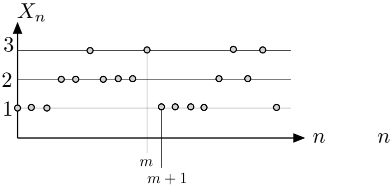




n

Distribution of X_n





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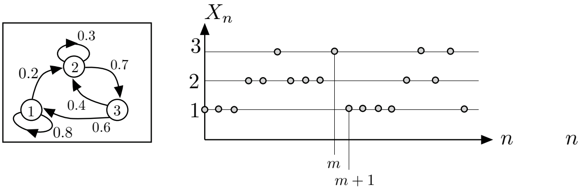
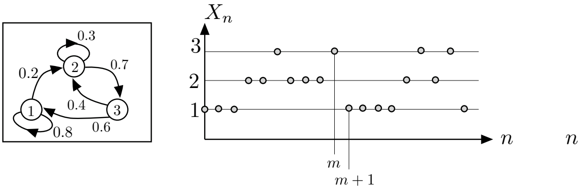


Recall π_n is a distribution over states for X_n.

Distribution of X_n



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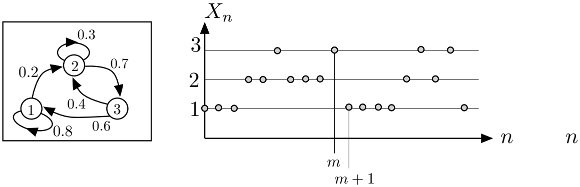

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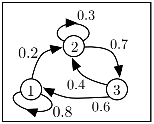

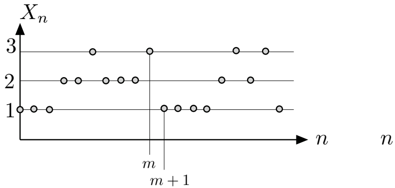

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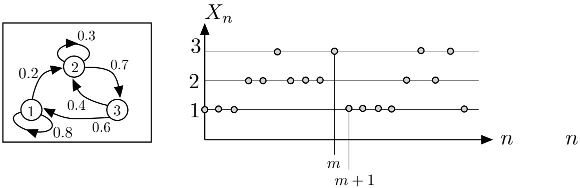

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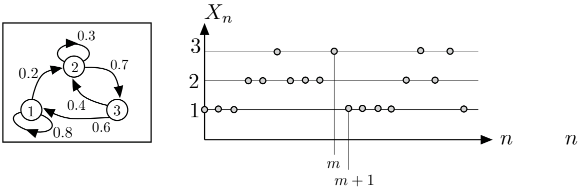

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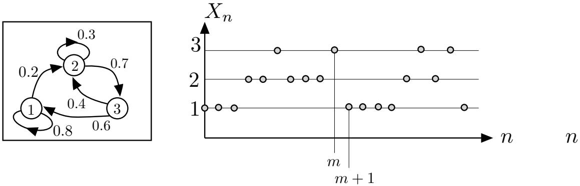

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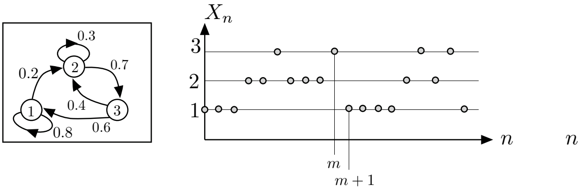

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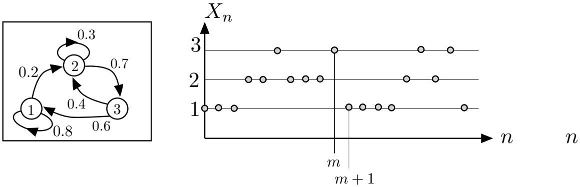

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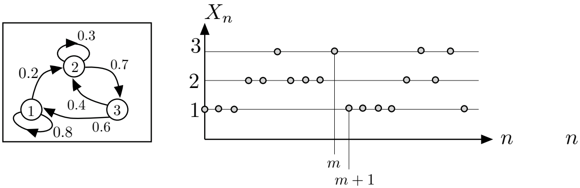

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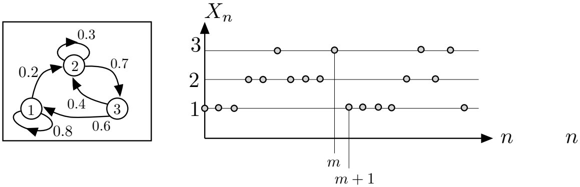

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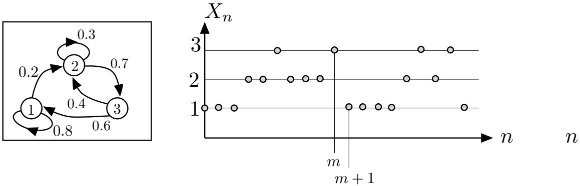

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Sometimes the distribution as $n \to \infty$

Stationary: Example

Example 1:

Stationary: Example

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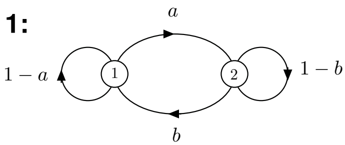

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Stationary: Example

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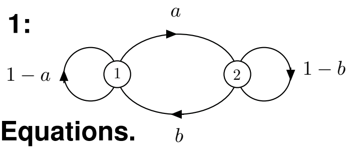

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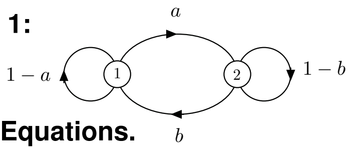

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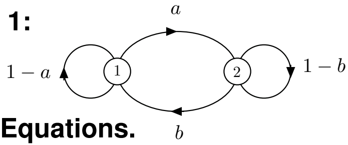

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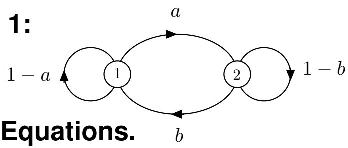

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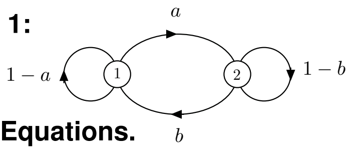

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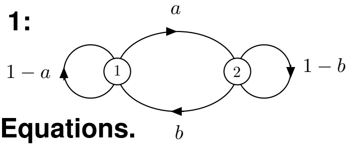

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These equations are redundant!

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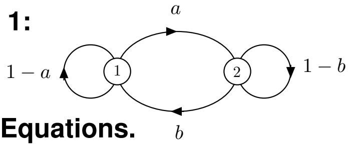

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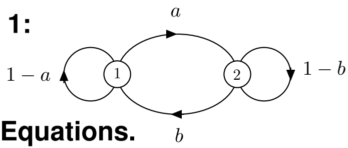

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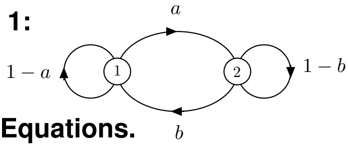

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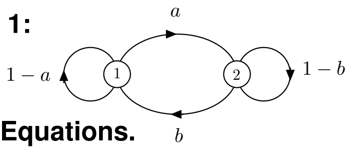

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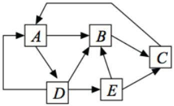

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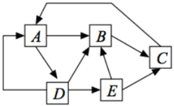

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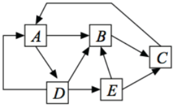

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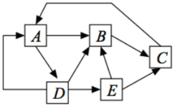

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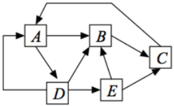

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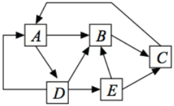

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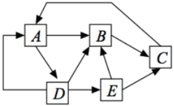

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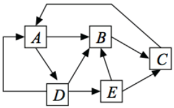

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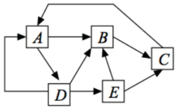

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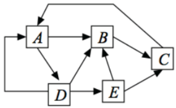

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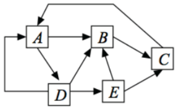

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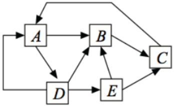

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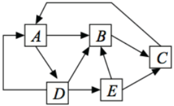

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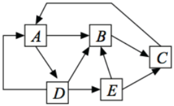

$$P = \begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & 0 & .5 & 0 & .5 & 0 \\ B & 0 & 0 & 1 & 0 & 0 \\ C & 1 & 0 & 0 & 0 & 0 \\ D & 1/3 & 1/3 & 0 & 0 & 1/3 \\ E & 0 & .5 & .5 & 0 & 0 \end{array}$$

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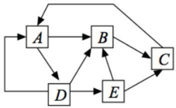
Verify: adds to 1. $\pi(A) = \pi(C) + 1/3\pi(D) \propto_{39} 10 + 1/3 \times 6 = 12$.

Stationary: Example 2



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Stationary distributions: Example 3

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Definition A Markov chain is irreducible if it can go from every state i to every state j

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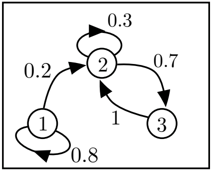
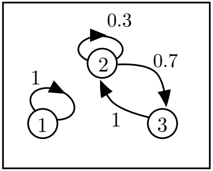
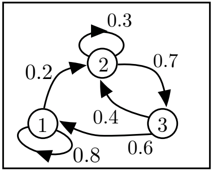


[A]



[B]



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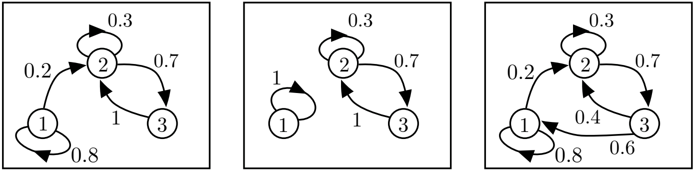

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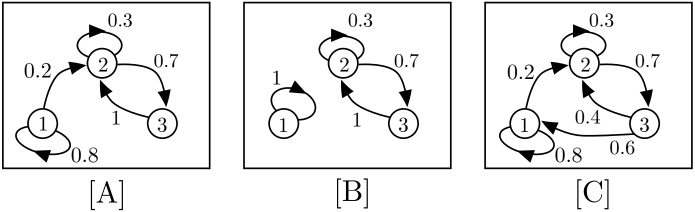

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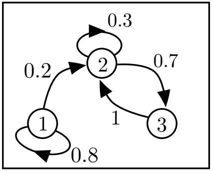

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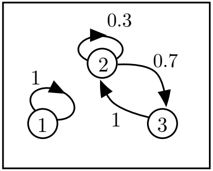

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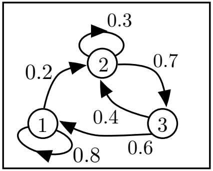

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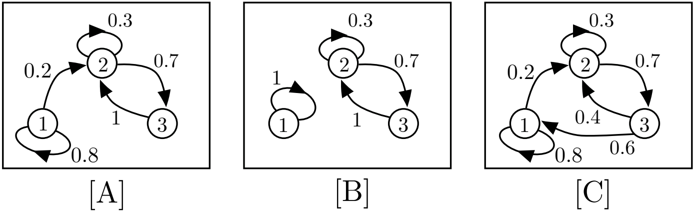

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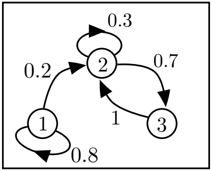

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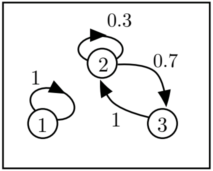

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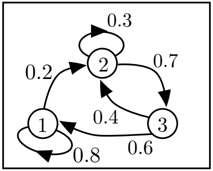

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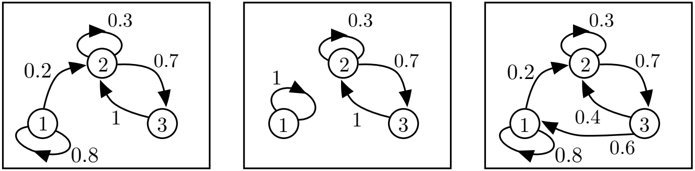

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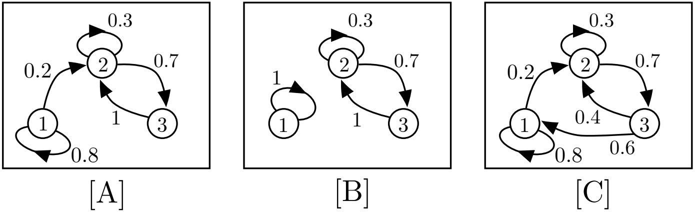

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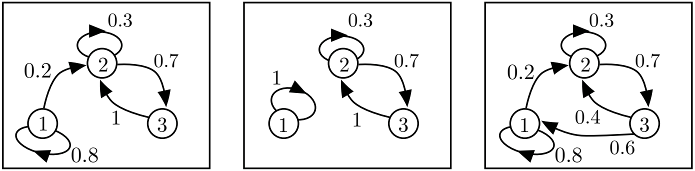

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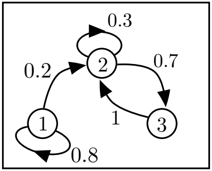

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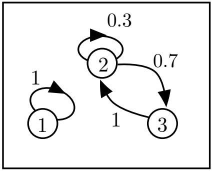

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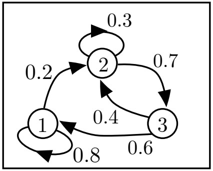

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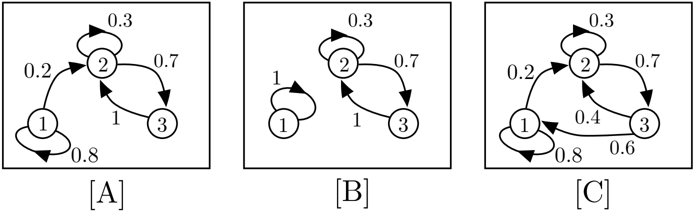

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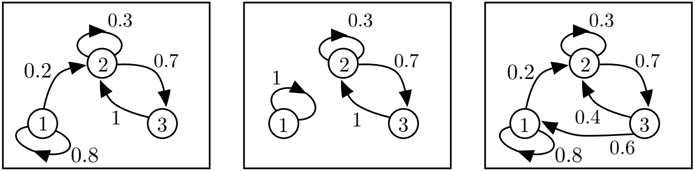

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If you consider the graph with arrows when $P(i,j) > 0$, irreducible means that there is a single (strongly) connected component.

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Only one stationary distribution if irreducible (or connected.)

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Proof: Lecture note 21 gives a plausibility argument.

□

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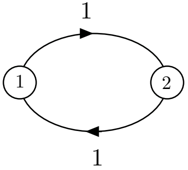

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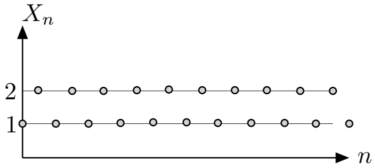

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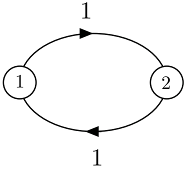

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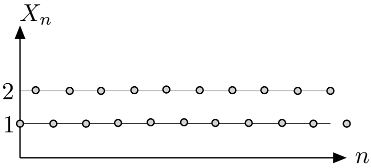

The fraction of time in state 1

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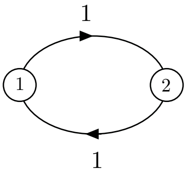

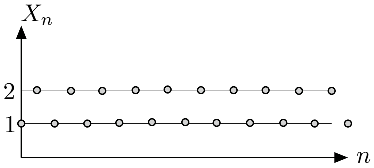

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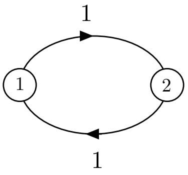

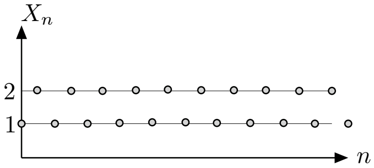

The fraction of time in state 1 converges to $1/2$, which is $\pi(1)$.

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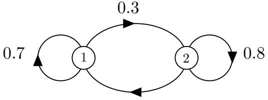

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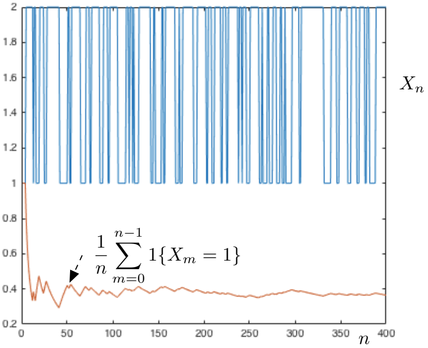

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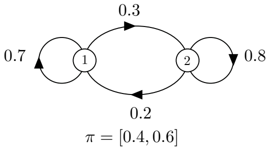

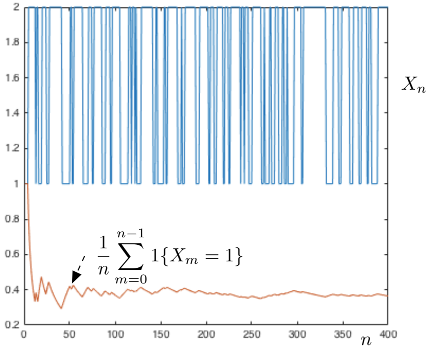

Convergence to Invariant Distribution

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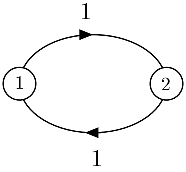

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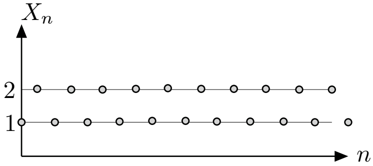

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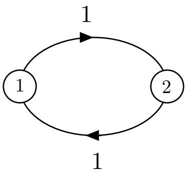

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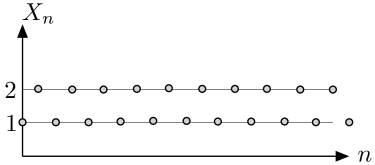

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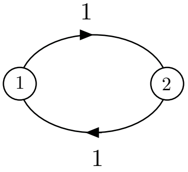

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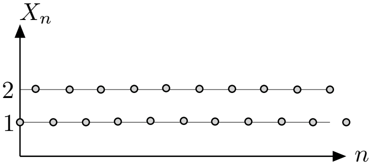

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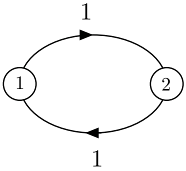

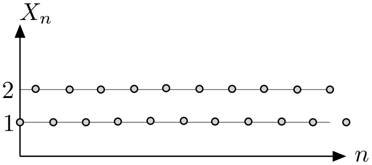

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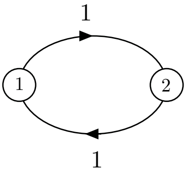

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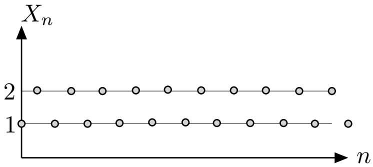

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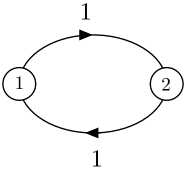

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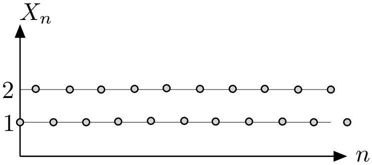

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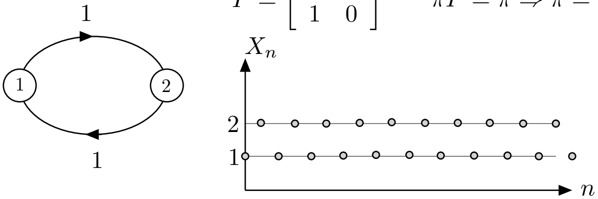

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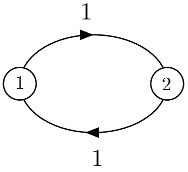

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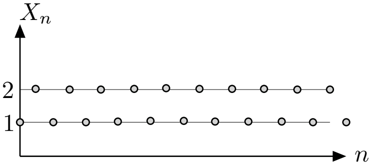

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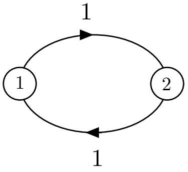

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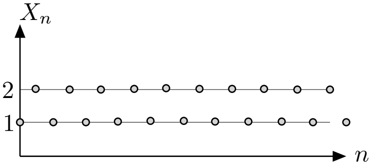

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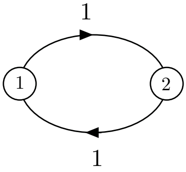

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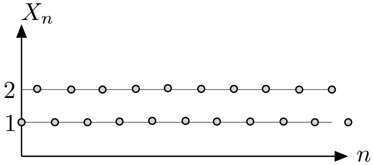

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Hence, π_n does not converge to $\pi = [1/2, 1/2]$.

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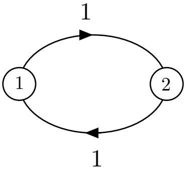

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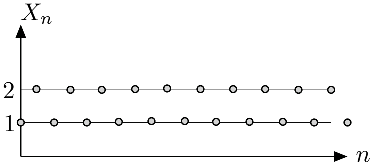

Assume $X_0 = 1$. Then $X_1 = 2, X_2 = 1, X_3 = 2, \ldots$.

Thus, if $\pi_0 = [1, 0]$, $\pi_1 = [0, 1], \pi_2 = [1, 0], \pi_3 = [0, 1]$, etc.

Hence, π_n does not converge to $\pi = [1/2, 1/2]$.
Notice, all cycles or closed walks have even length.

Convergence to stationary distribution.

Theorem Let X_n be an irreducible Markov chain with invariant distribution π. Then, for all i, $\frac{1}{n}\sum_{m=0}^{n-1} 1\{X_m = i\} \to \pi(i)$, as $n \to \infty$.

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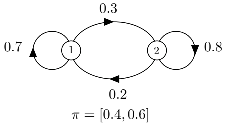

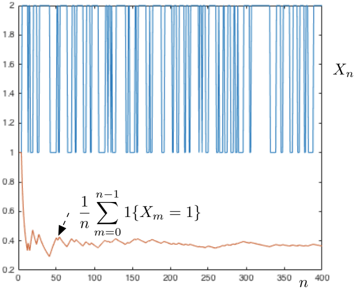

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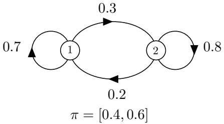

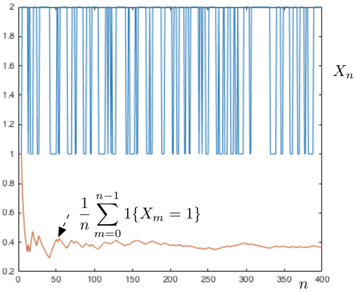

As n gets large the probability of being in state 1 approaches 0.4. (The stationary distribution.)

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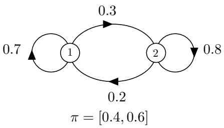

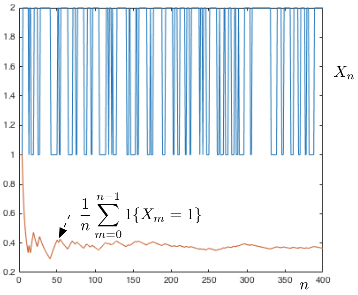

As n gets large the probability of being in state 1 approaches 0.4. (The stationary distribution.) Notice cycles of length 1 and 2.

Periodicity

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Definition: Periodicity is gcd of the lengths of all closed walks in irreducible chain.

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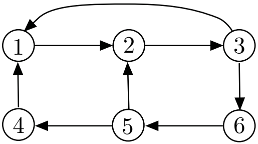
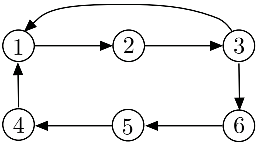


$[A]$



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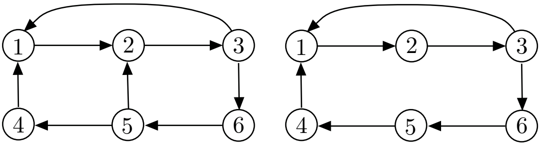

$[A]$ $[B]$

Which one converges to stationary?

(A) [A]

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(C) both

(D) neither.

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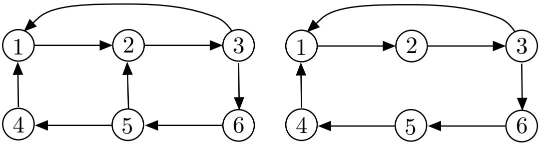

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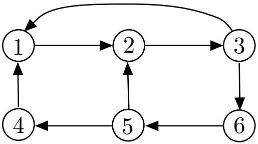

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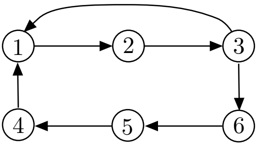

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Which one converges to stationary?
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[A]: Closed walks of length 3 and length 4

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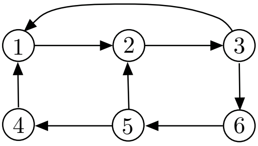

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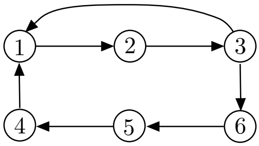

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Which one converges to stationary?

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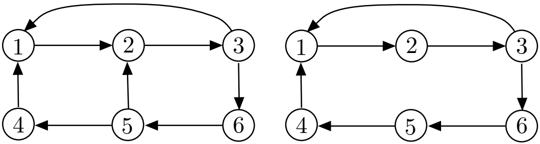

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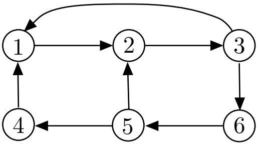

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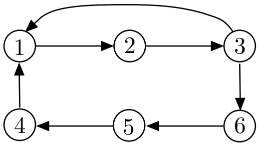

[B]

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[B]: All closed walks multiple of 3

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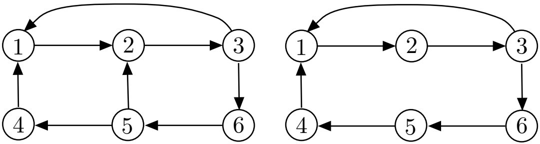

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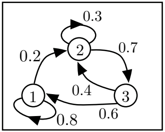

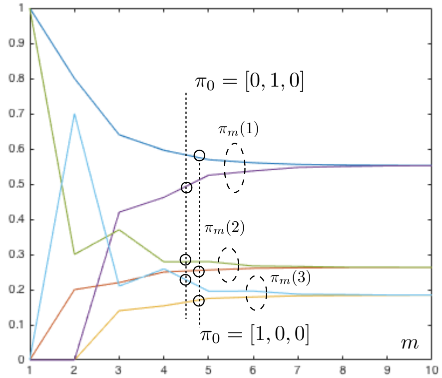

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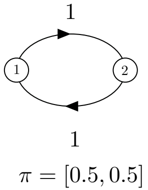
Non Example: periodic chain



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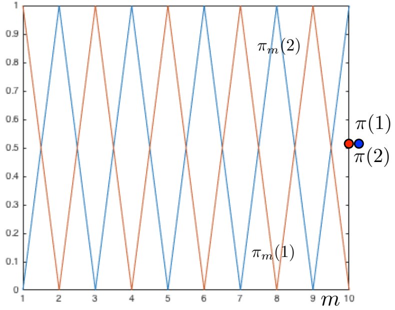
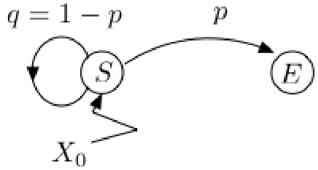


First Passage Time - Example 1. Poll

Let's flip a coin with $Pr[H] = p$ until we get H. How many flips, on average?



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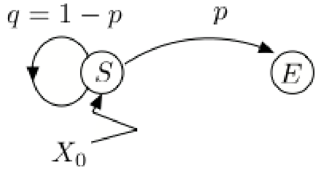

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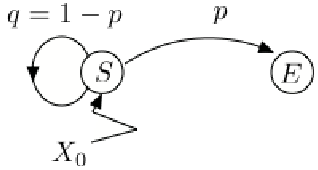

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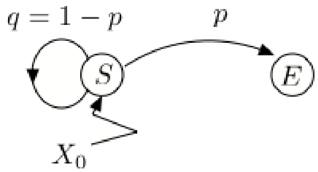

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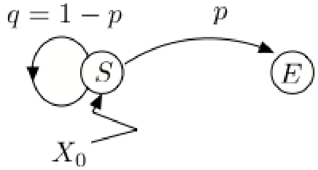

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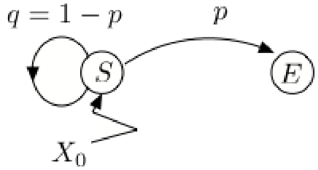

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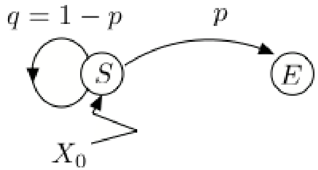

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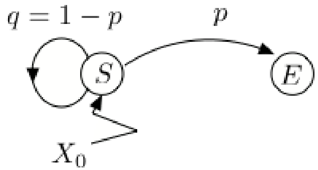

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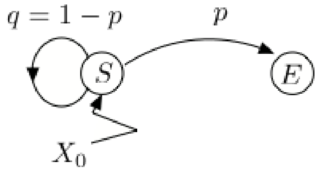

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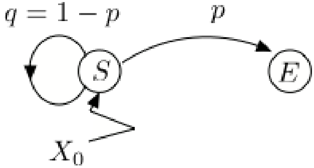

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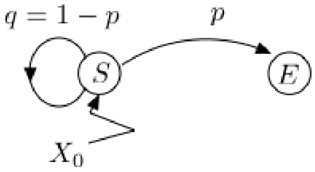

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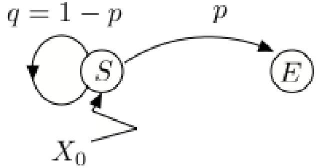

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(See next slide.)

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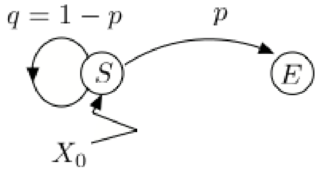

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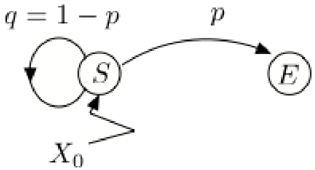

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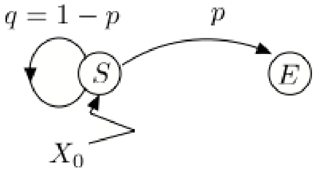

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Note: Time until E is $G(p)$.

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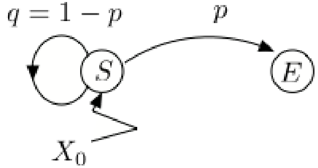

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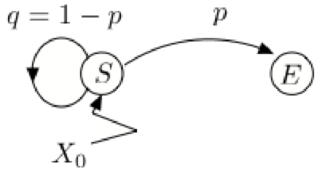
The mean of $G(p)$ is $1/p$!!!

First Passage Time - Example 1

Let's flip a coin with $Pr[H] = p$ until we get H. How many flips in expectation?



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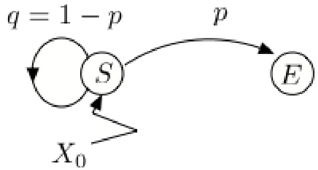

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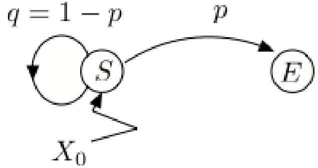

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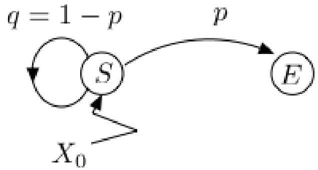

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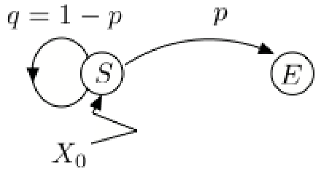

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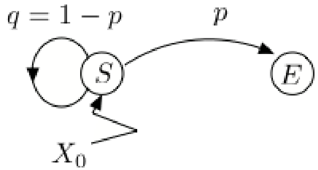

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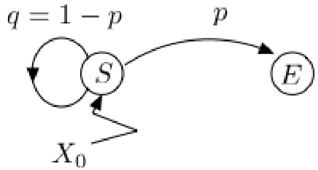

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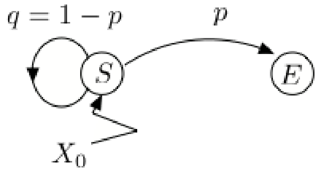

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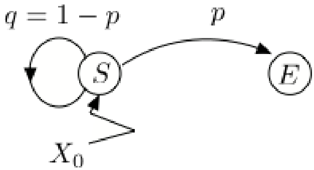

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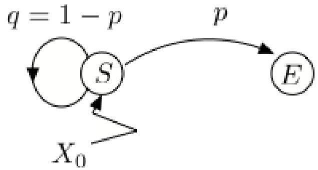

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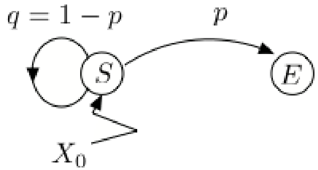

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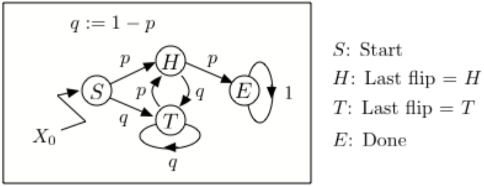

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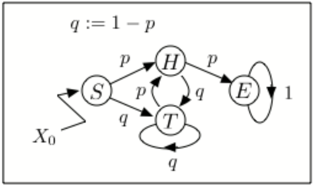

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Which one is correct?
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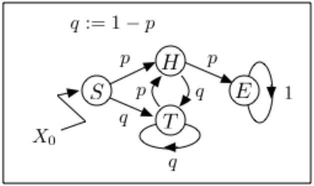

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(A) Expected time from S to E.

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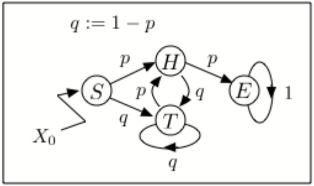

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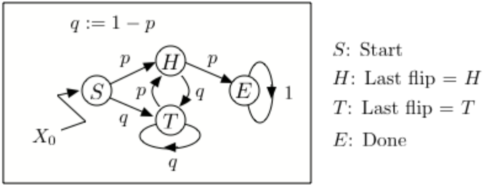

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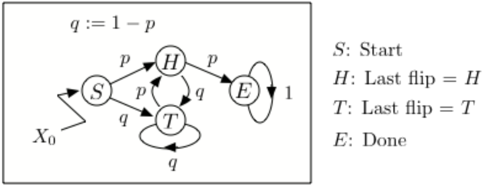

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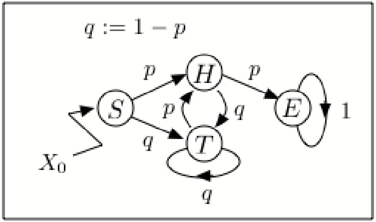

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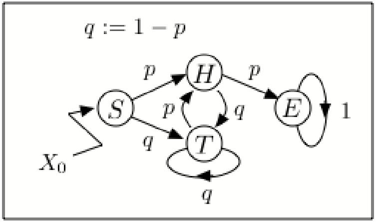

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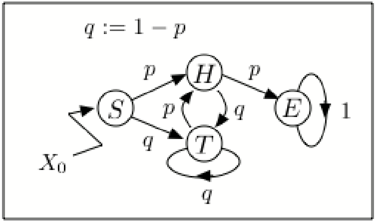

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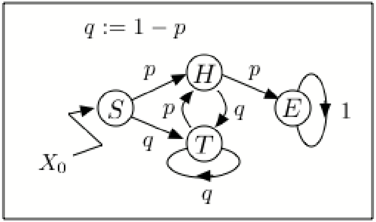

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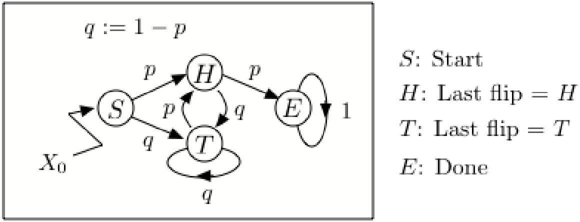

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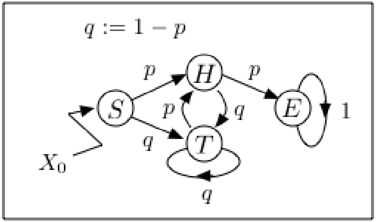

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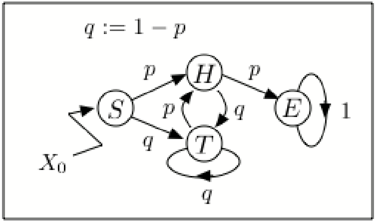

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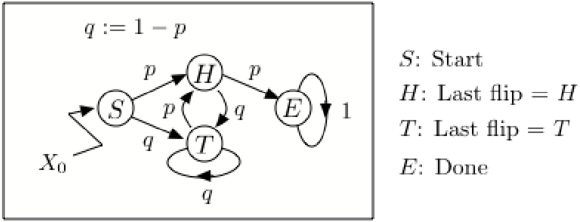

Let $\beta(i)$ be the average time from state i until the MC hits state E.

We claim that (these are called the first step equations)

$$\begin{aligned}\beta(S) &= 1 + p\beta(H) + q\beta(T)\\ \beta(H) &= 1 + p0 + q\beta(T)\\ \beta(T) &= 1 + p\beta(H) + q\beta(T).\end{aligned}$$

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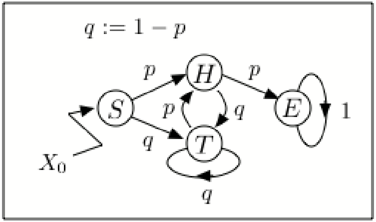

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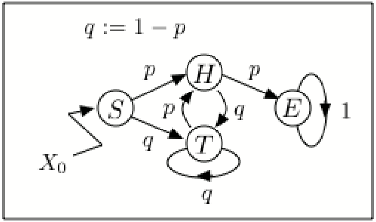

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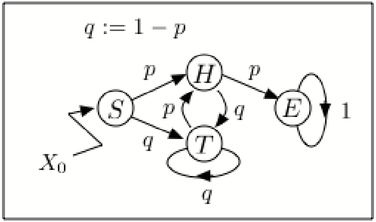

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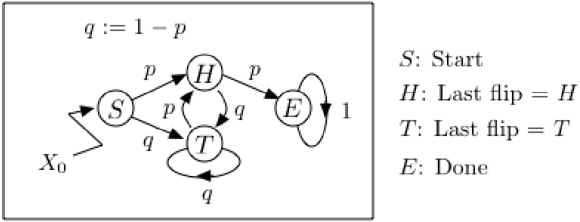

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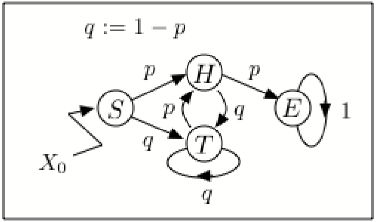

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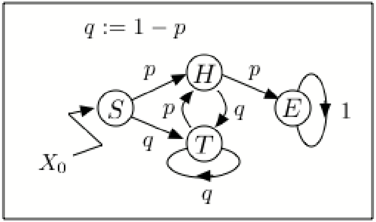

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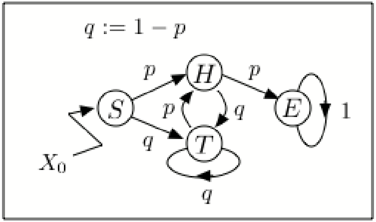

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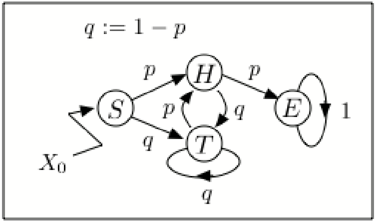

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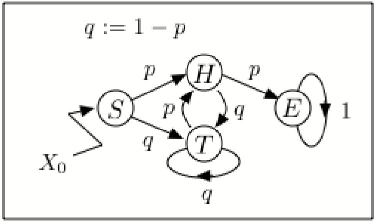

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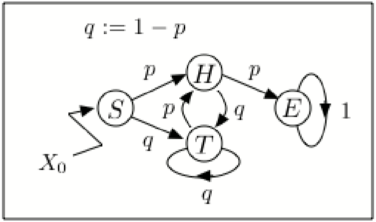

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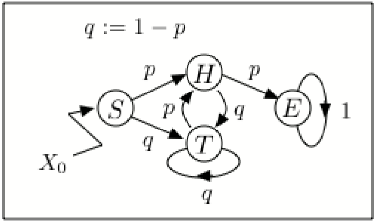

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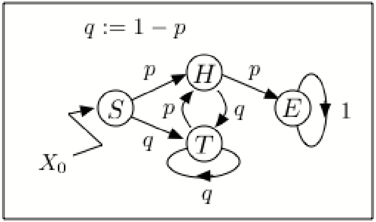

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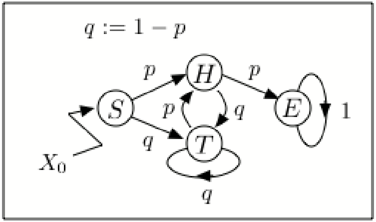

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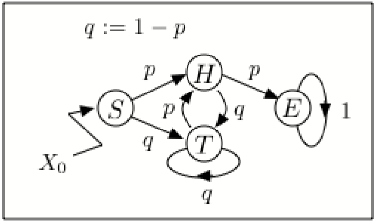

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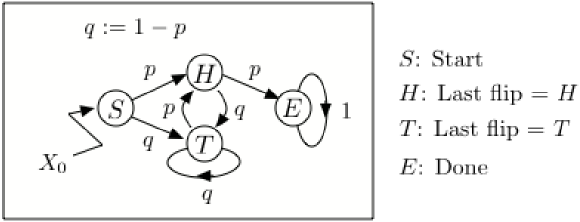

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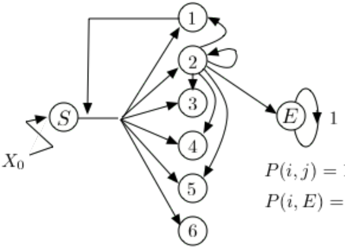

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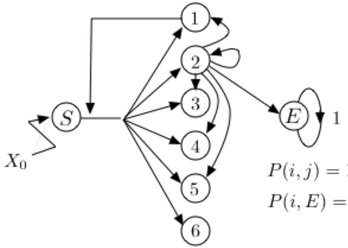

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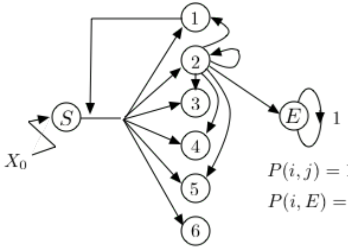

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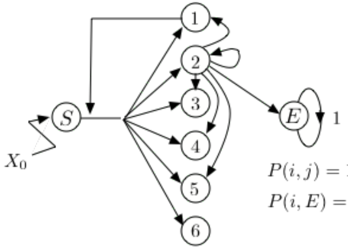

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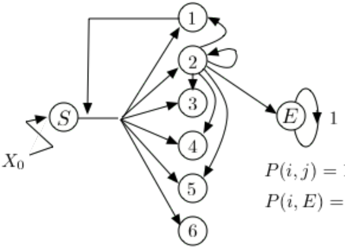

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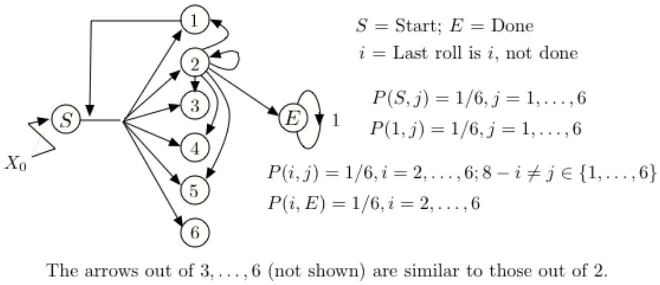

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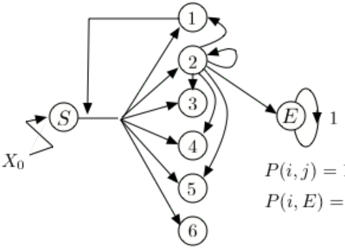

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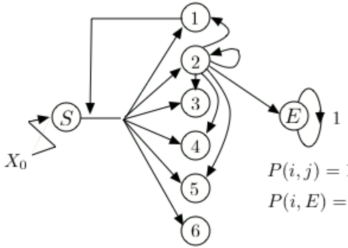

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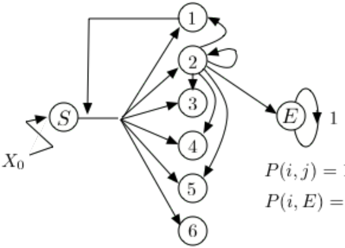

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$$\Rightarrow \cdots \beta(S) = 8.4.$$

Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$.

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Start with \$10.

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Each step, flip yields ‘heads’, earn \$1.

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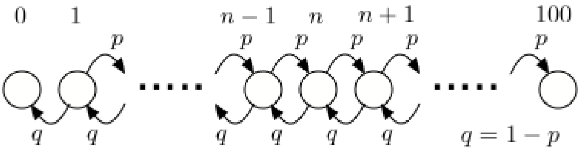

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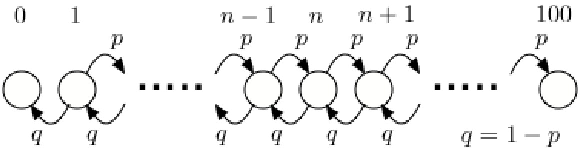

Let $\alpha(n)$ be the probability of reaching 100 before 0, starting from n, for $n = 0, 1, \ldots, 100$.

Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p < 0.5$.
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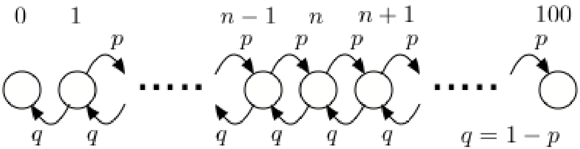

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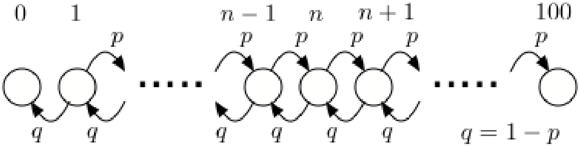

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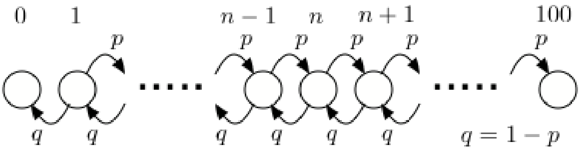

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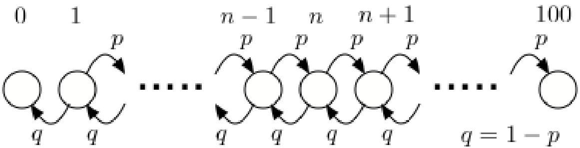

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(B) is incorrect, 0 is bad.
(D) is incorrect. Confuses expected hitting time with *A* before *B*.

Here before There - A before B

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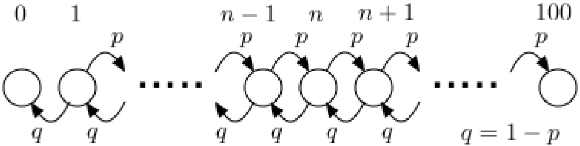

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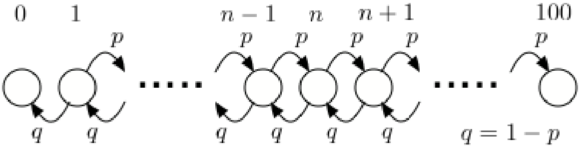

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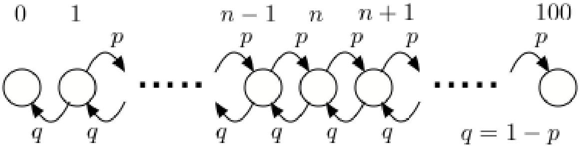

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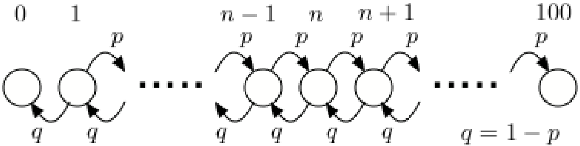

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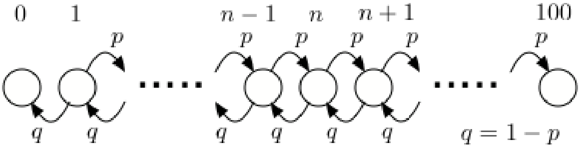

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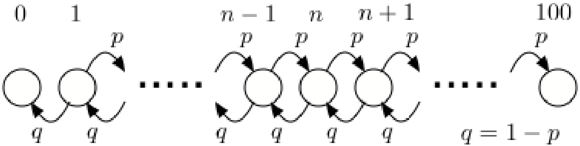

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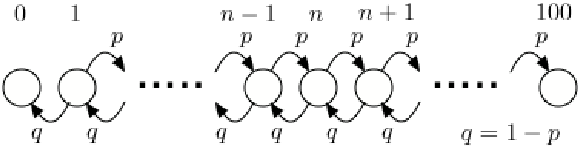

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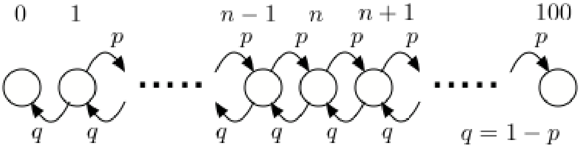

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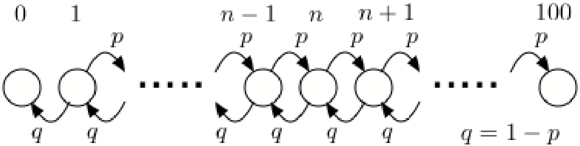

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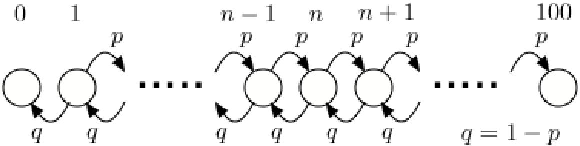

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Here before There - A before B

Game of “heads or tails” using coin with ‘heads’ probability $p = .48$.

Here before There - A before B

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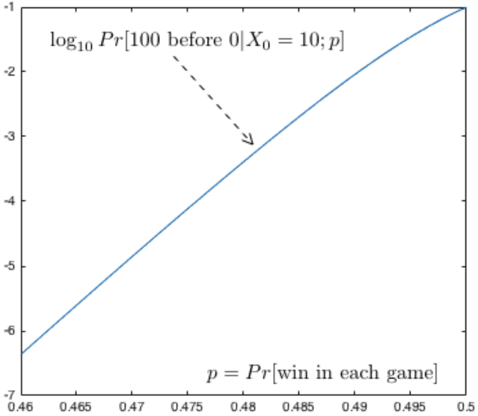

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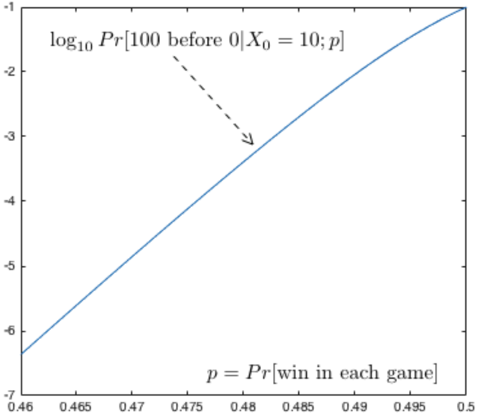

Less than 1 in a 1000.

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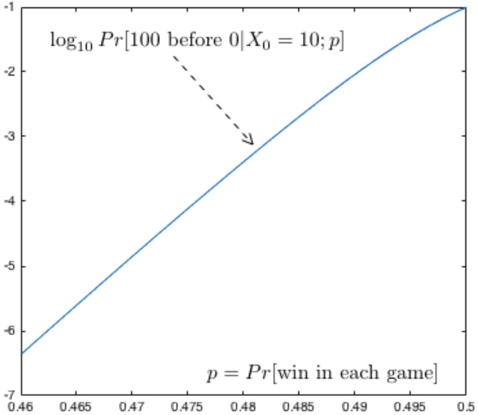

Less than 1 in a 1000. Moral of example:

Here before There - A before B

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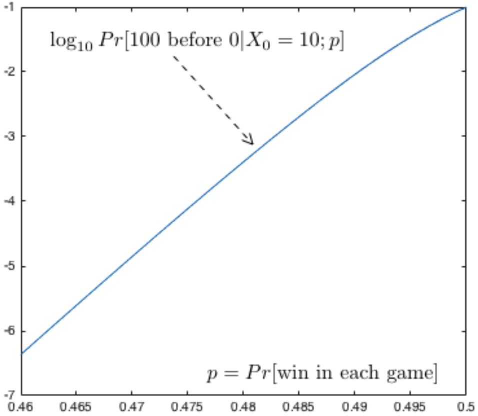

Less than 1 in a 1000. Moral of example: Money in Vegas

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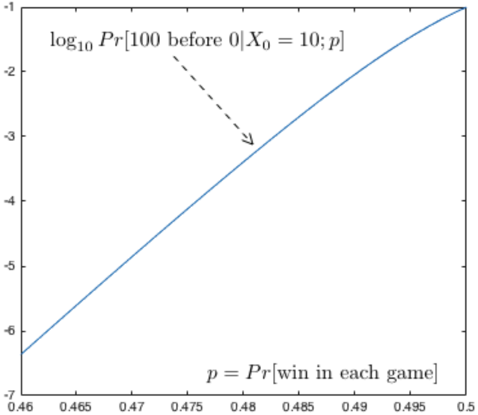
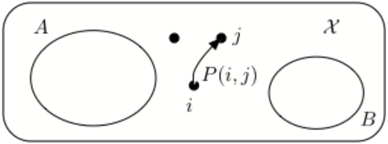


Less than 1 in a 1000. Moral of example: Money in Vegas stays in Vegas.

First Step Equations

First Step Equations



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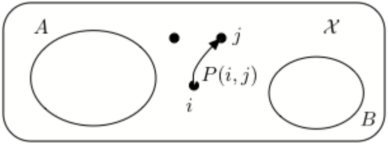

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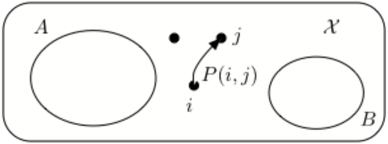

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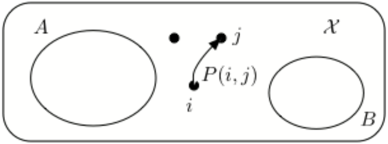

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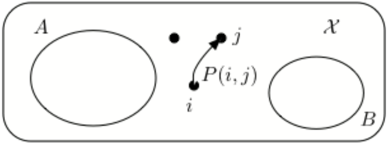

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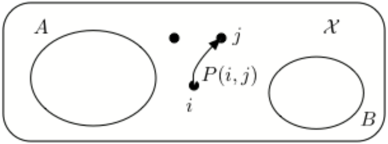

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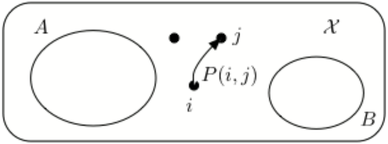

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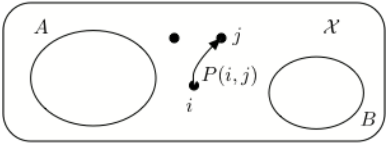

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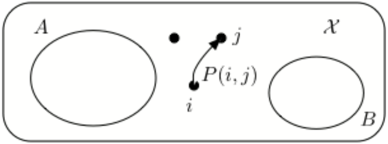

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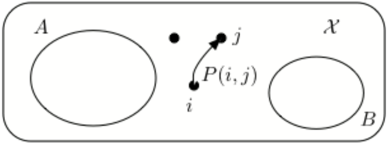

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Then

$$\gamma(i) = \begin{cases} g(i), & \text{if } i \in A \\ & \end{cases}$$

Accumulating Rewards

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Let also $g : \mathscr{X} \to \Re$ be some function.

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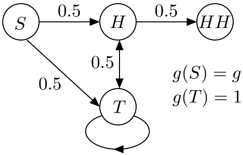

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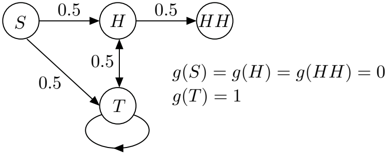

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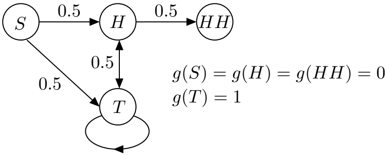

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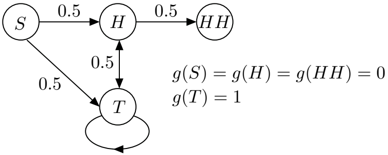

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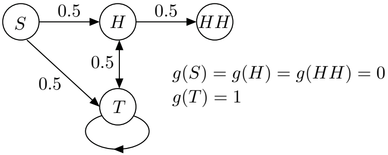

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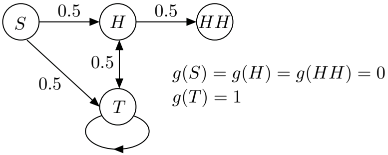

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Solving, we find $\gamma(S) = 2.5$.

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