Climb an infinite ladder?
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\[ P(0) \]

\[ P(1) \]

\[ P(2) \]

\[ P(3) \]

\[ \forall k, P(k) = \Rightarrow P(k+1) \]

\[ P(0) = \Rightarrow P(1) = \Rightarrow P(2) = \Rightarrow P(3) = \ldots \]

\[ \forall n \in \mathbb{N}, P(n) \]

Your favorite example of forever...
Climb an infinite ladder?

\[ P(0) \]
\[ \forall k, P(k) \implies P(k+1) \]
Climb an infinite ladder?

∀k, P(k) ⇒ P(k + 1)
P(0) ⇒ P(1) ⇒ P(2)

Your favorite example of forever or the natural numbers...
Climb an infinite ladder?

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P(0) \\
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∀k, P(k) ⟹ P(k + 1)
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\[ P(0) \implies P(1) \implies P(2) \implies P(3) \ldots \]

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Your favorite example of forever… or the natural numbers…
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\[ P(n), \quad P(n+1), \quad \ldots \]

\[ \forall k, P(k) \implies P(k+1) \]

\[ P(0) \implies P(1) \implies P(2) \implies P(3) \ldots \]
Climb an infinite ladder?

∀n ∈ N, P(n) ⇒ P(n+1)

P(0) ⇒ P(1) ⇒ P(2) ⇒ P(3) …
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∀n ∈ N P(n)

P(0) ⇒ P(1) ⇒ P(2) ⇒ P(3) ...
Climb an infinite ladder?

∀ \( n \in \mathbb{N} \), \( P(n) \) → \( P(n+1) \) → \( P(n+2) \) → \( P(n+3) \)...

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Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.
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Any islander who knows they have green eyes must “leave the island” that day.
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Any islander who knows they have green eyes must “leave the island” that day.
No islander knows there own eye color, but knows everyone elses.

Visitor: “I see someone has green eyes.”

Result: What happens?

(A) Nothing, no information was added.
(B) Information was added, maybe?
(C) They all leave the island.
(D) They all leave the island on day 100.

On day 100, they all leave.
Why?
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Why?
They know induction.

Thm: If there are \( n \) villagers with green eyes they leave on day \( n \).
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Induction hypothesis:
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Thm: If there are $n$ villagers with green eyes they leave on day $n$.

**Proof:**
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If $n$ people with green eyes, they would leave on day $n$. 

SAD.

Wait! Visitor added no information.
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**Proof:**

Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:

If $n$ people with green eyes, they would leave on day $n$.

Induction step:

On day $n + 1$, a green eyed person sees $n$ people with green eyes.
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But they didn’t leave.
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On day $n+1$, a green eyed person sees $n$ people with green eyes.
But they didn’t leave.
So there must be $n + 1$ people with green eyes.

I have to leave the island.
I like the island.
SAD.

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Wait! Visitor added no information.
Quick Poll.

If 66 villagers out of the 100 had green eyes, what would happen?
Quick Poll.

If 66 villagers out of the 100 had green eyes, what would happen?

(A) Everyone would leave on the first day.
(B) The villagers with green eyes would leave on the 66th day.
(C) All the villagers would leave on the 66th day.
(D) The green eyed villagers would leave on the 100th day.
(E) All the villagers would leave on the 100th day.
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(B)
Common Knowledge.

Using knowledge about what other people’s knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, everyone knows no one sees 98 since everyone knows everyone else does not see 97...

On day 100, uh oh!

Another example:

Emperor’s new clothes!

No one knows other people see that he has no clothes. Until kid points it out.

Political arguments?

Everyone knows the arguments, everyone knows everyone knows the arguments.......

The islanders didn’t talk.

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Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$. 

Instead of proof, let's write some code!

```python
def find_x_y(n):
    if (n==12) return (3,0)
    elif (n==13): return(2,1)
    elif (n==14): return(1,2)
    elif (n==15): return(0,3)
    else:
        (x',y') = find_x_y(n-4)
        return(x'+1,y')
```

Prove: Given $n$, returns $(x, y)$ where $n = 4x + 5y$, for $n \geq 12$. 

Base cases: $P(12), P(13), P(14), P(15)$. 

Yes. 

Strong Induction step: Recursive call is correct: $P(n-4) = \Rightarrow P(n)$. 

$n - 4 = 4x' + 5y' = \Rightarrow n = 4(x' + 1) + 5y'$. 

Slight differences: showed for all $n \geq 16$ that $n - 1_i = 4P(i) = \Rightarrow P(n)$. 

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def find-x-y(n):
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        (x',y') = find-x-y(n-4)
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Prove: Given $n$, returns $(x, y)$ where $n = 4x + 5y$, for $n \geq 12$.

Strong Induction and Recursion.

Thm: For every natural number \( n \geq 12 \), \( n = 4x + 5y \).

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$n - 4 = 4x' + 5y' \implies n = 4(x' + 1) + 5(y')$
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Base cases: \( P(12) \) , \( P(13) \) , \( P(14) \) , \( P(15) \). Yes.

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Recursive call is correct: \( P(n - 4) \Rightarrow P(n) \).
\[ n - 4 = 4x’ + 5y’ \Rightarrow n = 4(x’ + 1) + 5(y’) \]
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Slight differences: showed for all \( n \geq 16 \) that \( \land_{i=4}^{n-1} P(i) \implies P(n) \).
Tidying up induction.

The induction principle works on the natural numbers.
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Base Case: typically start at 3. Since \( \forall n \in \mathbb{N}, Q(n) \implies Q(n+1) \) is trivially true before 3.

Can you do induction over other things? Yes.

Any set where any subset of the set has a smallest element. In some sense, the natural numbers.
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Stable Matching Problem

- $n$ candidates and $n$ jobs.
- Each job has a ranked preference list of candidates.
- Each candidate has a ranked preference list of jobs.
- How should they be matched?
  - Maximize total satisfaction.
  - Maximize number of first choices.
  - Maximize worse off.
  - Minimize difference between preference ranks.
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The best laid plans..

Consider the pairs..

- Cal Bears and the Pac-12
- Wake Forest and the ACC
The best laid plans..

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- Cal Bears and the Pac-12
- Wake Forest and the ACC

Cal Bears prefers the ACC
The best laid plans..

Consider the pairs..

- Cal Bears and the Pac-12
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Cal Bears prefers the ACC
The ACC prefers Cal Bears.
Consider the pairs..

- Cal Bears and the Pac-12
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Cal Bears prefers the ACC
The ACC prefers Cal Bears.
Uh..oh.
The best laid plans..

Consider the pairs..

- Cal Bears and the Pac-12
- Wake Forest and the ACC

Cal Bears prefers the ACC
The ACC prefers Cal Bears.
Uh..oh. Sad Pac-12, (and Wake Forest.)
Produce a matching where there are no crazy moves!
So..

Produce a matching where there are no crazy moves!

**Definition:** A matching is disjoint set of $n$ job-candidate pairs.
So..

Produce a matching where there are no crazy moves!

**Definition:** A **matching** is disjoint set of $n$ job-candidate pairs.

Example: A matching

$S = \{(CalBears, Pac - 12); (WakeForest, ACC)\}.$
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**Definition:** A **matching** is disjoint set of $n$ job-candidate pairs.

Example: A matching $S = \{(\text{CalBears}, \text{Pac} - 12); (\text{WakeForest}, \text{ACC})\}$.

**Definition:** A **rogue couple** $b, g^*$ for a pairing $S$: $b$ and $g^*$ prefer each other to their partners in $S$. 
Produce a matching where there are no crazy moves!

**Definition:** A *matching* is disjoint set of *n* job-candidate pairs.

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Example: Cal Bears and the ACC are a rogue couple in \( S \).

Not a great example of stable matching, but interesting exercise in “selfish” incentives.
A stable matching??

Given a set of preferences.
A stable matching??

Given a set of preferences.
Is there a stable matching?
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How does one find it?
A stable matching??

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Consider a single type version: stable roommates.

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<tr>
<th></th>
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A   B   C   D
B   C   A   D
C   A   B   D
D   A   B   C

A ---- B
     |
     C ---- D
A stable matching??

Given a set of preferences.

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Consider a single type version: stable roommates.

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\begin{array}{c|cccc}
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A stable matching is found when there are no unmatched pairs who both prefer each other to their current partners. In this example, A and B are matched, as are C and D.
A stable matching??

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How does one find it?

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\[
\begin{array}{c|cccc}
A & B & C & D \\
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C & A & B & D \\
D & A & B & C \\
\end{array}
\]

\[
A \quad B
\]

\[
C \quad D
\]
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\[ C \rightarrow D \\ A \rightarrow B \]
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A stable matching is found when there are no unstable pairs. An unstable pair is one where one member prefers the other over their current partner, and the other member prefers the first over their current partner.
The Propose and Reject Algorithm.

Each Day:

1. Each job proposes to its favorite candidate on its list.
2. Each candidate rejects all but their favorite proposer (whom they put on a string.)
3. Rejected job crosses rejecting candidate off its list.

Stop when each job gets exactly one proposal.
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<th>Candidates</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1 2 3</td>
<td>1 C A B</td>
</tr>
<tr>
<td>B</td>
<td>1 2 3</td>
<td>2 A B C</td>
</tr>
<tr>
<td>C</td>
<td>2 1 3</td>
<td>3 A C B</td>
</tr>
</tbody>
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<thead>
<tr>
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<tbody>
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<tr>
<td>B</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
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<table>
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<tr>
<th></th>
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<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
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<tr>
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</table>
Example.

| Jobs | | Candidates |
|------| |------------|
| A    | 1 2 3 | 1 C A B |
| B    | 1 2 3 | 2 A B C |
| C    | 2 1 3 | 3 A C B |

<table>
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| Jobs | | Candidates | |
|------|------|-------------|
| A    | X    | 1           | C   |
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The Propose and Reject Algorithm.

Each Day:
1. Each job proposes to its favorite candidate on its list.
2. Each candidate rejects all but their favorite proposer (whom they put on a string.)
3. Rejected job crosses rejecting candidate off its list.

Stop when each job gets exactly one proposal.

Does this terminate?...produce a matching?...a stable matching?

Do jobs or candidates do “better”?
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Do jobs or candidates do “better”?
Termination.
Every non-terminated day a job crossed an item off the list.
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Every non-terminated day a job crossed an item off the list.
Total size of lists?
Every non-terminated day a job crossed an item off the list.
Total size of lists? $n$ jobs, $n$ length list.
Termination.

Every non-terminated day a job \textit{crossed} an item off the list.

Total size of lists? \( n \) jobs, \( n \) length list. \( n^2 \)
Termination.

Every non-terminated day a job crossed an item off the list.
Total size of lists? $n$ jobs, $n$ length list. $n^2$
Terminates in $\leq n^2$ steps!
It gets better every day for candidates.
It gets better every day for candidates.

Improvement Lemma: It just gets better for candidates

If on day $t$ a candidate $g$ has a job $b$ on a string, any job, $b'$, on candidate $g$'s string for any day $t'$ > $t$ is at least as good as $b$.

Example: Candidate "Alice" has job "Amalmagated Concrete" on string on day 5. She has job "Amalmagated Asphalt" on string on day 7. Does Alice prefer "Amalmagated Asphalt" or "Amalmagated Concrete"?

Improvement Lemma says she prefers "Amalmagated Asphalt".

Day 10: Can Alice have "Amalmagated Asphalt" on her string? Yes. Alice prefers day 10 job as much as day 7 job. Here, $b = b'$.

Why is lemma true?

Proof Idea: She can always keep the previous job on the string.
It gets better every day for candidates.

**Improvement Lemma**: It just gets better for candidates

If on day $t$ a candidate $g$ has a job $b$ on a string,
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**Improvement Lemma: It just gets better for candidates**
If on day $t$ a candidate $g$ has a job $b$ on a string, any job, $b'$, on candidate $g$'s string for any day $t' > t$
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If on day $t$ a candidate $g$ has a job $b$ on a string, any job, $b'$, on candidate $g$'s string for any day $t' > t$ is at least as good as $b$.

Example: Candidate “Alice” has job “Amalgamated Concrete” on string on day 5.

She has job “Amalgamated Asphalt” on string on day 7.

Does Alice prefer “Amalgamated Asphalt” or “Amalgamated Concrete”?

$g$ - ’Alice’, $b$ - ’Am. Con.’, $b'$ - ’Am. Asph.’, $t = 5$, $t' = 7$.

Improvement Lemma says she prefers ’Amalgamated Asphalt’.

Day 10: Can Alice have “Amalgamated Asphalt” on her string? Yes.

Alice prefers day 10 job as much as day 7 job.  Here, $b = b'$.

Why is lemma true?

Proof Idea:
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Proof:

$P(0)$ – true. Candidate has $b$ on string.

Assume $P(k)$. Let $b'$ be job on string on day $t + k$. On day $t + k + 1$, job $b'$ comes back.

Candidate $g$ can choose $b'$, or do better with another job, $b''$. That is, $b' \geq b$ by induction hypothesis.

And $b''$ is better than $b'$ by algorithm.

$\Rightarrow$ Candidate does at least as well as with $b$.

$P(k) = \Rightarrow P(k+1)$.

And by principle of induction, lemma holds for every day after $t$. 

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Improvement Lemma

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Poll

Question: It just gets better for candidates,

(A) By induction on days.

(B) When the economy is good.

(C) The candidate can always keep the job on the string.

Sure on (B), but that’s Econ not CS.

To be sure, stable matching is from Econ. Professor’s: Gale and Shapley.

Economics: Study of choice.
Poll

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(A) and (C).

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Matching when done.

**Lemma:** Every job is matched at end.
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**Lemma**: Every job is matched at end.

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If not, a job $b$ must have been rejected $n$ times. Every candidate has been proposed to by $b$, and Improvement lemma
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$n$ candidates and $n$ jobs.
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$\implies b$ must be on some candidate’s string!
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\[ \Rightarrow \text{each candidate has a job on a string}. \]

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Contradiction.
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Poll: The argument for termination ...

(A) Implies: no unmatched job at end.
(B) Uses Improvement Lemma: every candidate matched.
(C) From Algorithm: unmatched job would ask everyone.
(D) Implies: every one gets their favorite job.
Matching is Stable.

**Lemma:** There is no rogue couple for the matching formed by traditional marriage algorithm.
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\(b\) proposes to \(g^*\) before proposing to \(g\).

So \(g^*\) rejected \(b\) (since he moved on).

By improvement lemma, \(g^*\) prefers \(b^*\) to \(b\).

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Question: Proof of Job Propose and Reject a stable pairing uses?

(A) Contradiction.

(B) Uses the improvement lemma.

(C) Induction.

(D) Direct.

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Good for jobs? candidates?

Is the Job-Proposes better for jobs?
Good for jobs? candidates?

Is the Job-Proposes better for jobs? for candidates?

Definition: A matching is $x$-optimal if $x$’s partner is its best partner in any stable pairing.

Definition: A matching is $x$-pessimal if $x$’s partner is its worst partner in any stable pairing.

Definition: A matching is job optimal if it is $x$-optimal for all jobs $x$.

...and so on for job pessimal, candidate optimal, candidate pessimal.

Claim: The optimal partner for a job must be first in its preference list.

True? False?

False!

Subtlety here: Best partner in any stable matching. As well as you can be in a globally stable solution!

Question: Is there a job or candidate optimal matching?

Is it possible: $b$-optimal pairing different from the $b'$-optimal matching!

Yes? No?
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..and so on for job pessimal, candidate optimal, candidate pessimal.

Claim: The optimal partner for a job must be first in its preference list.

True? False?
Good for jobs? candidates?

Is the Job-Proposes better for jobs? for candidates?

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Yes? No?
Good for jobs? candidates?

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Yes? No?
Understanding Optimality: by example.

<table>
<thead>
<tr>
<th></th>
<th>1,2</th>
<th>1:</th>
<th>2:</th>
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</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>A,B</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td>B,A</td>
</tr>
</tbody>
</table>

Consider pairing: (A,1), (B,2).

Stable?
Yes.

Optimal for B?
Notice: only one stable pairing.
If (A,2) are pair, (A,1) is rogue couple.
So this is the best B can do in a stable pairing.
So optimal for B.
Also optimal for A, 1 and 2.
Also pessimal for A, B, 1 and 2.

Pairing S: (A,1), (B,2).
Stable?
Yes.

Pairing T: (A,2), (B,1).
Also Stable.
Which is optimal for A?
S
Which is optimal for B?
S
Which is optimal for 1?
T
Which is optimal for 2?
T
Pessimality?
Understanding Optimality: by example.

A: 1,2  
1: A,B
B: 1,2  
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Consider pairing: (A,1), (B,2).

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Pairing S: (A,1), (B,2).

Stable? Yes.

Pairing T: (A,2), (B,1).

Also Stable.

Which is optimal for A? S
Which is optimal for B? S
Which is optimal for 1? T
Which is optimal for 2? T

Pessimality?
Understanding Optimality: by example.

A: 1,2  1: A,B
B: 1,2  2: B,A

Consider pairing: (A,1), (B,2).

Stable?
Understanding Optimality: by example.

A: 1,2  
1: A,B

B: 1,2  
2: B,A

Consider pairing: (A,1), (B,2).

Stable? Yes.
Understanding Optimality: by example.

A: 1,2 1: A,B
B: 1,2 2: B,A

Consider pairing: (A,1),(B,2).

Stable? Yes.

Optimal for B?
Understanding Optimality: by example.

Consider pairing: 

\((A, 1), (B, 2)\).

Stable? Yes.

Optimal for \(B\)?

Notice: only one stable pairing.

\(\text{Pairing S: } \{(A, 1), (B, 2)\}\).

\(\text{Pairing T: } \{(A, 2), (B, 1)\}\).

Which is optimal for \(A\)?

Which is optimal for \(B\)?

Which is optimal for \(1\)?

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Which is optimal for 1?

Which is optimal for 2?

Pessimality?
Understanding Optimality: by example.

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B:  1,2  2:  B, A

Consider pairing: (A, 1), (B, 2).

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Understanding Optimality: by example.

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B: 1,2

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Pairing \(S\): \((A, 1), (B, 2)\). Stable? Yes.
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A: 1,2  
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Pairing \(S\): \((A, 1), (B, 2)\). Stable? Yes.

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Which is optimal for \(A\)?
Understanding Optimality: by example.

A: 1,2  
B: 1,2

Consider pairing: (A, 1), (B, 2).

Stable? Yes.

Optimal for B?
Notice: only one stable pairing. If (A, 2) are pair, (A, 1) is rogue couple.
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So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A, B, 1 and 2.

A: 1,2  
B: 2,1

Pairing S: (A, 1), (B, 2). Stable? Yes.
Pairing T: (A, 2), (B, 1). Also Stable.
Which is optimal for A? S
Understanding Optimality: by example.

A: 1,2  
1: A,B

B: 1,2  
2: B,A

Consider pairing: (A,1), (B,2).

Stable? Yes.

Optimal for B? 
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A: 1,2  
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B: 2,1  
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Pairing T: (A,2), (B,1). Also Stable.

Which is optimal for A? S  Which is optimal for B?
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A: 1,2 1: A,B
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Which is optimal for 1? T
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Pairing \(T\): \((A, 2), (B, 1)\). Also Stable.

Which is optimal for \(A\)? \(S\) \nWhich is optimal for \(B\)? \(S\)
Which is optimal for 1? \(T\) \nWhich is optimal for 2?
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Which is optimal for A? S
Which is optimal for B? S
Which is optimal for 1? T
Which is optimal for 2? T

Pessimality?
Job Propose and Candidate Reject is optimal!

For jobs?

Theorem:

Job Propose and Reject produces a job-optimal pairing.

Proof:

Assume not:

A job $a$ is not paired with optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be the first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^* -$ knocks $b$ off of $g$'s string on day $t$.

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.

So $S$ is not a stable pairing.

Contradiction.

Notes:

$S$ - stable.

$S(b^*, g^*)$. 

But $(b^*, g^*)$ is a rogue couple!

Used Well-Ordering principle... Induction.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

Theorem:
Job Propose and Reject produces a job-optimal pairing.

Proof:
Assume not: a job $b$ is not paired with optimal candidate, $g$.

There is a stable pairing $S$ where $b$ and $g$ are paired.

Let $t$ be first day job $b$ gets rejected by its optimal candidate $g$ who it is paired with in stable pairing $S$.

$b^*$ knocks $b$ off of $g$'s string on day $t = \Rightarrow g$ prefers $b^*$ to $b$.

By choice of $t$, $b^*$ likes $g$ at least as much as optimal candidate.

$\Rightarrow b^*$ prefers $g$ to its partner $g^*$ in $S$.

Rogue couple for $S$.

So $S$ is not a stable pairing.

Notes:
$S$ - stable.
$(b^*, g^*) \in S$.
But $(b^*, g)$ is a rogue couple!

Used Well-Ordering principle...
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Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem:** Job Propose and Reject produces a job-optimal pairing.

**Proof:**
Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

**Theorem**: Job Propose and Reject produces a job-optimal pairing.

**Proof**: Assume not:

Let \( t \) be first day job \( b \) gets rejected by its optimal candidate \( g \) who it is paired with in stable pairing \( S \). \( b \)∗ knocks \( b \) off of \( g \)'s string on day \( t \). By choice of \( t \), \( b \)∗ prefers \( g \) at least as much as optimal candidate. \( b \)∗ \( \Rightarrow \) \( b \)∗ prefers \( g \) to its partner \( g \)∗ in \( S \). Rogue couple for \( S \). So \( S \) is not a stable pairing. Contradiction.

Notes:

\( S - \) stable.

\( (b \ast, g \ast) \in S \).

But \( (b \ast, g) \) is a rogue couple!

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Job Propose and Candidate Reject is optimal!

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Rogue couple for $S$. 

Notes:

- $S$ - stable.
- $(b^*, g^*) \in S$.
- But $(b^*, g)$ is rogue couple!

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How about for candidates?

Theorem:

Job Propose and Reject produces candidate-pessimal pairing.

$T$ – pairing produced by JPR.

$S$ – worse stable pairing for candidate $g$.

In $T$, $(g, b)$ is pair.

In $S$, $(g, b^*)$ is pair.

$g$ prefers $b$ to $b^*$.

$T$ is job optimal, so $b$ prefers $g$ to its partner in $S$.

$(g, b)$ is Rogue couple for $S$.

$S$ is not stable.

Contradiction.

Notes:

Not really induction.

Structural statement: Job optimality $\Rightarrow$ Candidate pessimality.
How about for candidates?

**Theorem:** Job Propose and Reject produces candidate-pessimal pairing.

T – pairing produced by JPR.
S – worse stable pairing for candidate g.

In T, (g, b) is pair. In S, (g, b*) is pair. g prefers b to b*. T is job optimal, so b prefers g to its partner in S. S is not stable. Contradiction.

Notes: Not really induction. Structural statement: Job optimality $\Rightarrow$ Candidate pessimality.
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How does one make it better for candidates?
Quick Questions.

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Propose and Reject - stable matching algorithm. One side proposes.
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Residency Matching..
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The method was used to match residents to hospitals.
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Takeaways.

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Stability:
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Job Optimality:
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