Stable Matching Problem

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- Maximize number of first choices.
- Minimize difference between preference ranks.

Objectives

Produce a matching that one cannot improve upon!

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Definition: A **matching** is disjoint set of n job-candidate pairs.

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Definition: A **matching** is disjoint set of n job-candidate pairs.

Definition: A **rogue couple** j, c^* for a pairing S:
j and c^* prefer each other to their partners in S

A stable matching??

Given a set of preferences.

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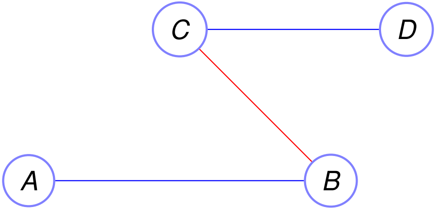
How does one find it?

Consider a single type version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



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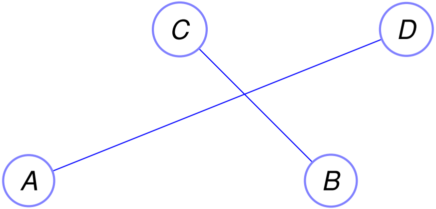

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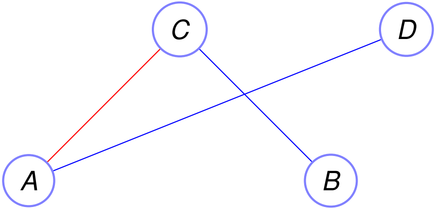

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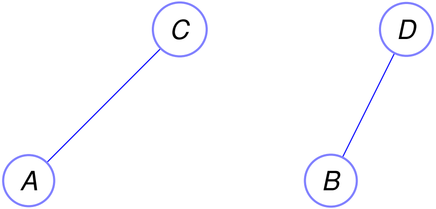

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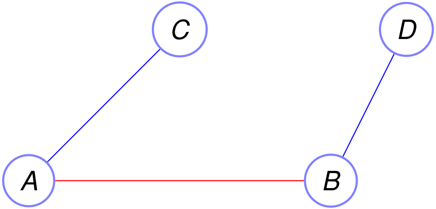

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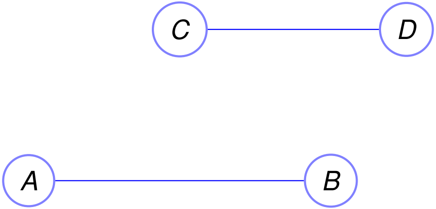

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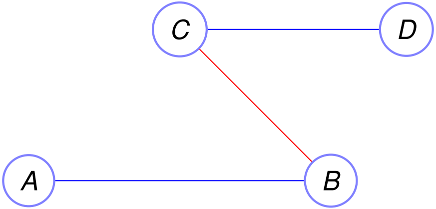

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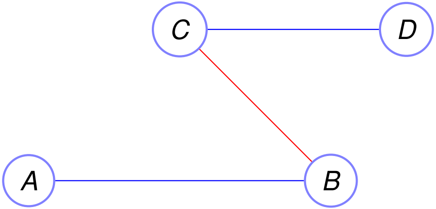

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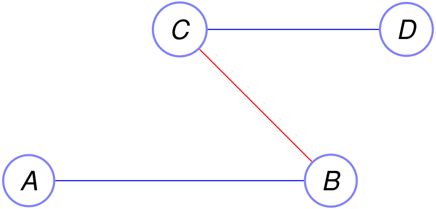

The Propose and Reject Algorithm.

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Stop when each job gets exactly one proposal (candidate).

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	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

Jobs			
A	1	2	3
B	~~1~~ X	2	3
C	2	1	3

Candidates			
1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
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Terminates in $\leq n^2$ steps!

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Does 1 prefer "C" or "A"?

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$P(0)$– true. Candidate has j on string.

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If on day t a candidate c has a job j on a string, any job, j', on c's string for any day $t' > t$ is at least as good as j.

Proof:
$P(k)$- - "job on c's string is at least as good as j on day $t+k$"

$P(0)$– true. Candidate has j on string.

Assume $P(k)$. Let j' be job **on string** on day $t+k$.

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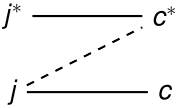

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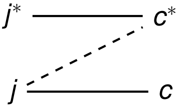

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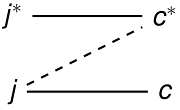

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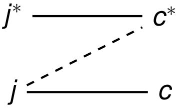

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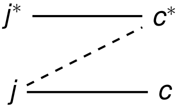

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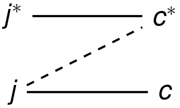

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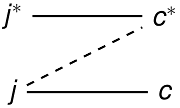

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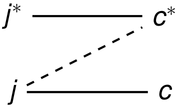

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Is the Job-Proposes better for jobs?

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B:	1,2	2:	B,A

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Notice: only one stable pairing.

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Notice: only one stable pairing. If (A,2) are pair, (A,1) is rogue couple.

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Consider pairing: $(A,1),(B,2)$.

Stable? Yes.

Optimal for B?
Notice: only one stable pairing. If (A,2) are pair, (A,1) is rogue couple.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

Understanding Optimality: by example.

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Pairing S: $(A,1),(B,2)$.

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Pairing *S*: $(A,1),(B,2)$. Stable?

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A:	1,2	1:	B,A
B:	2,1	2:	A,B

Pairing S: $(A, 1), (B, 2)$. Stable? Yes.

Pairing T: $(A, 2), (B, 1)$.

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Pairing *S*: $(A,1),(B,2)$. Stable? Yes.

Pairing *T*: $(A,2),(B,1)$. Also Stable.

Which is optimal for *A*?

Understanding Optimality: by example.

A:	1,2	1:	A,B
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Stable? Yes.

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Which is optimal for *A*? *S*

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Pairing *S*: $(A,1),(B,2)$. Stable? Yes.

Pairing *T*: $(A,2),(B,1)$. Also Stable.

Which is optimal for *A*? *S* Which is optimal for *B*?

Understanding Optimality: by example.

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Which is optimal for A? S Which is optimal for B? S

Understanding Optimality: by example.

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Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A:	1,2	1:	B,A
B:	2,1	2:	A,B

Pairing S: $(A, 1), (B, 2)$. Stable? Yes.

Pairing T: $(A, 2), (B, 1)$. Also Stable.

Which is optimal for A? S Which is optimal for B? S
Which is optimal for 1?

Understanding Optimality: by example.

A:	1,2	1:	A,B
B:	1,2	2:	B,A

Consider pairing: $(A, 1), (B, 2)$.

Stable? Yes.

Optimal for B?
Notice: only one stable pairing. If (A,2) are pair, (A,1) is rogue couple.
So this is the best B can do in a stable pairing.
So optimal for B.

Also optimal for A, 1 and 2. Also pessimal for A,B,1 and 2.

A:	1,2	1:	B,A
B:	2,1	2:	A,B

Pairing S: $(A, 1), (B, 2)$. Stable? Yes.

Pairing T: $(A, 2), (B, 1)$. Also Stable.

Which is optimal for A? S Which is optimal for B? S
Which is optimal for 1? T

Understanding Optimality: by example.

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Pairing S: $(A,1),(B,2)$. Stable? Yes.

Pairing T: $(A,2),(B,1)$. Also Stable.

Which is optimal for A? S Which is optimal for B? S

Which is optimal for 1? T Which is optimal for 2?

Understanding Optimality: by example.

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Which is optimal for 1? T Which is optimal for 2? T

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Pairing T: $(A,2),(B,1)$. Also Stable.

Which is optimal for A? S Which is optimal for B? S
Which is optimal for 1? T Which is optimal for 2? T

Pessimality?

Job Propose and Candidate Reject is optimal!

For jobs?

Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

Theorem: Job Propose and Reject produces a job-optimal pairing.

Job Propose and Candidate Reject is optimal!

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Theorem: Job Propose and Reject produces a job-optimal pairing.

Proof:

Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

Theorem: Job Propose and Reject produces a job-optimal pairing.

Proof:
Assume not:

Job Propose and Candidate Reject is optimal!

For jobs? For candidates?

Theorem: Job Propose and Reject produces a job-optimal pairing.

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Assume not: some job is not paired with its optimal candidate.

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Let t be first day some job j gets rejected by its optimal candidate c.

There is a stable pairing S where j and c are paired.

Job Propose and Candidate Reject is optimal!

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j^* - knocks j off of c's string on day t

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(j^*, c) – Rogue couple for S.

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So S is not a stable pairing.

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So S is not a stable pairing. Contradiction.

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Notes:

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Notes: S - stable. $(j^*, c^*) \in S$.

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Used Well-Ordering principle.

How about for candidates?

How about for candidates?

Theorem: Job Propose and Reject produces candidate-pessimal pairing.

T – pairing produced by JPR.
S – worse stable pairing for candidate c.
In T, (c, j) is pair.
In S, (c, j^*) is pair.
c prefers j to j^*.
T is job optimal, so j prefers c to its partner in S.
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Contradiction. □

Quick Questions.

How does one make it better for candidates?

Quick Questions.

How does one make it better for candidates?

Propose and Reject - stable matching algorithm. One side proposes.

Quick Questions.

How does one make it better for candidates?

Propose and Reject - stable matching algorithm. One side proposes.

Jobs Propose $\implies$ job optimal.

Quick Questions.

How does one make it better for candidates?

Propose and Reject - stable matching algorithm. One side proposes.

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Candidates propose.

Quick Questions.

How does one make it better for candidates?

Propose and Reject - stable matching algorithm. One side proposes.

Jobs Propose $\implies$ job optimal.

Candidates propose. $\implies$ optimal for candidates.

Residency Matching..

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The method was used to match residents to hospitals.

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...

Residency Matching..

The method was used to match residents to hospitals.

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..until 1990's...Resident optimal.

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

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Another variation: couples.

Residency Matching..

The method was used to match residents to hospitals.

Hospital optimal....

..until 1990's...Resident optimal.

Another variation: couples.

Takeaways.

Analysis of cool algorithm with interesting goal: stability.

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Induction over steps of algorithm.

Proofs carefully use definition:
Stability:
Improvement Lemma plus every day the job gets to choose.
Optimality proof:
Job Optimality:
contradiction of the existence of a better *stable* pairing.
that is, no rogue couple by improvement, job choice, and well ordering principle.
Candidate Pessimality:
contradiction plus job optimality implies better pairing.