Lecture Outline

Strengthening Induction Hypothesis.
Strengthening Induction Hypothesis.
Strong Induction
Lecture Outline

- Strengthening Induction Hypothesis.
- Strong Induction
- Well ordered principle.
Theorem: The sum of the first $n$ odd numbers is a perfect square.
Theorem: The sum of the first $n$ odd numbers is a perfect square.

Theorem: The sum of the first $n$ odd numbers is $n^2$. 

Strengthening Induction Hypothesis.

Base Case 1 (1st odd number) is $1^2$.

Induction Hypothesis: Sum of first $k$ odds is perfect square $a^2 = k^2$.

Induction Step: To prove that sum of first $k+1$ odds is $(k+1)^2$.

1. The $(k+1)$st odd number is $2(k+1) - 1 = 2k+1$.
2. Sum of the first $k+1$ odds is $a^2 + 2k + 1 = k^2 + 2k + 1$.
3. $k^2 + 2k + 1 = (k+1)^2$. 

... P($k+1$)!
Strengthening Induction Hypothesis.

**Theorem:** The sum of the first \( n \) odd numbers is a perfect square.

**Theorem:** The sum of the first \( n \) odd numbers is \( n^2 \).

\( k \)th odd number is \( 2k - 1 \) for \( k \geq 1 \).
Theorem: The sum of the first $n$ odd numbers is a perfect square.

Theorem: The sum of the first $n$ odd numbers is $n^2$.

$k$th odd number is $2k - 1$ for $k \geq 1$.

Base Case 1 (1st odd number) is $1^2$. 
Strengthening Induction Hypothesis.

**Theorem:** The sum of the first $n$ odd numbers is a perfect square.

**Theorem:** The sum of the first $n$ odd numbers is $n^2$.

The $k$th odd number is $2k - 1$ for $k \geq 1$.

**Base Case** 1 (1st odd number) is $1^2$.

**Induction Hypothesis** Sum of first $k$ odds is perfect square $a^2 = k^2$. 
**Theorem:** The sum of the first \( n \) odd numbers is a perfect square.

**Theorem:** The sum of the first \( n \) odd numbers is \( n^2 \).

\[ k \text{th odd number is } 2k - 1 \text{ for } k \geq 1. \]

**Base Case** 1 (1st odd number) is \( 1^2 \).

**Induction Hypothesis** Sum of first \( k \) odds is perfect square \( a^2 = k^2 \).

**Induction Step** To prove that sum of first \( k + 1 \) odds is \( (k + 1)^2 \).
Theorem: The sum of the first $n$ odd numbers is a perfect square.\[ n^2. \]

Theorem: The sum of the first $n$ odd numbers is $n^2$.\[ k \text{th odd number is } 2k - 1 \text{ for } k \geq 1. \]

Base Case 1 (1st odd number) is $1^2$.\[ \text{Induction Hypothesis} \quad \text{Sum of first } k \text{ odds is perfect square } a^2 = k^2. \]

Induction Step To prove that sum of first $k + 1$ odds is $(k + 1)^2$.\[ 1. \text{ The } (k + 1)\text{st odd number is } 2(k + 1) - 1 = 2k + 1. \]
Strengthening Induction Hypothesis.

**Theorem:** The sum of the first \( n \) odd numbers is a perfect square.

**Theorem:** The sum of the first \( n \) odd numbers is \( n^2 \).

\[
\text{kth odd number is } 2k - 1 \text{ for } k \geq 1.
\]

**Base Case** \( 1 \) (1st odd number) is \( 1^2 \).

**Induction Hypothesis** Sum of first \( k \) odds is perfect square \( a^2 = k^2 \).

**Induction Step** To prove that sum of first \( k + 1 \) odds is \((k + 1)^2\).

1. The \((k + 1)\)st odd number is \( 2(k + 1) - 1 = 2k + 1 \).
2. Sum of the first \( k + 1 \) odds is \( a^2 + 2k + 1 = k^2 + 2k + 1 \)
Strengthening Induction Hypothesis.

**Theorem:** The sum of the first \( n \) odd numbers is a perfect square.

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**Induction Hypothesis** Sum of first \( k \) odds is perfect square \( a^2 = k^2 \).

**Induction Step** To prove that sum of first \( k + 1 \) odds is \( (k + 1)^2 \).

1. The \( (k + 1) \)st odd number is \( 2(k + 1) - 1 = 2k + 1 \).
2. Sum of the first \( k + 1 \) odds is
   \[ a^2 + 2k + 1 = k^2 + 2k + 1 \]
3. \( k^2 + 2k + 1 = (k + 1)^2 \)
   ... \( P(k+1) \)!  

\( \Box \)
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.
Tiling Cory Hall Courtyard.

Use these L-tiles.

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Tiling Cory Hall Courtyard.

Use these $L$-tiles.

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To Tile this $4 \times 4$ courtyard.
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles.
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles with a center hole.
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles with a center hole.

Can we tile any $2^n \times 2^n$ with $L$-tiles (with a hole)
Tiling Cory Hall Courtyard.

Use these $L$-tiles.

To Tile this $4 \times 4$ courtyard.

Alright!

Tiled $4 \times 4$ square with $2 \times 2$ $L$-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with $L$-tiles (with a hole) for every $n$!
Hole have to be there? Maybe just one?

**Theorem**: Any tiling of $2^n \times 2^n$ square has to have one hole.
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. 
**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$
Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

Ind Hyp: $n = k$. $2^{2k} = 3a + 1$ for integer $a$. 

Hole have to be there? Maybe just one?
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

Ind Hyp: $n = k$. $2^{2k} = 3a + 1$ for integer $a$.

$$2^{2(k+1)}$$
Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

Ind Hyp: $n = k$. $2^{2k} = 3a + 1$ for integer $a$.

$$2^{2(k+1)} = 2^{2k} \times 2^2$$
Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

Ind Hyp: $n = k$. $2^{2k} = 3a + 1$ for integer $a$.

\[
\begin{align*}
2^{2(k+1)} &= 2^{2k} \times 2^2 \\
&= 4 \times 2^{2k}
\end{align*}
\]
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

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2^{2(k+1)} = 2^{2k} \times 2^2 \\
= 4 \times 2^{2k} \\
= 4 \times (3a + 1)
\]
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

Ind Hyp: $n = k$. $2^{2k} = 3a + 1$ for integer $a$.

\[
2^{2(k+1)} = 2^{2k} \times 2^2 = 4 \times 2^{2k} = 4(3a + 1) = 12a + 3 + 1
\]
Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

Ind Hyp: $n = k$. $2^{2k} = 3a + 1$ for integer $a$.

\[
\begin{align*}
2^{2(k+1)} &= 2^{2k} \times 2^2 \\
&= 4 \times 2^{2k} \\
&= 4 \times (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{align*}
\]
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

**Proof:** Each tile covers 3 squares. The remainder of $2^{2n}$ divided by 3 is 1.

Base case: true for $n = 0$. $2^0 = 1$

Ind Hyp: $n = k$. $2^{2k} = 3a + 1$ for integer $a$.

\[
\begin{align*}
2^{2(k+1)} &= 2^{2k} \cdot 2^2 \\
&= 4 \cdot 2^{2k} \\
&= 4 \cdot (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{align*}
\]

$a$ integer
Hole have to be there? Maybe just one?

**Theorem:** Any tiling of $2^n \times 2^n$ square has to have one hole.

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2^{2(k+1)} = 2^{2k} \times 2^2 \\
= 4 \times 2^{2k} \\
= 4 \times (3a + 1) \\
= 12a + 3 + 1 \\
= 3(4a + 1) + 1
$$

$a$ integer $\implies (4a + 1)$ is an integer.
Hole have to be there? Maybe just one?

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2^{2(k+1)} = 2^{2k} \cdot 2^2 \\
= 4 \cdot 2^{2k} \\
= 4 \cdot (3a + 1) \\
= 12a + 3 + 1 \\
= 3(4a + 1) + 1
\]

$a$ integer $\implies (4a + 1)$ is an integer.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.
**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.

The hole is adjacent to the center of the $2 \times 2$ square.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.

The hole is adjacent to the center of the $2 \times 2$ square.

Induction Hypothesis:
Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.
   The hole is adjacent to the center of the $2 \times 2$ square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.

The hole is adjacent to the center of the $2 \times 2$ square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.
Hole in center?

**Theorem:** Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

**Proof:**

Base case: A single tile works fine.
   - The hole is adjacent to the center of the $2 \times 2$ square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

\[
2^{n+1}
\]

\[
\begin{array}{cc}
\text{hole} & \text{hole} \\
\text{hole} & \text{hole}
\end{array}
\]

What to do now???
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

**Better theorem**
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... stronger induction hypothesis!
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... stronger induction hypothesis!

Base case: Sure. A tile is fine.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere*.

Better theorem ... stronger induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

**Better theorem ... stronger induction hypothesis!**

Base case: Sure. A tile is fine.
Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... stronger induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole anywhere.”

Consider $2^{n+1} \times 2^{n+1}$ square.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... stronger induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole anywhere.”

Consider $2^{n+1} \times 2^{n+1}$ square.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

**Better theorem ... stronger induction hypothesis!**

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**”

Consider $2^{n+1} \times 2^{n+1}$ square.

Use induction hypothesis in each.
**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... stronger induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole anywhere.”

Consider $2^{n+1} \times 2^{n+1}$ square.

Use induction hypothesis in each.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... stronger induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis: “Any $2^n \times 2^n$ square can be tiled with a hole anywhere.” Consider $2^{n+1} \times 2^{n+1}$ square.

Use induction hypothesis in each.

Use L-tile and ...
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent anywhere.

Better theorem ... stronger induction hypothesis!

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole anywhere.”

Consider $2^{n+1} \times 2^{n+1}$ square.

Use induction hypothesis in each.

Use L-tile and ... we are done.
Hole can be anywhere!

**Theorem:** Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere*.

**Better theorem ... stronger induction hypothesis!**

Base case: Sure. A tile is fine. Flipping the orientation can leave hole anywhere.

Induction Hypothesis:
“Any $2^n \times 2^n$ square can be tiled with a hole *anywhere*.”

Consider $2^{n+1} \times 2^{n+1}$ square.

Use induction hypothesis in each.

Use L-tile and ... we are done.
Strong Induction: Example

**Theorem:** Every natural number $n > 1$ is either a prime or can be written as a product of primes.

**Fact:** A prime $n$ has exactly 2 factors 1 and $n$.

**Base Case:** $n = 2$.

**Induction Step:** $P(n) =$ "$n$ is either a prime or a product of primes." Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$.

$P(n)$ says nothing about $a, b$!

**Strong Induction Principle:** If $P(0)$ and $(\forall k \in \mathbb{N})(P(0) \land \ldots \land P(k)) \Rightarrow P(k + 1)$, then $(\forall k \in \mathbb{N})(P(k))$.

$P(0) \Rightarrow P(1) \Rightarrow P(2) \Rightarrow P(3) = \ldots$

**Strong induction hypothesis:** "$a$ and $b$ are products of primes" $\Rightarrow$ "$n + 1 = a \cdot b$ = (factorization of $a$)(factorization of $b$)"

$n + 1$ can be written as the product of the prime factors!
Strong Induction: Example

**Theorem:** Every natural number $n > 1$ is either a prime or can be written as a product of primes.

**Fact:** A prime $n$ has exactly 2 factors 1 and $n$. 
Strong Induction: Example

**Theorem:** Every natural number \( n > 1 \) is either a prime or can be written as a product of primes.

**Fact:** A prime \( n \) has exactly 2 factors 1 and \( n \).

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Strong Induction: Example

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**Strong Induction: Example**

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**Induction Step:**

\( P(n) = \text{“}n \text{ is either a prime or a product of primes. “} \)
Strong Induction: Example

**Theorem:** Every natural number \( n > 1 \) is either a prime or can be written as a product of primes.

**Fact:** A prime \( n \) has exactly 2 factors 1 and \( n \).

**Base Case:** \( n = 2 \).

**Induction Step:**

Let \( P(n) = \text{"n is either a prime or a product of primes. "} \)

Either \( n + 1 \) is a prime or \( n + 1 = a \cdot b \) where \( 1 < a, b < n + 1 \).
Strong Induction: Example

**Theorem:** Every natural number $n > 1$ is either a prime or can be written as a product of primes.

**Fact:** A prime $n$ has exactly 2 factors 1 and $n$.

**Base Case:** $n = 2$.

**Induction Step:**

$P(n)$ = “$n$ is either a prime or a product of primes. “

Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$.

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Strong Induction: Example

**Theorem:** Every natural number \( n > 1 \) is either a prime or can be written as a product of primes.

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**Induction Step:**

\[ P(n) = \text{"} n \text{ is either a prime or a product of primes. "} \]

Either \( n + 1 \) is a prime or \( n + 1 = a \cdot b \) where \( 1 < a, b < n + 1 \).

\( P(n) \) says nothing about \( a, b \)!

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**Strong Induction Principle:** If \( P(0) \) and

\[
(\forall k \in N)((P(0) \land \ldots \land P(k)) \implies P(k+1)),
\]

then \((\forall k \in N)(P(k))\).
**Strong Induction: Example**

**Theorem:** Every natural number \( n > 1 \) is either a prime or can be written as a product of primes.

**Fact:** A prime \( n \) has exactly 2 factors 1 and \( n \).

**Base Case:** \( n = 2 \).

**Induction Step:**

\( P(n) = \) “\( n \) is either a prime or a product of primes. “

Either \( n + 1 \) is a prime or \( n + 1 = a \cdot b \) where \( 1 < a, b < n + 1 \).

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(\forall k \in N)((P(0) \land \ldots \land P(k)) \Rightarrow P(k+1)),
\]

then \( (\forall k \in N)(P(k)) \).

\[
P(0) \Rightarrow P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \ldots
\]
Strong Induction: Example

**Theorem:** Every natural number \( n > 1 \) is either a prime or can be written as a product of primes.

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**Induction Step:**

\( P(n) = \) “\( n \) is either a prime or a product of primes. “

Either \( n + 1 \) is a prime or \( n + 1 = a \cdot b \) where \( 1 < a, b < n + 1 \).

\( P(n) \) says nothing about \( a, b \)!

---

**Strong Induction Principle:** If \( P(0) \) and

\[
(\forall k \in \mathbb{N})(P(0) \land \ldots \land P(k)) \Rightarrow P(k+1),
\]

then \( (\forall k \in \mathbb{N})(P(k)) \).

\[
P(0) \Rightarrow P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \ldots
\]

---

Strong induction hypothesis: “\( a \) and \( b \) are products of primes”
Strong Induction: Example

**Theorem:** Every natural number $n > 1$ is either a prime or can be written as a product of primes.

**Fact:** A prime $n$ has exactly 2 factors 1 and $n$.

**Base Case:** $n = 2$.

**Induction Step:**

$P(n) =$ “$n$ is either a prime or a product of primes. “

Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$.

$P(n)$ says nothing about $a, b$!

---

**Strong Induction Principle:** If $P(0)$ and

$$(\forall k \in N)((P(0) \land \ldots \land P(k)) \implies P(k + 1)),$$

then $(\forall k \in N)(P(k))$.

$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots$

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Strong induction hypothesis: “$a$ and $b$ are products of primes”

$\implies \text{“} n + 1 = a \cdot b \text{“}$
**Strong Induction: Example**

**Theorem:** Every natural number $n > 1$ is either a prime or can be written as a product of primes.

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**Induction Step:**
$P(n) =$ “$n$ is either a prime or a product of primes. “
Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$.
$P(n)$ says nothing about $a, b$!

---

**Strong Induction Principle:** If $P(0)$ and

$$(\forall k \in \mathbb{N})(P(0) \land \ldots \land P(k)) \implies P(k + 1),$$

then $(\forall k \in \mathbb{N})(P(k))$.

$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots$

---

Strong induction hypothesis: “$a$ and $b$ are products of primes”

$\implies “n + 1 = a \cdot b = (\text{factorization of } a)(\text{factorization of } b)”$

$n + 1$ can be written as the product of the prime factors!
Strong Induction: Example

Theorem: Every natural number $n > 1$ is either a prime or can be written as a product of primes.

Fact: A prime $n$ has exactly 2 factors 1 and $n$.

Base Case: $n = 2$.

Induction Step: $P(n) = \text{“} n \text{ is either a prime or a product of primes. “}$
Either $n + 1$ is a prime or $n + 1 = a \cdot b$ where $1 < a, b < n + 1$. $P(n)$ says nothing about $a, b$!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in \mathbb{N})(P(0) \land \ldots \land P(k)) \implies P(k + 1),$$

then $(\forall k \in \mathbb{N})(P(k))$.

$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \ldots$

Strong induction hypothesis: “$a$ and $b$ are products of primes”
$\implies \text{“} n + 1 = a \cdot b = (\text{factorization of } a)(\text{factorization of } b)\text{“}$

$n + 1$ can be written as the product of the prime factors!
Strong Induction is a form of (regular) Induction.

Let $Q(k) = P(0) \land P(1) \cdots P(k)$.
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Well Ordering Principle and Induction.

If $\forall n \ P(n)$ is not true, then $(\exists n) \neg P(n)$.
Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest $m$, with $\neg P(m)$,

The Well ordering principle states that for any subset of the natural numbers there is a smallest element.
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If \((\forall n)P(n)\) is not true, then \((\exists n)\neg P(n)\).

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\[ P(m - 1) \implies P(m) \] must be false (assuming \(P(0)\) holds.)
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This is a proof of the induction principle!

I.e.,

$$\neg(\forall nP(n)) \implies ((\exists n)\neg(P(n - 1) \implies P(n)).$$
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The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.
Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$. 
Strong Induction and Recursion.

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Instead of proof, let’s write some code!

```python
def find_x_y(n):
    if (n==12) return (3,0)
    elif (n==13): return(2,1)
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    elif (n==15): return(0,3)
    else:
        (x,y) = find_x_y(n-4)
        return(x+1,y)
```

Base cases: \( P(12), P(13), P(14), P(15) \).

Strong Induction step: Recursive call is correct: \( P(n-4) \Rightarrow P(n) \).

Slight differences: showed for all \( n \geq 16 \) that \( n-1 = 4P(i) \Rightarrow P(n) \).
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Holds for all.

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Horses of the same color...

**Theorem:** All horses have the same color.

Base Case: $P(1)$ - trivially true.

New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any $k$ horses have the same color.

Induction step $P(k+1)$?

First $k$ have same color by $P(k)$.

Second $k$ have same color by $P(k)$.

A horse in the middle in common!

All $k$ must have the same color.

How about $P(1) \Rightarrow P(2)$?

Fix base case. ...Still doesn't work!! (There are two horses is $\not\equiv$ For all two horses!!!)

Of course it doesn't work. As we will see, it is more subtle to catch errors in proofs of correct theorems!!
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Base Case: \( P(1) \) - trivially true.

**New Base Case:** \( P(2) \): there are two horses with same color.

Induction Hypothesis: \( P(k) \) - Any \( k \) horses have the same color.

Induction step \( P(k + 1) \)?

First \( k \) have same color by \( P(k) \).
Second \( k \) have same color by \( P(k) \).
A horse in the middle in common!

Fix base case.
...Still doesn’t work!!
(There are two horses is \( \neq \) For all two horses!!!)

Of course it doesn’t work.
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As we will see, it is more subtle to catch errors in proofs of correct theorems!!
Summary: principle of induction.

Today: More induction.
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Today: More induction.

\( P(0) \)

Statement to prove: \( P(n) \) for \( n \) starting from \( n_0 \).

Base Case: Prove \( P(n_0) \).

Ind. Step: Prove for all values, \( n \geq n_0 \), \( P(n) = \Rightarrow P(n+1) \).

Statement is proven!

Strong Induction: \( (P(0) \land (\forall n \leq k P(n))) = \Rightarrow P(k+1) \)\n
\( \Rightarrow (\forall n \in \mathbb{N}) (P(n)) \)

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.

Sum of first \( n \) odds is \( n^2 \).

Hole anywhere.

Not same as strong induction.

Induction \( \equiv \) Recursion.
Summary: principle of induction.

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\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k+1))))\]
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\[(P(0) \land ((\forall k \in N)(P(k) \implies P(k + 1)))) \implies (\forall n \in N)(P(n))\]
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**Strengthen theorem statement.**

- Sum of first \(n\) odds is \(n^2\).
- Hole anywhere.

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Induction \(\equiv\) Recursion.