More on Graphs

Types of graphs.
- Complete Graphs.
- Trees.
- Planar Graphs.
- Hypercubes.

Complete Graph.

A complete graph $K_n$ on $n$ vertices.
All edges are present.
Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n-1$ edges.
Sum of degrees is $n(n-1)$.
$\Rightarrow$ Number of edges is $n(n-1)/2$.

Remember sum of degree is $2|E|$.

$K_4$ and $K_5$

$K_5$ is not planar.
Cannot be drawn in the plane without an edge crossing!
We will prove this later!

Trees!

Graph $G = (V, E)$.
Binary Tree!

More generally.

Trees: Definitions

Definitions:
- A connected graph without a cycle.
- A connected graph with $|V| - 1$ edges.
- A connected graph where any edge removal disconnects it.
- A connected graph where any edge addition creates a cycle.

Some trees.

Equivalence of Definitions

Thm:
“$G$ connected and has $|V| - 1$ edges” $\equiv$ “$G$ is connected and has no cycles.”

Proof of $\Rightarrow$ (only if):
By induction on $|V|$.
Base Case: $|V| = 1$. 0 = $|V| - 1$ edges and has no cycles.
Induction Step: Assume for $G$ with up to $k$ vertices. Prove for $k+1$
Consider some vertex $v$ in $G$. How is it connected to the rest of $G$?
Might it be connected by just 1 edge?
Is there a Degree 1 vertex?
Is the rest of $G$ connected?
**Planar graphs.**

A graph that can be drawn in the plane without edge crossings.

- **Planar?** Yes for Triangle.
- Four node complete $K_4$? Yes.
- Five node complete or $K_5$? No! Why? Later.

Two to three nodes, bipartite? Yes.

Three to three nodes, complete/bipartite or $K_{3,3}$? No. Why? Later.

**Euler’s Formula.**

Faces: connected regions of the plane.

How many faces for triangle? 2

- complete on four vertices or $K_4$? 4
- bipartite, complete two/three or $K_{3,3}$? 3

$v$ is number of vertices, $e$ is number of edges, $f$ is number of faces.

Euler’s Formula: Connected planar graph has $v + f - e = 2$.

- Triangle: $3 + 2 - 3 + 2!$
- $K_4$: $4 + 4 = 6 + 2!$
- $K_{3,3}$: $5 + 3 = 6 + 2!$
- Examples = 3! Proven! Not!!!
Euler and non-planarity of $K_5$ and $K_{3,3}$

Euler: $v + f = e + 2$ for connected planar graph.
We consider simple graphs where $v \geq 3$.
Consider Face edge Adjacencies

Each face is adjacent to at least three edges ($v > 2$).
$\geq 3f$ face-edge adjacencies.
Each edge is adjacent to two faces.
$= 2e$ face-edge adjacencies.
$\Rightarrow 3f \leq 2e$ for any planar graph with $v > 2$.
Or $e \leq \frac{3}{2}v$.

Plug into Euler: $v + \frac{3}{2}v \geq e + 2 \Rightarrow e \leq \frac{3}{2}v - 6$

$10 \leq \frac{3}{2}(5) - 6 = 9$. $\Rightarrow K_5$ is not planar.

Proof of Euler's formula.

**Theorem (Euler):** Connected planar graph has $v + f = e + 2$.

**Proof:** Induction on $e$.
Base: $e = 0$, $v = f = 1$.
Induction Step:
First, if it is a tree: $e = v - 1$, $f = 1$, $v + 1 = (v - 1) + 2$. Done.
Suppose it is NOT a tree. Assume holds for $e \leq n$. Consider $e = n + 1$.
Find a cycle. Remove edge.

Joins two faces.
$\Rightarrow v + (f - 1) = (e - 1) + 2$ by induction hypothesis.
Therefore $v + f = e + 2$.

Proving non-planarity for $K_{3,3}$

$e \leq 3(v) - 6$ for planar graphs.
$9 \leq 3(6) - 6$? Sure!
Need a different approach! See notes for details.

Summary: Planarity and Euler

These graphs cannot be drawn in the plane without edge crossings.

Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.
An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n - 1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges ($0x$, $1x$).

<table>
<thead>
<tr>
<th>0</th>
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$2^n$ vertices. number of $n$-bit strings!
$n2^{n-1}$ edges.
$2^n$ vertices each of degree $n$
  total degree is $n2^n$ and half as many edges!
Thm: Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$, $|E \cap (V - S)| \geq |S|$.

Terminology:
- $(S, V - S)$ is cut.
- $(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

**Proof of Large Cuts.**

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**

**Base Case:** $n = 1 V = \{0, 1\}$.

- $S = \{0\}$ has one edge leaving.
- $S = \emptyset$ has 0.

**Induction Step.**

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

**Induction Step. Case 2.**

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof:** **Induction Step. Case 2.** $|S_0| \geq |V_0|/2$.

Recall Case 1: $|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

| $|S_0| \geq |V_0|/2$ | $|S_0| \geq |V_0| - |S_1| \leq |V_1|/2$ | $|E_0| \geq |V_0| - |S_0|$ | $|E_0| \geq |S_1|$ |
|---------------|-----------------|----------------|----------------|
| $|S_0| \geq |V_0|/2$ | $|S_0| \geq |V_0| - |S_1| \leq |V_1|/2$ | $|E_0| \geq |V_0| - |S_0|$ | $|E_0| \geq |S_1|$ |
| Edges in $E_0$, connect corresponding nodes. | Edges cut in $E_0$, connect corresponding nodes. | Edges cut in $E_0$, connect corresponding nodes. | Edges cut in $E_0$, connect corresponding nodes. |

Total edges cut:

- $|S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$
- $|V_0| = |V|/2 \geq |S|$

Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

**Induction Step.**

Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.