Lecture Outline

Strengthening Induction Hypothesis.

Strong Induction

Well ordered principle.

Strengthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2k-1$ for $k \geq 1$.

Base Case 1 (1st odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step To prove that sum of first $k+1$ odds is $(k+1)^2$.

1. The $(k+1)$st odd number is $2(k+1)-1 = 2k+1$.
2. Sum of the first $k+1$ odds is $a^2+2k+1 = k^2+2k+1$
3. $k^2+2k+1 = (k+1)^2$

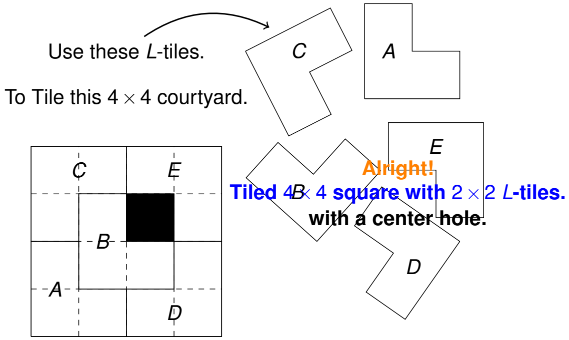
   ... P(k+1)!

□

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: Each tile covers 3 squares. The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

Ind Hyp: $n = k$. $2^{2k} = 3a+1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a+1) \\ &= 12a+3+1 \\ &= 3(4a+1)+1 \end{aligned}$$

a integer $\Longrightarrow$ $(4a+1)$ is an integer. □

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

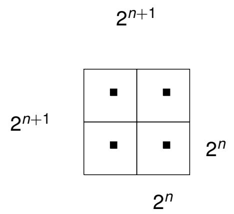
Proof:

Base case: A single tile works fine.

The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:

Any $2^n \times 2^n$ square can be tiled with a hole at the center.



What to do now???

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ... stronger induction hypothesis!

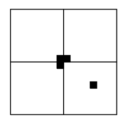
Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.

Induction Hypothesis:

"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"

Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done. □

Strong Induction: Example

Theorem: Every natural number $n > 1$ is either a prime or can be written as a product of primes.
Fact: A prime n has exactly 2 factors 1 **and** n**.**
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n is either a prime or a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)(factorization of b)"
$n+1$ can be written as the product of the prime factors!

□

Strong Induction is a form of (regular) Induction.

Let $Q(k) = P(0) \wedge P(1) \cdots P(k)$.

By the induction principle:
"If $Q(0)$, and $(\forall k \in N)(Q(k) \implies Q(k+1))$ then $(\forall k \in N)(Q(k))$"

Also, $Q(0) \equiv P(0)$, and $(\forall k \in N)(Q(k)) \equiv (\forall k \in N)(P(k))$

$(\forall k \in N)(Q(k) \implies Q(k+1))$
$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies (P(0) \cdots P(k) \wedge P(k+1)))$
$\equiv (\forall k \in N)((P(0) \cdots \wedge P(k)) \implies P(k+1))$

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$,

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a proof of the induction principle!
I.e.,

$$\neg(\forall n P(n)) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

```
def find-x-y(n):
   if (n==12) return (3,0)
   elif (n==13): return(2,1)
   elif (n==14): return(1,2)
   elif (n==15): return(0,3)
   else:
      (x,y) = find-x-y(n-4)
      return(x+1,y)
```

Base cases: P(12) , P(13) , P(14) , P(15). Holds for all.

Strong Induction step:
Recursive call is correct: $P(n-4) \implies P(n)$.

Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$. $1, 1, 2, 2, 3, \ldots, k, k+1$
Second k have same color by $P(k)$. $1, 1, 2, 2, 3, \ldots, k, k+1$
A horse in the middle in common! $1, 1, 2, 2, 3, \ldots, k, k+1$
All k must have the same color. No horse in common! $1, 2, 3, \ldots, k, k+1$
How about $P(1) \implies P(2)$?
Fix base case.
...Still doesn't work!!
(There are two horses is $\not\equiv$ For all two horses!!!)

Of course it doesn't work.

As we will see, it is more subtle to catch errors in proofs of correct theorems!!

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \leq k P(n)) \implies P(k+1))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction.

Induction $\equiv$ Recursion.