The natural numbers.

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.
Gauss: It’s \(100 \times 101\) or 5050!

Five year old Gauss Theorem: \(\forall n \in \mathbb{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\).

It is a statement about all natural numbers.

\(\forall (n \in \mathbb{N}) : P(n)\).

\(P(n)\) is \(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\).

Principle of Induction:

▶ Prove \(P(0)\).
▶ Assume \(P(k)\), “Induction Hypothesis”
▶ Prove \(P(k + 1)\). “Induction Step.”

Gauss induction proof.

Theorem: For all natural numbers \(n\), \(0 + 1 + 2 \cdots n = \frac{n(n+1)}{2}\)

Base Case: Does \(0 = \frac{0 \times (0+1)}{2}\)? Yes.

Induction Step: Show \(\forall k \geq 0, P(k) \implies P(k + 1)\)

Induction Hypothesis: \(P(k) = 1 + \cdots + k = \frac{k(k+1)}{2}\)

\[
1 + \cdots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1)
= \frac{k^2 + k + 2(k + 1)}{2}
= \frac{k^2 + 3k + 2}{2}
= \frac{(k + 1)(k + 2)}{2}
\]

P\((k + 1)!\) By principle of induction...

Induction

The canonical way of proving statements of the form

\((\forall k \in \mathbb{N})(P(k))\)

▶ For all natural numbers \(n\), \(1 + 2 \cdots n = \frac{n(n+1)}{2}\)
▶ For all \(n \in \mathbb{N}\), \(n^2 - n\) is divisible by 3.
▶ The sum of the first \(n\) odd integers is a perfect square.

The basic form

▶ Prove \(P(0)\). “Base Case”.
▶ \(P(k) \implies P(k + 1)\)
▶ Assume \(P(k)\), “Induction Hypothesis”
▶ Prove \(P(k + 1)\). “Induction Step.”

P\((n)\) true for all natural numbers!!!

Get to use \(P(k)\) to prove \(P(k + 1)!\) !!!
Proof by Induction.

Base Case. 

\[ P(0) \text{ is } \sum_{i=0}^{0} i = 0(0+1)/2 = 0. \]

Is predicate, \( P(n) \) true for \( n = k + 1? \)

\[ \sum_{i=0}^{k+1} i = \sum_{i=0}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}. \]

How about \( k + 2 \). Same argument starting at \( k + 1 \) works!

\textbf{Induction Step.} \( P(k) \Rightarrow P(k+1). \)

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. \( P(0) \text{ is } \sum_{i=0}^{0} i = 0(0+1)/2 \text{ Base Case.} \)

Statement is true for \( n = 0 \) \( P(0) \) is true

plus inductive step \( \Rightarrow \) true for \( n = 1 \) \( P(0)+P(1) \Rightarrow P(1) \)

plus inductive step \( \Rightarrow \) true for \( n = 2 \) \( P(1)+P(2) \Rightarrow P(2) \)

...\n
Predicate, \( P(n) \). True for all natural numbers! \quad \text{Proof by Induction.}

\[ \text{Predicate, } P(n), \text{ True for all natural numbers!} \]

Another Induction Proof.

\textbf{Theorem:} For every \( n \in N, n^3 - n \text{ is divisible by } 3. (3 | (n^3 - n) ) \).

\textbf{Proof:} By induction.

\textbf{Base Case:} \( P(0) \text{ is } 0^3 - 0 \text{ is divisible by } 3. \text{ Yes!} \)

\textbf{Induction Step:} \( (\forall k \in N)(P(k) \Rightarrow P(k+1)) \)

\textbf{Induction Hypothesis:} \( k^3 - k \text{ is divisible by } 3. \)

or \( k^3 - k = 3q \) for some integer \( q \).

\( (k + 1)^3 - (k + 1) = k^3 + 3k^2 + 3k + 1 - (k + 1) \)

\( = k^3 + 3k^2 + 2k \)

\( = (k^3 - k) + 3k^2 + 3k \)

\( \Rightarrow 3q + 3(k^2 + k) \quad \text{Induction Hyp. Factor.} \)

\( \Rightarrow 3(q + k^2 + k) \quad \text{(Un)Distributive over } \times \)

\( \text{Or } (k + 1)^3 - (k + 1) = 3(q + k^2 + k). \)

\( (q + k^2 + k) \text{ is integer (closed under addition and multiplication).} \rightarrow (k + 1)^3 - (k + 1) \text{ is divisible by } 3. \)

Thus, \( (\forall k \in N)(P(k) \Rightarrow P(k+1)) \)

Thus, theorem holds by induction. \quad \square

\[ \text{Two color theorem: example.} \]

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

Proper coloring: for each line segment the regions on the two sides have different colors. 1

\textbf{Fact:} Switching red and blue gives another valid colors.

\[ \text{Algorithm gives } P(k) \Rightarrow P(k + 1). \] \quad \square

\[ \text{Poll: What did Gauss use in the proof?} \]

\begin{itemize}
  \item [(A)] Every natural number has a next number.
  \item [(B)] The recursive leap of faith.
  \item [(C)] \( \exists k \in N, \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2} \)
  \item [(D)] \( \forall k \in N, \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2} \)
\end{itemize}

\[ \text{Poll: what did we use in the proof.} \]

\begin{itemize}
  \item [(A)] Switching a 2-coloring is a valid coloring.
  \item [(B)] Definition of 2-coloring.
  \item [(C)] Definition of adjacent.
  \item [(D)] Definition of region.
  \item [(E)] The four color theorem.
\end{itemize}
Strengthening Induction Hypothesis.

**Theorem:** The sum of the first $n$ odd numbers is a perfect square.

**Theorem:** The sum of the first $n$ odd numbers is $n^2$.

Base Case: 1 (first odd number) is $1^2$.

Induction Hypothesis: Sum of first $k$ odd numbers is $k^2$.

Induction Step:
1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odd numbers is $k^2 + 2k + 1 = k^2 + 2k + 1$
3. $k^2 + 2k + 1 = (k + 1)^2$
   ... $P(k+1)$

Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Tiled $8 \times 4$ square with $2 \times 2$ $L$-tiles. with a center hole.

Can we tile any $2^n \times 2^n$ with $L$-tiles (with a hole) for every $n$?

Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** The remainder of $2^n$ divided by 3 is 1.

Base case: true for $k = 0$, $2^0 = 1$

Ind Hyp: $2^{2k} - 3a + 1$ for integer $a$.

$$2^{2(k+1)} = 2^{2k} \times 2^2 = 4 \times 2^{2k} = 4 \times (3a + 1) = 12a + 3 + 1 = 3(4a + 1) + 1$$

$a$ integer $\implies (4a + 1)$ is an integer.

Hole can anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

**Better theorem ... better induction hypothesis!**

Base case: Sure. A tile is fine.

Induction Step:
1. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
   *Any $2^n \times 2^n$ square can be tiled with a hole anywhere.*

   Consider $2^{n+1} \times 2^{n+1}$ square.

   Use induction hypothesis in each.

   Use $L$-tile and ... we are done.

Strong Induction.

**Theorem:** Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

**Definition:** A prime $n$ has exactly 2 factors: 1 and $n$.

**Base Case:** $n = 2$.

**Induction Step:** $P(n) = "n can be written as a product of primes."

Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$.

$P(n)$ says nothing about $a, b$!

**Strong Induction Principle:** If $P(0)$ and

$$\forall k \in N \left( P(0) \land \ldots \land P(k) \right) \implies P(k+1)$$

then $\forall k \in N \left( P(k) \right)$.

$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots$

**Strong induction hypothesis:** "$a$ and $b$ are products of primes"

$\implies n + 1 = a \cdot b = \text{(factorization of a)} \cdot \text{(factorization of b)}$.

$n + 1$ can be written as the product of the prime factors.
Tournaments have long paths.

**Def:** A round robin tournament on $n$ players: every player $p$ plays every other player $q$, and either $p \rightarrow q$ ($p$ beats $q$) or $q \rightarrow p$ ($q$ beats $p$).

**Tournaments have short cycles**

**Def:** A cycle: a sequence of $p_1, \ldots, p_k$, $p_1 \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$.

**Theorem:** Any tournament that has a cycle has a cycle of length 3.

**Tournaments have short cycles**

There are two horses of the same color. Theorem: All horses have the same color.

**Base Case:** $P(1)$ - trivially true.

**New Base Case:** $P(k)$ for any $k$.

**Induction Hypothesis:** $P(k)$ - any $k$ horses have the same color.

**Induction step $P(k+1)$?**

- If $k$ have same color by $P(k)$: 1, 2, 2, 3, ..., $k$ $k+1$
- Second $k$ have same color by $P(k)$: 1, 2, 2, 3, ..., $k$ $k+1$
- A horse in the middle in common! 1, 2, 3, ..., $k$, $k+1$

All $k$ must have the same color! 1, 2, 3, ..., $k$, $k+1$

**How about $P(1) \Rightarrow P(2)$?**

Fix base case. There are two horses of the same color. ...Still doesn’t work!! (There are two horses is $\neq$ For all two horses!!!)

Of course it doesn’t work. As we will see, it is more subtle to catch errors in proofs of correct theorems!!
Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.
Any islander who knows they have green eyes must do an unpleasant ritual that day.
No islander knows there own eye color, but knows everyone elses.
All islanders have green eyes!
First rule of island: Don’t talk about eye color!
Visitor: “I see someone has green eyes.”
Result: Poll.
On day 100, they all do the ritual.
Why?

They know induction.

Thm: If there are \( n \) villagers with green eyes they do ritual on day \( n \).

Proof:
Base: \( n = 1 \). Person with green eyes does ritual on day 1.
Induction hypothesis:
If \( n \) people with green eyes, they would do ritual on day \( n \).
Induction step:
On day \( n + 1 \), a green eyed person sees \( n \) people with green eyes.
But they didn’t do the ritual.
So there must be \( n + 1 \) people with green eyes.
One of them, is me.
Sad.
Wait! Visitor added no information.

Common Knowledge.

Using knowledge about what other people’s knowledge (your eye color) is.
On day 1, everyone knows everyone sees more than zero.
On day 2, everyone knows everyone sees more than one.
... On day 99, everyone knows no one sees 98 since everyone knows everyone else does not see 97... On day 100, ...uh oh!
Another example:
Emperor’s new clothes!
No one knows other people see that he has no clothes.
Until kid points it out.

Summary: principle of induction.

Today: More induction.

\[
(\forall n)(\forall k \in \mathbb{N})(P(k) \Rightarrow P(k + 1)) \Rightarrow (\forall n \in \mathbb{N})(P(n))
\]
Statement to prove: \( P(n) \) for \( n \) starting from \( n_0 \)
Base Case: Prove \( P(n_0) \).
Ind. Step: Prove. For all values, \( n \geq n_0, P(n) \Rightarrow P(n + 1) \).
Statement is proven!

Strong Induction:

\[
(\forall n)(\forall k \in \mathbb{N})(P(n) \Rightarrow P(n + 1))) \Rightarrow (\forall n \in \mathbb{N})(P(n))
\]
Also Today: strengthened induction hypothesis.

Tiling Cory Hall Courtyard.

Summary: principle of induction.

Induction \equiv Recursion.

Not same as strong induction. E.g., used in product of primes proof.
Induction \equiv Recursion.