Some quibbles.

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What if the statement is only for $n \geq 3$?
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What if the statement is only for \( n \geq 3 \)?

\[ \forall n \in \mathbb{N}, (n \geq 3) \implies P(n) \]
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The induction principle works on the natural numbers. Proves statements of form: $\forall n \in \mathbb{N}, P(n)$. Yes.

What if the statement is only for $n \geq 3$?

$\forall n \in \mathbb{N}, (n \geq 3) \implies P(n)$

Restate as:
Some quibbles.

The induction principle works on the natural numbers.

Proves statements of form: \( \forall n \in \mathbb{N}, P(n) \).

Yes.

What if the statement is only for \( n \geq 3 \)?

\[ \forall n \in \mathbb{N}, (n \geq 3) \implies P(n) \]

Restate as:

\[ \forall n \in \mathbb{N}, Q(n) \text{ where } Q(n) = "(n \geq 3) \implies P(n)". \]
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What if the statement is only for \( n \geq 3 \)?

\[
\forall n \in \mathbb{N}, (n \geq 3) \implies P(n)
\]

Restate as:
\[
\forall n \in \mathbb{N}, Q(n) \text{ where } Q(n) = "(n \geq 3) \implies P(n)".
\]

Base Case: typically start at 3.
Some quibbles.

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Yes.

What if the statement is only for $n \geq 3$?

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$$\forall n \in \mathbb{N}, Q(n) \text{ where } Q(n) = "(n \geq 3) \implies P(n)".$$  

Base Case: typically start at 3.

Since $\forall n \in \mathbb{N}, Q(n) \implies Q(n+1)$ is trivially true before 3.
Some quibbles.

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Proves statements of form: $\forall n \in \mathbb{N}, P(n)$.
Yes.

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Base Case: typically start at 3.
Since $\forall n \in \mathbb{N}, Q(n) \implies Q(n + 1)$ is trivially true before 3.

Can you do induction over other things? Yes.
Some quibbles.

The induction principle works on the natural numbers. Proves statements of form: $\forall n \in \mathbb{N}, P(n)$.

Yes.

What if the statement is only for $n \geq 3$?

$\forall n \in \mathbb{N}, (n \geq 3) \implies P(n)$

Restate as:

$\forall n \in \mathbb{N}, Q(n)$ where $Q(n) = "(n \geq 3) \implies P(n)"$.

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Any set where any subset of the set has a smallest element.
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Any set where any subset of the set has a smallest element.

In some sense, the natural numbers.
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$. 

Instead of proof, let's write some code!

```python
def find_x_y(n):
    if n == 12:
        return (3, 0)
    elif n == 13:
        return (2, 1)
    elif n == 14:
        return (1, 2)
    elif n == 15:
        return (0, 3)
    else:
        (x_prime, y_prime) = find_x_y(n - 4)
        return (x_prime + 1, y_prime)
```

Prove: Given $n$, returns $(x, y)$ where $n = 4x + 5y$, for $n \geq 12$.

Base cases: $P(12)$, $P(13)$, $P(14)$, $P(15)$.

Yes.

Strong Induction step: Recursive call is correct: $P(n - 4) = \Rightarrow P(n)$.

$n - 4 = 4x' + 5y' = \Rightarrow n = 4(x' + 1) + 5y'$.

Slight differences: showed for all $n \geq 16$ that $\wedge n - 1 = 4P(i) = \Rightarrow P(n)$. 
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Base cases: P(12)
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Slight differences: showed for all \( n \geq 16 \) that \( \land_{i=4}^{n-1} P(i) \implies P(n) \).
Stable Matching Problem
Stable Matching Problem

- $n$ candidates and $n$ jobs.
Stable Matching Problem

- \( n \) candidates and \( n \) jobs.
- Each job has a ranked preference list of candidates.
Stable Matching Problem

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- Each candidate has a ranked preference list of jobs.
Stable Matching Problem

- $n$ candidates and $n$ jobs.
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.

How should they be matched?
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.
Lakers prefer Davis.
The best laid plans..

Consider the pairs..

▶ (Anthony) Davis and Pelicans
▶ (Lonzo) Ball and Lakers

Davis prefers the Lakers.
Lakers prefer Davis.
Uh..oh.
The best laid plans..

Consider the pairs..

- (Anthony) Davis and Pelicans
- (Lonzo) Ball and Lakers

Davis prefers the Lakers.
Lakers prefer Davis.
Uh..oh. Sad Lonzo and Pelicans.
So..

Produce a matching where there are no crazy moves!
So..

Produce a matching where there are no crazy moves!

**Definition:** A matching is disjoint set of $n$ job-candidate pairs.
So..

Produce a matching where there are no crazy moves!

**Definition:** A matching is disjoint set of \( n \) job-candidate pairs.

Example: A matching \( S = \{(Lakers, Ball); (Pelicans, Davis)\} \).
Produce a matching where there are no crazy moves!

**Definition:** A **matching** is disjoint set of $n$ job-candidate pairs.

Example: A matching $S = \{(\text{Lakers}, \text{Ball}); (\text{Pelicans}, \text{Davis})\}$.

**Definition:** A **rogue couple** $b, g^*$ for a pairing $S$: $b$ and $g^*$ prefer each other to their partners in $S$. 
Produce a matching where there are no crazy moves!

Definition: A matching is disjoint set of $n$ job-candidate pairs.
Example: A matching $S = \{(\text{Lakers, Ball}); (\text{Pelicans, Davis})\}$.

Definition: A rogue couple $b, g^*$ for a pairing $S$: $b$ and $g^*$ prefer each other to their partners in $S$.
Example: Davis and Lakers are a rogue couple in $S$. 
Example.

<table>
<thead>
<tr>
<th>Jobs</th>
<th>Candidates</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1 2 3</td>
</tr>
<tr>
<td></td>
<td>C A B</td>
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<tr>
<td>B</td>
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The Propose and Reject Algorithm.

Each Day:
1. Each job proposes to its favorite candidate on its list.
2. Each candidate rejects all but their favorite proposer (whom they put on a string).
3. Rejected job crosses rejecting candidate off its list.

Stop when each job gets exactly one proposal.

Does this terminate?

...produce a matching?

...a stable matching?

Do jobs or candidates do "better"?
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Do jobs or candidates do “better”?
Termination.
Termination.

Every non-terminated day a job crossed an item off the list.
Termination.

Every non-terminated day a job **crossed** an item off the list.

Total size of lists?
Termination.

Every non-terminated day a job crossed an item off the list.
Total size of lists? $n$ jobs, $n$ length list.
Termination.

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Total size of lists? $n$ jobs, $n$ length list. $n^2$
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Total size of lists? $n$ jobs, $n$ length list. $n^2$
Terminates in $\leq n^2$ steps!
It gets better every day for candidates.

Improvement Lemma: It just gets better for candidates.

If on day $t$, a candidate has a job $b$ on a string, any job, $b'$, on candidate $g$'s string for any day $t'$ where $t' > t$ is at least as good as $b$.

Example: Candidate "Alice" has job "Amalgamated Concrete" on string on day 5. She has job "Amalgamated Asphalt" on string on day 7. Does Alice prefer "Amalgamated Asphalt" or "Amalgamated Concrete"?

$g$ - 'Alice', $b$ - 'Am. Con.', $b'$ - 'Am. Asph.', $t = 5$, $t' = 7$.

Improvement Lemma says she prefers "Amalgamated Asphalt".

Day 10: Can Alice have "Amalgamated Asphalt" on her string? Yes. Alice prefers day 10 job as much as day 7 job. Here, $b = b'$.

Why is lemma true? Proof Idea: She can always keep the previous job on the string.
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Improvement Lemma: It just gets better for candidates
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**Improvement Lemma:** It just gets better for candidates

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$P(k)$ - “job on $g$’s string is at least as good as $b$ on day $t + k$”
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\( P(0) \)- true. Candidate has \( b \) on string.
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\( P(k) \)- “job on \( g \)'s string is at least as good as \( b \) on day \( t + k \)”

\( P(0) \)- true. Candidate has \( b \) on string.

Assume \( P(k) \). Let \( b' \) be job **on string** on day \( t + k \).
Improvement Lemma

Improvement Lemma: It just gets better for candidates.

If on day $t$ a candidate $g$ has a job $b$ on a string, any job, $b'$, on $g$’s string for any day $t' > t$ is at least as good as $b$.

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Assume $P(k)$. Let $b'$ be job **on string** on day $t + k$.

On day $t + k + 1$, job $b'$ comes back.
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On day \( t + k + 1 \), job \( b' \) comes back.

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That is,
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That is, $b' \geq b$ by induction hypothesis.
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That is, $b' \geq b$ by induction hypothesis.

And $b''$ is better than $b'$ by algorithm.
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$\implies$ Candidate does at least as well as with $b$. 
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$P(k) \implies P(k+1)$. 
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And by principle of induction, lemma holds for every day after $t$. 
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\( P(k) \implies P(k + 1) \).

And by principle of induction, lemma holds for every day after \( t \).
Poll

Question: It just gets better for candidates, because?

(A) Induction on days.
(B) When the economy is good.
(C) The candidate can always keep the job on the string.
Matching when done.

**Lemma:** Every job is matched at end.
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**Lemma:** Every job is matched at end.

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If not, a job $b$ must have been rejected $n$ times. Every candidate has been proposed to by $b$, and Improvement lemma

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and each job is on at most one string.
Lemma: Every job is matched at end.

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$n$ candidates and $n$ jobs.
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Contradiction.
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If not, a job $b$ must have been rejected $n$ times.
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$$\Rightarrow \text{ each candidate has a job on a string.}$$

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$n$ candidates and $n$ jobs. Same number of each.

$$\Rightarrow b \text{ must be on some candidate’s string!}$$

Contradiction.
Question: The argument for termination uses.

(A) Implies: no unmatched job at end.

(B) Improvement Lemma: every candidate matched.

(C) Algorithm: unmatched job would ask everyone.

(D) Implies: every one gets their favorite job.
Matching is Stable.

**Lemma:** There is no rogue couple for the matching formed by traditional marriage algorithm.
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\begin{align*}
  b^* & \quad \quad g^* \\
  b & \quad \quad g
\end{align*}
\]

Job \(b\) proposes to \(g^*\) before proposing to \(g\).
So \(g^*\) rejected \(b\) (since he moved on)
By improvement lemma, \(g^*\) prefers \(b^*\) to \(b\).

**Contradiction!**
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\hline
 g^* \\
\hline
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Assume there is a rogue couple; $(b, g^*)$

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\text{b}^* & \quad \quad \quad \text{g}^* \\
\text{b} & \quad \quad \quad \text{g}
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\]

$b$ prefers $g^*$ to $g$.

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Contradiction!
Good for jobs? candidates?

Is the Job-Proposes better for jobs?
Good for jobs? candidates?

Is the Job-Proposes better for jobs? for candidates?

Definition: A matching is $x$-optimal if $x'$'s partner is its best partner in any stable pairing.

Definition: A matching is $x$-pessimal if $x'$'s partner is its worst partner in any stable pairing.

Definition: A matching is job optimal if it is $x$-optimal for all jobs $x$.

..and so on for job pessimal, candidate optimal, candidate pessimal.

Claim: The optimal partner for a job must be first in its preference list. True? False?

Subtlety here: Best partner in any stable matching. As well as you can be in a globally stable solution!

Question: Is there a job or candidate optimal matching? Is it possible: $b$-optimal pairing different from the $b'$-optimal matching! Yes? No?
Good for jobs? candidates?

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**Definition:** A matching is **x-optimal** if x’s partner is its best partner in any **stable** pairing.

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Yes?  
No?

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Claim: The optimal partner for a job must be first in its preference list.

True? False? False!

Subtlety here: Best partner in any stable matching.

As well as you can be in a globally stable solution!

**Question:** Is there a job or candidate optimal matching?

Is it possible:

$b$-optimal pairing different from the $b'$-optimal matching!

Yes? No?
Question: The SMA produces a stable pairing is a proof by?

(A) Contradiction.
(B) Uses the improvement lemma.
(C) Induction.
(D) Direct.
Understanding Optimality: by example.

A: 1,2  
1: A,B
B: 1,2  
2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.
Optimal for B? Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.
Also optimal for A, 1 and 2.

Also pessimal for A, B, 1 and 2.

Pairing S: (A, 1), (B, 2).
Stable? Yes.

Pairing T: (A, 2), (B, 1).
Also Stable.
Which is optimal for A? S
Which is optimal for B? S
Which is optimal for 1? T
Which is optimal for 2? T
Understanding Optimality: by example.

A: 1,2  
B: 1,2  

1: A,B  
2: B,A

Consider pairing: \((A, 1), (B, 2)\).
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable?
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.
Understanding Optimality: by example.

A: 1,2  
1: A,B

B: 1,2  
2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Understanding Optimality: by example.

A:  1,2  
B:  1,2  

Consider pairing: \((A, 1), (B, 2)\).

Stable? Yes.

Optimal for \(B\)?
Notice: only one stable pairing.
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: \((A, 1), (B, 2)\).

Stable? Yes.

Optimal for \(B\)?
Notice: only one stable pairing.
So this is the best \(B\) can do in a stable pairing.
Understanding Optimality: by example.

A: 1,2 1: A,B
B: 1,2 2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
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Understanding Optimality: by example.

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Consider pairing: (A,1), (B,2).

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So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.
Understanding Optimality: by example.

A:  1,2  1: A,B
B:  1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A:  1,2  1: B,A
B:  2,1  2: A,B
Understanding Optimality: by example.

A: 1,2  
B: 1,2  

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2  
B: 2,1  

Pairing S: (A, 1), (B, 2).
Understanding Optimality: by example.

A: 1,2  
B: 1,2  

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2  
B: 2,1  

Pairing S: (A, 1), (B, 2). Stable?
Understanding Optimality: by example.

A: 1,2 1: A,B
B: 1,2 2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
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Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2 1: B,A
B: 2,1 2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Understanding Optimality: by example.

A: 1,2  
B: 1,2

Consider pairing: \((A, 1), (B, 2)\).

Stable? Yes.

Optimal for \(B\)?
Notice: only one stable pairing.
So this is the best \(B\) can do in a stable pairing.
So optimal for \(B\).

Also optimal for \(A\), 1 and 2. Also pessimal for \(A, B, 1\) and 2.

A: 1,2  
B: 2,1

Pairing \(S\): \((A, 1), (B, 2)\). Stable? Yes.
Understanding Optimality: by example.

A: 1,2  
B: 1,2

1: A,B 
2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2  
B: 2,1

1: B,A 
2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1).
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2  1: B,A
B: 2,1  2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.
Understanding Optimality: by example.

A: 1,2 1: A,B
B: 1,2 2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A, B, 1 and 2.

A: 1,2 1: B,A
B: 2,1 2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.
Which is optimal for A?
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?

Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A: 1,2  1: B,A
B: 2,1  2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2  1: B,A
B: 2,1  2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S
Which is optimal for B?
Understanding Optimality: by example.

A:  1,2  1:  A,B
B:  1,2  2:  B,A

Consider pairing: \((A, 1), (B, 2)\).

Stable? Yes.

Optimal for \(B\)?
Notice: only one stable pairing.
So this is the best \(B\) can do in a stable pairing.
So optimal for \(B\).

Also optimal for \(A\), 1 and 2. Also pessimal for \(A,B,1\) and 2.

A:  1,2  1:  B,A
B:  2,1  2:  A,B

Pairing \(S\): \((A, 1), (B, 2)\). Stable? Yes.

Pairing \(T\): \((A, 2), (B, 1)\). Also Stable.

Which is optimal for \(A\)? \(S\) Which is optimal for \(B\)? \(S\)
Understanding Optimality: by example.

A:  1,2  1:  A,B
B:  1,2  2:  B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B, 1 and 2.

A:  1,2  1:  B,A
B:  2,1  2:  A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S
Which is optimal for B? S
Which is optimal for 1?
Understanding Optimality: by example.

A: 1,2  
B: 1,2

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B? Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2  
B: 2,1

Pairing S: (A, 1), (B, 2). Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S Which is optimal for B? S
Which is optimal for 1? T
Understanding Optimality: by example.

A: 1,2 1: A,B
B: 1,2 2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
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So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2 1: B,A
B: 2,1 2: A,B

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S
Which is optimal for B? S
Which is optimal for 1? T
Which is optimal for 2?
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A: 1,2  1: B,A
B: 2,1  2: A,B

Pairing S: (A, 1), (B, 2).  Stable? Yes.

Pairing T: (A, 2), (B, 1). Also Stable.

Which is optimal for A? S  Which is optimal for B? S
Which is optimal for 1? T  Which is optimal for 2? T
Job Propose and Candidate Reject is optimal!

For jobs?

Theorem:
Job Propose and Reject produces a job-optimal pairing.

Proof:
Assume not: there is a job $b$ does not get optimal candidate, $g$.
There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^* -$ knocks $b$ off of $g$'s string on day $t$.

By choice of $t$, $b^*$ prefers $g$ to $b$.

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.

So $S$ is not a stable pairing.

Notes:
$S$ - stable.
$(b^*, g^*) \in S$.

But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...

Induction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

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Job Propose and Reject produces a job-optimal pairing.

Proof:
Assume not: there is a job \( b \) does not get optimal candidate, \( g \).
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Let \( t \) be first day job \( b \) gets rejected by its optimal candidate \( g \) who it is paired with in stable pairing \( S \).

\( b^* \) knocks \( b \) off of \( g \)'s string on day \( t \) \( \Rightarrow \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.

\( \Rightarrow \) \( b^* \) prefers \( g \) to its partner \( g^* \) in \( S \).

Rogue couple for \( S \).

So \( S \) is not a stable pairing.

Contradiction.

Notes:
\( S \) - stable.
\( (b^*, g^*) \) \( \in \) \( S \).

But \( (b^*, g) \) is rogue couple!

Used Well-Ordering principle...

Induction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not:

There is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^*$ knocks $b$ off of $g$'s string on day $t$.

By choice of $t$, $b^*$ prefers $g$ to $b$.

$b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.

So $S$ is not a stable pairing.

Contradiction.

Notes:
$S$ - stable.
$S$ - stable.
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Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

Notes:
- $S$ - stable.
- $(b^*, g^*) \in S$.
- But $(b^*, g^*)$ is rogue couple!
- Used Well-Ordering principle...
- Induction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job \( b \) does not get optimal candidate, \( g \).

There is a stable pairing \( S \) where \( b \) and \( g \) are paired.

Let \( t \) be first day job \( b \) gets rejected
   by its optimal candidate \( g \) who it is paired with
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
by its optimal candidate $g$ who it is paired with
in stable pairing $S$. 
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
by its optimal candidate $g$ who it is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t$
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job \( b \) does not get optimal candidate, \( g \).

There is a stable pairing \( S \) where \( b \) and \( g \) are paired.

Let \( t \) be first day job \( b \) gets rejected by its optimal candidate \( g \) who it is paired with in stable pairing \( S \).

\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.
**Job Propose and Candidate Reject is optimal!**

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t$ $\implies$ $g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$. 

Notes:
$S$ - stable.
$(b^*, g^*) \in S$.
But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...
Induction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
by its optimal candidate $g$ who it is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t$ $\implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$. 
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

Theorem: Job Propose and Reject produces a job-optimal pairing.

Proof:
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
by its optimal candidate $g$ who it is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t$ $\implies$ $g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
by its optimal candidate $g$ who it is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$'s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.
**Job Propose and Candidate Reject is optimal!**

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
  by its optimal candidate $g$ who it is paired with
  in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job \( b \) does not get optimal candidate, \( g \).

There is a stable pairing \( S \) where \( b \) and \( g \) are paired.

Let \( t \) be first day job \( b \) gets rejected
by its optimal candidate \( g \) who it is paired with
in stable pairing \( S \).

\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.

\( \implies \) \( b^* \) prefers \( g \) to its partner \( g^* \) in \( S \).

Rogue couple for \( S \).
So \( S \) is not a stable pairing. Contradiction.

Notes:
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job \( b \) does not get optimal candidate, \( g \).

There is a stable pairing \( S \) where \( b \) and \( g \) are paired.

Let \( t \) be first day job \( b \) gets rejected by its optimal candidate \( g \) who it is paired with in stable pairing \( S \).

\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.

\( \implies \) \( b^* \) prefers \( g \) to its partner \( g^* \) in \( S \).

Rogue couple for \( S \).
So \( S \) is not a stable pairing. Contradiction.

Notes: \( S \) - stable.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected
by its optimal candidate $g$ who it is paired with
in stable pairing $S$.

$b^*$ - knocks $b$ off of $g$’s string on day $t$ $\implies$ $g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^*, g^*) \in S$. 
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job \( b \) does not get optimal candidate, \( g \).

There is a stable pairing \( S \) where \( b \) and \( g \) are paired.

Let \( t \) be first day job \( b \) gets rejected
   by its optimal candidate \( g \) who it is paired with
   in stable pairing \( S \).

\( b^* \) - knocks \( b \) off of \( g \)'s string on day \( t \) \( \implies \) \( g \) prefers \( b^* \) to \( b \)

By choice of \( t \), \( b^* \) likes \( g \) at least as much as optimal candidate.

\( \implies \) \( b^* \) prefers \( g \) to its partner \( g^* \) in \( S \).

Rogue couple for \( S \).
So \( S \) is not a stable pairing. Contradiction. \( \square \)

Notes: \( S \) - stable. \((b^*, g^*) \in S\). But \((b^*, g)\)
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Assume not: there is a job $b$ does not get optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^\ast$ - knocks $b$ off of $g$’s string on day $t \implies g$ prefers $b^\ast$ to $b$

By choice of $t$, $b^\ast$ likes $g$ at least as much as optimal candidate.

$\implies b^\ast$ prefers $g$ to its partner $g^\ast$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^\ast, g^\ast) \in S$. But $(b^\ast, g)$ is rogue couple!
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
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by its optimal candidate $g$ who it is paired with
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$b^*$ - knocks $b$ off of $g$’s string on day $t$ $\implies$ $g$ prefers $b^*$ to $b$

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\implies b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.
So $S$ is not a stable pairing. Contradiction.

Notes: $S$ - stable. $(b^*, g^*) \in S$. But $(b^*, g)$ is rogue couple!

Used Well-Ordering principle...
Job Propose and Candidate Reject is optimal!

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Notes: \( S \) - stable. \( (b^*, g^*) \in S \). But \( (b^*, g) \) is rogue couple!

Used Well-Ordering principle...Induction.
How about for candidates?

Theorem: Job Propose and Reject produces candidate-pessimal pairing. 

\( T \) – pairing produced by JPR. 
\( S \) – worse stable pairing for candidate \( g \).

In \( T \), \((g, b)\) is pair. 
In \( S \), \((g, b^*)\) is pair. 
\( g \) prefers \( b \) to \( b^* \).

\( T \) is job optimal, so \( b \) prefers \( g \) to its partner in \( S \). 
\( (g, b) \) is Rogue couple for \( S \). 
\( S \) is not stable. 
Contradiction.

Notes: Not really induction. 
Structural statement: Job optimality \( \Rightarrow \) Candidate pessimality.
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  - In \( T \), \((g, b)\) is pair.
  - In \( S \), \((g, b^∗)\) is pair.
  - \( g \) prefers \( b \) to \( b^∗ \).
  - \( T \) is job optimal, so \( b \) prefers \( g \) to its partner in \( S \).
  - \((g, b)\) is Rogue couple for \( S \).
  - \( S \) is not stable.

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Structural statement: Job optimality $\implies$ Candidate pessimality.
Quick Questions.

How does one make it better for candidates?
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Propose and Reject - stable matching algorithm. One side proposes.
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- Jobs Propose $\implies$ job optimal.
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Hospital optimal....
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The method was used to match residents to hospitals. Hospital optimal.... ..until 1990’s...Resident optimal. Another variation: couples.
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