Graphs!
Definitions: model.
Fact!
Planar graphs.
Euler Again!!!

Map Coloring.
Fewer Colors?
Yes! Three colors.
Fewer Colors?
Four colors required!
Theorem: Four colors enough.

Scheduling: coloring.

Graphs: formally.

Directed Graphs

Graph Concepts and Definitions.

The sum over vertices of degrees!

\[ \sum \text{degree of vertex} \]

Length of cycle? tours 2.

Length of path? No!

No repeated vertex!

\[ \{ \text{2} \} \]

\[ \{ \text{2} \} \]

\[ \{ \text{2} \} \]

For triangle number of edges is 3, the sum of degrees is 6.

Could sum always be...

\[ \{ \text{A} \} \]

\[ \{ \text{B} \} \]

\[ \{ \text{A} \} \]

(A) 2|E|? ..

(B) 2|V|?

(A) is true.

Paths, walks, cycles, tour.

A path in a graph is a sequence of edges.

Path: \{ 1, 10 \}, \{ 8, 5 \}, \{ 4, 11 \}? No!

Path: \{ 1, 10 \}, \{ 10, 5 \}, \{ 5, 4 \}, \{ 4, 11 \}? Yes!

Path: \( (v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k) \).

Quick Check! Length of path? \( k \) vertices or \( k - 1 \) edges.

Cycle: Path from \( v_i \) to \( v_{i+1} \), + edge \( (v_{i+1}, v_i) \). Length of cycle? \( k - 1 \) vertices and edges!

Path is usually simple. No repeated vertex!

Walk is sequence of edges with possible repeated vertex or edge.

Tour is walk that starts and ends at the same node.

Quick Check!

Path is to Walk as Cycle is to ?? Tour!

Quick Proof of an Equality.

The sum of the vertex degrees is equal to ??

Recall: edge, \((u, v)\), is incident to endpoints, \( u \) and \( v \).

degree of \( u \) number of edges incident to \( u \)

Let’s count incidences in two ways.

How many incidences does each edge contribute? 2.

Total Incidences? \(|E| \) edges, 2 each. \( \rightarrow 2|E| \)

What is degree \( v \)? Incidences corresponding to \( v \!\)!

Total Incidences? The sum over vertices of degrees!

Thm: Sum of vertex degrees is \( 2|E| \).

Directed Paths.

Path: \( (v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k) \).

Paths, walks, cycles, tours ... are analogous to undirected now.
Eulerian cycles

An Eulerian tour visits each edge exactly once.

Theorem: Any undirected graph has an Eulerian tour if and only if all vertices have even degree and is connected.

Proof of if: Even + connected \( \Rightarrow \) Eulerian Tour.

We will give an algorithm. First by picture.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).
2. Remove cycle, \( C \), from \( G \).
3. Let \( G_1, \ldots, G_k \) be connected components. Each is touched by \( C \).
4. Why? \( G \) was connected.
5. Let \( v \) be first vertex of \( C \) that is in \( G_i \).
6. Recurse on \( G_i \) starting from \( v \).
7. Splice together.

Proof of only if: Eulerian \( \Rightarrow \) connected and all even degree.

Eulerian Tour is connected so graph is connected.

Uses two incident edges per visit. Tours uses all accident edges. Therefore \( v \) has even degree.

When you enter, you can leave.

For starting node, tour leaves first then enters at end.

Not The Hotel California.

Recursive/Inductive Algorithm.

1. Take a walk from arbitrary node \( v \), until you get back to \( v \).
2. Remove cycle, \( C \), from \( G \).
3. Find tour \( T \) of \( G \) starting/ending at \( v \).
4. Splice \( T \), into \( C \) where \( v \) first appears in \( C \).

Visits every edge once:

Visits edges in \( C \) exactly once.

By induction for all edges in each \( G_i \).
Poll: Euler concepts.

Mark correct statements for a connected graph where all vertices have even degree. (Below, we use tour to mean uses an edge exactly once, but may involve a vertex several times.

(A) Removing a tour leaves a graph of even degree.
(B) A tour connecting a set of connected components, each with a Eulerian tour is really cool! Eulerian even.
(C) There is no hotel california in this graph.
(D) After removing a set of edges $E'$ in a connected graph, every connected component is incident to an edge in $E'$
(E) If one walks on new edges, starting at $v$, one must eventually get back to $v$.  
(F) Removing a tour leaves a connected graph.

Only (F) is false.

A Tree, a tree.

Graph $G = (V, E)$.

Binary Tree!

More generally.

Proof of only if.

Thm:  
"$G$ connected and has $|V| - 1$ edges" $\implies$  
"$G$ is connected and has no cycles."

Proof of $\implies$: By induction on $|V|$.

Base Case: $|V| = 1$. 0 = $|V| - 1$ edges and has no cycles.

Induction Step:
Claim: There is a degree 1 node.

Proof of Claim:
Can't visit more than once since no cycle.

Entered. Didn't leave. Only one incident edge.

Removing node doesn't create cycle.

New graph is connected.

Removing degree 1 node doesn't disconnect from Degree 1 lemma.

By induction $G - v$ has $|V| - 2$ edges. By induction $G - v$ has no cycle.

And no cycle in $G$ since degree 1 cannot participate in cycle.

Proof of if

Thm:  
"$G$ is connected and has no cycles" $\implies$  
"$G$ connected and has $|V| - 1$ edges"

Proof:  
Walk from a vertex using untraversed edges. Until get stuck.

Claim: Degree 1 vertex.

Proof of Claim:  
Can't visit more than once since no cycle.

Entered. Didn't leave. Only one incident edge.

Removing node doesn't create cycle.

New graph is connected.

Removing degree 1 node doesn't disconnect from Degree 1 lemma.

By induction $G - v$ has $|V| - 2$ edges. By induction $G - v$ has no cycle.

$G$ has one more or $|V| - 1$ edges.
Let $G$ be a connected graph with $|V| - 1$ edges.

(A) Removing a degree 1 vertex can disconnect the graph.
(B) One can use induction on smaller objects.
(C) The average degree is $2 - 2/|V|$.
(D) There is a hotel california: a degree 1 vertex.
(E) Everyone can be bigger than average.

(B), (C), (D) are true.