Lecture 7. Outline.

1. Modular Arithmetic.
Lecture 7. Outline.

1. Modular Arithmetic.
   Clock Math!!!
Lecture 7. Outline.

1. Modular Arithmetic.  
   Clock Math!!

2. Inverses for Modular Arithmetic: Greatest Common Divisor.
Lecture 7. Outline.

1. Modular Arithmetic.
   Clock Math!!!

2. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!
Lecture 7. Outline.

1. Modular Arithmetic.  
   Clock Math!!!

2. Inverses for Modular Arithmetic: Greatest Common Divisor.  
   Division!!!

3. Euclid’s GCD Algorithm.
Lecture 7. Outline.

1. Modular Arithmetic.
   Clock Math!!!

2. Inverses for Modular Arithmetic: Greatest Common Divisor.
   Division!!!

3. Euclid’s GCD Algorithm.
   A little tricky here!
Hypercubes.

Complete graphs, really connected!
Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V| - 1)/2$
Hypercubes.

Complete graphs, really connected! But lots of edges.

\[ |V|(|V| - 1)/2 \]

Trees,
Hypercubes.

Complete graphs, really connected! But lots of edges.
\[|V|(|V| - 1)/2\]
Trees, few edges. (|V| - 1)
Hypercubes.

Complete graphs, really connected! But lots of edges.
\[ |V|(|V| - 1)/2 \]
Trees, few edges. \((|V| - 1)\)

but just falls apart!
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Hypercubes. Really connected. $|V| \log |V|$ edges!
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Also represents bit-strings nicely.
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\[ G = (V, E) \]
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\[ G = (V, E) \]
\[ |V| = \{0, 1\}^n, \]
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2\(^n\) vertices.
Hypercubes.

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2\(^n\) vertices. number of \(n\)-bit strings!
\(n2^{n-1}\) edges.
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2\(^n\) vertices. number of n-bit strings!
n2\(^{n-1}\) edges.
2\(^n\) vertices each of degree \(n\)
Hypercubes.

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$2^n$ vertices. number of $n$-bit strings!
$n2^{n-1}$ edges.
$2^n$ vertices each of degree $n$
total degree is $n2^n$
Hypercubes.

Complete graphs, really connected! But lots of edges.
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2\(^n\) vertices. number of \(n\)-bit strings!
\(n2^{n-1}\) edges.

2\(^n\) vertices each of degree \(n\)
total degree is \(n2^n\) and half as many edges!
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n\(2^{n-1}\) edges.

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Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.
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An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ $(1x)$ with the additional edges $(0x, 1x)$. 
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An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$. 
Thm: Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$.

Terminology: $(S, V - S)$ is cut. $(E \cap S \times (V - S))$ - cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.
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Terminology:
Hypercube: Can’t cut me!

**Thm:** Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|

Terminology:
(S, V − S) is cut.
**Thm:** Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

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Proof:
Base Case: \(n = 1\)
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Proof:
Base Case: \(n = 1\) \(V\) is \(\{0,1\}\).
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Base Case: \(n = 1\) \(V\) = \(\{0,1\}\).
\(S = \{0\}\) has one edge leaving.
Proof of Large Cuts.

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Base Case: $n = 1$ $V = \{0, 1\}$.
$S = \{0\}$ has one edge leaving. $|S| = \phi$ has 0.
Proof of Large Cuts.

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Induction Step Idea

Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.
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**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.
Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.
Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.
Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

![Diagram of two connected cubes with edges labeled](image)
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).
Induction Step

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof:** Induction Step.
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**Proof:** Induction Step.
Recursive definition:
**Induction Step**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**
Recursive definition:

\[ H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.} \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

Recursive definition:
\[
H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.}
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\[
H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)
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Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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S = S_0 \cup S_1 \text{ where } S_0 \text{ in first, and } S_1 \text{ in other.}
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Induction Step

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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

Recursive definition:
- \(H_0 = (V_0, E_0), H_1 = (V_1, E_1)\), edges \(E_x\) that connect them.
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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)
Both \(S_0\) and \(S_1\) are small sides.
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2

Both \(S_0\) and \(S_1\) are small sides. So by induction.
Induction Step

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof: Induction Step.**

Recursive definition:

\[ H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.} \]

\[ H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x) \]

\[ S = S_0 \cup S_1 \text{ where } S_0 \text{ in first, and } S_1 \text{ in other.} \]

**Case 1:** $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$

Both $S_0$ and $S_1$ are small sides. So by induction.

Edges cut in $H_0 \geq |S_0|$. 
**Induction Step**

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof: Induction Step.**

Recursive definition:

- $H_0 = (V_0, E_0), H_1 = (V_1, E_1)$, edges $E_x$ that connect them.
- $H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
- $S = S_0 \cup S_1$ where $S_0$ in first, and $S_1$ in other.

**Case 1:** $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$

Both $S_0$ and $S_1$ are small sides. So by induction.
- Edges cut in $H_0 \geq |S_0|$.
- Edges cut in $H_1 \geq |S_1|$.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

Recursive definition:

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Both \(S_0\) and \(S_1\) are small sides. So by induction.

- Edges cut in \(H_0 \geq |S_0|\).
- Edges cut in \(H_1 \geq |S_1|\).
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

Recursive definition:

- \(H_0 = (V_0, E_0), H_1 = (V_1, E_1)\), edges \(E_x\) that connect them.
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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)

Both \(S_0\) and \(S_1\) are small sides. So by induction.

- Edges cut in \(H_0 \geq |S_0|\).
- Edges cut in \(H_1 \geq |S_1|\).

Total cut edges \(\geq |S_0| + |S_1| = |S|\).
**Induction Step**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step.

Recursive definition:

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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)

Both \(S_0\) and \(S_1\) are small sides. So by induction.

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Total cut edges \(\geq |S_0| + |S_1| = |S|\).
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[|S_0| \geq |V_0|/2.\]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

\[|S_0| \geq \frac{|V_0|}{2}.\]

Recall Case 1: \(|S_0|, |S_1| \leq \frac{|V|}{2}\)

\(|S_1| \leq \frac{|V_1|}{2}\) since \(|S| \leq \frac{|V|}{2}\).
**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]

\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]

\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]

\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]
Induction Step. Case 2.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]

- Recall Case 1: \(|S_0|, |S_1| \leq |V|/2 \]
- \(|S_1| \leq |V_1|/2 \) since \(|S| \leq |V|/2.\)
  \[ \implies \geq |S_1| \text{ edges cut in } E_1. \]

\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]
  \[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).


Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2\).

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Edges in \(E_x\) connect corresponding nodes.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

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Total edges cut:
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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|S_0| \geq |V_0|/2.
\]

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Total edges cut:
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\geq |S_1| + |V_0| - |S_0|
\]
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\[ |V_0| = \frac{|V|}{2} \geq |S|. \]

Also, case 3 where \(|S_1| \geq \frac{|V|}{2}\) is symmetric.
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \(\{0, 1\}^n\).
Hypercubes and Boolean Functions.

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Central area of study in computer science!
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Central area of study in computer science!

Yes/No Computer Programs \( \equiv \) Boolean function on \( \{0, 1\}^n \)
Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \( \{0, 1\}^n \).

Central area of study in computer science!

Yes/No Computer Programs \( \equiv \) Boolean function on \( \{0, 1\}^n \)

Central object of study.
Summary.

Euler: $v + f = e + 2$.
Tree. Plus adding edge adds face.
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Count face-edge incidences to get $2e \leq 3f$.
Replace $f$ in Euler.
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Coloring:
  degree $d$ vertex can be colored if $d + 1$ colors.
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Recolor separate and planarity: 5 color theorem.
Euler: \( v + f = e + 2 \).
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Count face-edge incidences to get \( 2e \leq 3f \).
Replace \( f \) in Euler.

Coloring:
degree \( d \) vertex can be colored if \( d + 1 \) colors.
Small degree vertex in planar graph: 6 color theorem.
Recolor separate and planarity: 5 color theorem.

Graphs:
Trees: sparsest connected.
Complete: densest
Hypercube: middle.
Modular Arithmetic.

Applications: cryptography, error correction.
Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y - x)$.
Key ideas for modular arithmetic.

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Proof:
Key ideas for modular arithmetic.

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$x = ad, \ y = bd,$
Key ideas for modular arithmetic.

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Proof:

$x = ad$, $y = bd$,

$(x - y) = (ad - bd) = d(a - b) \implies d|(x - y)$. 

\[
\begin{array}{c}
\end{array}
\]
Key ideas for modular arithmetic.

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$x = ad$, $y = bd$,
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Theorem: Every number $n \geq 2$ can be represented as a product of primes.
Key ideas for modular arithmetic.

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Proof:
\[ x = ad, \quad y = bd, \]
\[ (x - y) = (ad - bd) = d(a - b) \implies d|(x - y). \]

Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Proof: Either prime, or $n = a \times b$, and use strong induction. (Uniqueness? Later.)
What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler’s formula.
(C) The definition of a prime number.
(D) Euclid’s Lemma.
What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler’s formula.
(C) The definition of a prime number.
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(A) and (C)
Next Up.

Modular Arithmetic.
Clock Math

If it is 1:00 now.
If it is 1:00 now.
What time is it in 2 hours?
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours?
If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours?

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to addition/subtraction of 12.
101 = 12 \times 8 + 5.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.
Custom is only to use the representative in \{12, 1, ... , 11\} (Almost remainder, except for 12 and 0 are equivalent.)
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
Clock Math

If it is 1:00 now.
    What time is it in 2 hours? 3:00!
    What time is it in 5 hours? 6:00!
    What time is it in 15 hours? 16:00!
        Actually 4:00.
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
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If it is 1:00 now.
    What time is it in 2 hours? 3:00!
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Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
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   Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours?
If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system. 
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00!
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
  Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
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16 is the “same as 4” with respect to a 12 hour clock system.
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What time is it in 100 hours? 101:00! or 5:00.
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5 is the same as 101 for a 12 hour clock system.
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
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What time is it in 100 hours? 101:00! or 5:00.
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  What time is it in 2 hours? 3:00!
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Custom is only to use the representative in \{12, 1, \ldots, 11\}
(Almost remainder, except for 12 and 0 are equivalent.)
Day of the week.

This is Thursday is September 14, 2023.
Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now?
Day of the week.

This is Thursday is September 14, 2023. What day is it a year from now? on September 14, 2023?
Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
Day of the week.

This is Thursday is September 14, 2023. What day is it a year from now? on September 14, 2023? Number days. 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.
This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.
Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
  5 days from then.

11 days from then
  is day 1
  which is Monday!

What day is it a year from then?
Next year is not a leap year.
So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
  subtract 7 until smaller than 7.
  divide and get remainder.
  369/7
leaves quotient of 52 and remainder 5.
369
= 7(52) + 5
or September 15, 2022 is a Friday.
Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9
Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2
Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
  5 days from then. day 9 or day 2 or Tuesday.

11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year.
So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
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369/7 leaves quotient of 52 and remainder 5.
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Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
   0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
   5 days from then. day 9 or day 2 or Tuesday.
   25 days from then.

\[
\begin{align*}
11 & \equiv 4 \pmod{7} \\
29 & \equiv 1 \pmod{7} \\
369 & \equiv 5 \pmod{7}
\end{align*}
\]

or September 15, 2023 is a Friday.
Day of the week.

This is Thursday is September 14, 2023. What day is it a year from now? on September 14, 2023? Number days.

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Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29
Day of the week.

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25 days from then. day 29 or day 1.
Day of the week.

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  5 days from then. day 9 or day 2 or Tuesday.
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Day of the week.

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  5 days from then. day 9 or day 2 or Tuesday.
  25 days from then. day 29 or day 1. $29 = (7)4 + 1$
  two days are equivalent up to addition/subtraction of multiple of 7.
Day of the week.

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25 days from then. day 29 or day 1. \(29 = (7)4 + 1\)
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11 days from then
Day of the week.

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11 days from then is day 1
Day of the week.

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  11 days from then is day 1 which is Monday!
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two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
This is Thursday is September 14, 2023. What day is it a year from now? on September 14, 2023? Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. \(29 = (7)4 + 1\) two days are equivalent up to addition/subtraction of multiple of 7. 11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year.
Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
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Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. 29 = (7)4 + 1
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
  5 days from then. day 9 or day 2 or Tuesday.
  25 days from then. day 29 or day 1. 29 = (7)4 + 1
two days are equivalent up to addition/subtraction of multiple of 7.
  11 days from then is day 1 which is Monday!

What day is it a year from then?
  Next year is not a leap year. So 365 days from then.
  Day 4+365 or day 369.
This is Thursday is September 14, 2023. What day is it a year from now on September 14, 2023? Number days. 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
- 5 days from then. day 9 or day 2 or Tuesday.
- 25 days from then. day 29 or day 1. $29 = (7)4 + 1$
  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
 5 days from then. day 9 or day 2 or Tuesday.
 25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
 11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
  subtract 7 until smaller than 7.
Day of the week.

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What day is it a year from now? on September 14, 2023?
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Today: day 4.
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25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7
Day of the week.

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Today: day 4.
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two days are equivalent up to addition/subtraction of multiple of 7.
 11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
  subtract 7 until smaller than 7.
  divide and get remainder.
  $369/7$ leaves quotient of 52 and remainder 5.
Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
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0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

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11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7 leaves quotient of 52 and remainder 5. 369 = 7(52) + 5
Day of the week.

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Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$
or September 15, 2022 is a Friday.
This is Thursday is September 14, 2023.
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What day is it a year from then?
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Smallest representation:
  subtract 7 until smaller than 7.
  divide and get remainder.
369/7 leaves quotient of 52 and remainder 5. 369 = 7(52) + 5
  or September 15, 2022 is a Friday.
Years and years...

80 years?
Years and years...

80 years? 20 leap years.
Years and years...

80 years? 20 leap years. $366 \times 20$ days
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years.
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 4.
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. 
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Get Day: $4 + 2 \times 20 + 1 \times 60$.
Remainder when dividing by 7?

$104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!
Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $4 + 2 \times 6 + 1 \times 4$.
Or Day 6.
September 14, 2103 is Saturday.
"Reduce" at any time in calculation!

$17 / 52$
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?
Hmm.
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7?

What is remainder of 365 when dividing by 7?

Today is day 4.
Get Day: $4 + 366 \times 20 + 365 \times 60$.
Remainder when dividing by 7?

Or September 15, 2102 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $4 + 2 \times 6 + 1 \times 4$.

Or Day 6.
September 14, 2103 is Saturday.

"Reduce" at any time in calculation!

17 / 52
Years and years...

80 years? 20 leap years. \(366 \times 20\) days
60 regular years. \(365 \times 60\) days

Today is day 4.
It is day \(4 + 366 \times 20 + 365 \times 60\). Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? \(52 \times 7 + 2\).
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days

Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7?

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.

Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.
Or Day 6.

September 14, 2103 is Saturday.

"Reduce" at any time in calculation!
Years and years...

80 years? 20 leap years. 366 × 20 days
   60 regular years. 365 × 60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
   What is remainder of 366 when dividing by 7? 52 × 7 + 2.
   What is remainder of 365 when dividing by 7? 1
Years and years...

80 years? 20 leap years. $366 \times 20$ days
   60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
   What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
   What is remainder of 365 when dividing by 7? 1
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
  What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
  What is remainder of 365 when dividing by 7? 1
Today is day 4.
  Get Day: $4 + 2 \times 20 + 1 \times 60$
Years and years...

80 years?  20 leap years.  $366 \times 20$ days
    60 regular years.  $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
    What is remainder of 366 when dividing by 7?  $52 \times 7 + 2$.
    What is remainder of 365 when dividing by 7?  1
Today is day 4.
    Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Years and years...

80 years? 20 leap years. 366 × 20 days
60 regular years. 365 × 60 days
Today is day 4.
It is day 4 + 366 × 20 + 365 × 60. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? 52 × 7 + 2.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: 4 + 2 × 20 + 1 × 60 = 104
Remainder when dividing by 7?
Years and years...

80 years? 20 leap years. $366 \times 20$ days 
60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7$
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$. 
Years and years...

80 years? 20 leap years. $366 \times 20$ days
   60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!

"Reduce" at any time in calculation!
80 years? 20 leap years. $366 \times 20$ days  
60 regular years. $365 \times 60$ days

Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1

Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
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Further Simplify Calculation:
Years and years...

80 years? 20 leap years. $366 \times 20$ days  
60 regular years. $365 \times 60$ days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Or September 15, 2102 is Saturday!

Further Simplify Calculation:

20 has remainder 6 when divided by 7.
Years and years...

80 years? 20 leap years. 366 × 20 days
60 regular years. 365 × 60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Years and years...

80 years? 20 leap years. $366 \times 20$ days
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Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1

Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!

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20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$. 
Years and years...

80 years? 20 leap years. $366 \times 20$ days
   60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
   Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
   Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
   Or September 15, 2102 is Saturday!

Further Simplify Calculation:
   20 has remainder 6 when divided by 7.
   60 has remainder 4 when divided by 7.
Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.
   Or Day 6.
Years and years...

80 years?  20 leap years.  $366 \times 20$ days
    60 regular years.  $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$.  Equivalent to?

Hmm.
    What is remainder of 366 when dividing by 7?  $52 \times 7 + 2$.
    What is remainder of 365 when dividing by 7?  1
Today is day 4.
    Get Day:  $4 + 2 \times 20 + 1 \times 60 = 104$
    Remainder when dividing by 7?  $104 = 14 \times 7 + 6$.
    Or September 15, 2102 is Saturday!

Further Simplify Calculation:
    20 has remainder 6 when divided by 7.
    60 has remainder 4 when divided by 7.
Get Day:  $4 + 2 \times 6 + 1 \times 4 = 20$.
    Or Day 6.  September 14, 2103 is Saturday.
80 years? 20 leap years. $366 \times 20$ days
   60 regular years. $365 \times 60$ days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
   What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
   What is remainder of 365 when dividing by 7? 1
Today is day 4.
   Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
   Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
   Or September 15, 2102 is Saturday!

Further Simplify Calculation:
   20 has remainder 6 when divided by 7.
   60 has remainder 4 when divided by 7.
Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.
   Or Day 6. September 14, 2103 is Saturday.

“Reduce” at any time in calculation!
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \mod m \)” if and only if \( (x - y) \) is divisible by \( m \).
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \) (mod \( m \))” if and only if \( (x - y) \) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \pmod{m} \)” if and only if \( (x - y) \) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).

...or \( x = y + km \) for some integer \( k \).
Modular Arithmetic: refresher.

*x is congruent to y modulo m* or “*x ≡ y (mod m)***” if and only if (x – y) is divisible by m.

...or x and y have the same remainder w.r.t. m.
...or x = y + km for some integer k.
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \pmod{m} \)” if and only if \( (x - y) \) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).

...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence or residue classes:
Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.
...or $x$ and $y$ have the same remainder w.r.t. $m$.
...or $x = y + km$ for some integer $k$.

Mod 7 equivalence or residue classes:
\[
\{\ldots, -7, 0, 7, 14, \ldots \}
\]
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \pmod{m} \)” if and only if \((x - y)\) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).

...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence or residue classes:
\[ \{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \]
Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.

...or $x$ and $y$ have the same remainder w.r.t. $m$.

...or $x = y + km$ for some integer $k$.

Mod 7 equivalence or residue classes:

$\{ \ldots, -7, 0, 7, 14, \ldots \}$ $\{ \ldots, -6, 1, 8, 15, \ldots \}$ ...
Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.

...or $x$ and $y$ have the same remainder w.r.t. $m$.

...or $x = y + km$ for some integer $k$.

Mod 7 equivalence or residue classes:

$\{\ldots, -7, 0, 7, 14, \ldots \}$  $\{\ldots, -6, 1, 8, 15, \ldots \}$  $\ldots$

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$. 

Proof:

If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.

If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.

Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.

$\implies a + b \equiv c + d \pmod{m}$.

Can calculate with representative in $\{0, \ldots, m-1\}$. 

Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or \( "x \equiv y \ (\text{mod } m)" \)
if and only if \((x - y)\) is divisible by \( m \).
...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence or residue classes:
\( \{\ldots, -7, 0, 7, 14, \ldots \} \quad \{\ldots, -6, 1, 8, 15, \ldots \} \quad \ldots \)

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or \( " a \equiv c \ (\text{mod } m) \) and \( b \equiv d \ (\text{mod } m) \)
Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or \(x \equiv y \pmod{m}\)
if and only if \((x - y)\) is divisible by \(m\).
...or \(x\) and \(y\) have the same remainder w.r.t. \(m\).
...or \(x = y + km\) for some integer \(k\).

Mod 7 equivalence or \textit{residue} classes:
\[
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots
\]

\textbf{Useful Fact:} Addition, subtraction, multiplication can be done with
any equivalent \(x\) and \(y\).

or \(a \equiv c \pmod{m}\) and \(b \equiv d \pmod{m}\)
\[
\implies a + b \equiv c + d \pmod{m}\) and \(a \cdot b \equiv c \cdot d \pmod{m}\)\)
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$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.
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Mod 7 equivalence or residue classes:
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.

or “$a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$
\[\implies a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m}\]

**Proof:** If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$. 
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \ (\text{mod } m) \)” if and only if \( (x - y) \) is divisible by \( m \).
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Mod 7 equivalence or residue classes:
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \ (\text{mod } m) \) and \( b \equiv d \ (\text{mod } m) \)
\[ \implies a + b \equiv c + d \ (\text{mod } m) \] and \( a \cdot b \equiv c \cdot d \ (\text{mod } m) \)"

**Proof:** If \( a \equiv c \ (\text{mod } m) \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \ (\text{mod } m) \), then \( b = d + jm \) for some integer \( j \).
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or \( "x \equiv y \pmod{m}\)"
if and only if \( (x - y) \) is divisible by \( m \).
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Mod 7 equivalence or residue classes:
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots

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**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore,
Modular Arithmetic: refresher.

\(x\) is congruent to \(y\) modulo \(m\) or “\(x \equiv y \pmod{m}\)”
if and only if \((x - y)\) is divisible by \(m\).
...or \(x\) and \(y\) have the same remainder w.r.t. \(m\).
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Mod 7 equivalence or residue classes:
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \quad \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \(x\) and \(y\).

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**Proof:** If \(a \equiv c \pmod{m}\), then \(a = c + km\) for some integer \(k\).
If \(b \equiv d \pmod{m}\), then \(b = d + jm\) for some integer \(j\).
Therefore, \(a + b = c + d + (k + j)m\)
Modular Arithmetic: refresher.

\( x \text{ is congruent to } y \text{ modulo } m \) or “\( x \equiv y \pmod{m} \)” if and only if \( (x - y) \) is divisible by \( m \).

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...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence or residue classes:
\{\ldots, -7, 0, 7, 14, \ldots \} \quad \{\ldots, -6, 1, 8, 15, \ldots \} \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)
\[ \implies a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m} \]”

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore, \( a + b = c + d + (k + j)m \) and since \( k + j \) is integer.
Modular Arithmetic: refresher.

*x is congruent to y modulo m* or “$x \equiv y \pmod{m}$”

if and only if $(x - y)$ is divisible by $m$.

...or $x$ and $y$ have the same remainder w.r.t. $m$.

...or $x = y + km$ for some integer $k$.

Mod 7 equivalence or *residue* classes:

$\{\ldots, -7, 0, 7, 14, \ldots\}$ $\{\ldots, -6, 1, 8, 15, \ldots\}$ ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.

or “$a \equiv c \pmod{m}$ and $b \equiv d \pmod{m}$

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**Proof:** If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.

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Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.

$\implies a + b \equiv c + d \pmod{m}$. 

Can calculate with representative in $\{0, \ldots, m-1\}$.
Modular Arithmetic: refresher.

**x is congruent to y modulo m** or “\(x \equiv y \pmod{m}\)”
if and only if \((x - y)\) is divisible by \(m\).
...or \(x\) and \(y\) have the same remainder w.r.t. \(m\).
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Mod 7 equivalence or *residue* classes:
\[
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots
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**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \(x\) and \(y\).

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**Proof:** If \(a \equiv c \pmod{m}\), then \(a = c + km\) for some integer \(k\).
If \(b \equiv d \pmod{m}\), then \(b = d + jm\) for some integer \(j\).
Therefore, \(a + b = c + d + (k + j)m\) and since \(k + j\) is integer.
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Mod 7 equivalence or residual classes:
$\{\ldots, -7, 0, 7, 14, \ldots\}$ $\{\ldots, -6, 1, 8, 15, \ldots\}$ ...

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**Proof:** If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.
If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.
Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.
$\implies a + b \equiv c + d \pmod{m}$.

Can calculate with representative in $\{0, \ldots, m - 1\}$. 
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
Notation

\[ x \ (\text{mod } m) \text{ or } \text{mod } (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).
Notation

\( x \ ( \text{mod} \ m) \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x,m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x,m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
Notation

\[ x \pmod{m} \text{ or } \text{mod } (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 \]
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x, m) = x - \lfloor \frac{x}{m} \rfloor m
\]

\( \lfloor \frac{x}{m} \rfloor \) is quotient.

\[
\text{mod} (29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12
\]
Notation

\[ x \ (\text{mod } m) \text{ or } \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \mod (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \mod (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 \]
Notation

\( x \pmod{m} \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4
\]
Notation

\[ x \pmod{m} \text{ or } \mod(x,m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \mod(x,m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod(29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \text{X} = 5 \]

Work in this system.
Notation

\( x \pmod{m} \) or \( \text{mod } (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod } (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4
\]

Work in this system.
\( a \equiv b \pmod{m} \)
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \\
\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}
\]

\[
\text{mod} \ (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = X = 5
\]

Work in this system.

\[ a \equiv b \pmod{m}. \]
 Says two integers \( a \) and \( b \) are equivalent modulo \( m \).
Notation

$x \pmod{m}$ or $\text{mod} (x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

\[
\text{mod}(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\[\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}\]

\[
\text{mod}(29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 = 5
\]

Work in this system.

\[a \equiv b \pmod{m}\]
Says two integers $a$ and $b$ are equivalent modulo $m$.

**Modulus** is $m$
Notation

\[x \ (\text{mod } m) \text{ or } \ \text{mod} \ (x, m)\]
- remainder of \(x\) divided by \(m\) in \(\{0, \ldots, m - 1\}\).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\(\left\lfloor \frac{x}{m} \right\rfloor\) is quotient.

\[
\text{mod} \ (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \not{4} = 5
\]

Work in this system.

\[a \equiv b \ (\text{mod } m)\]
Says two integers \(a\) and \(b\) are equivalent modulo \(m\).

**Modulus** is \(m\)

\(6 \equiv \)
Notation

$x \pmod{m}$ or $\text{mod} (x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m - 1\}$.

$$\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m$$

$\left\lfloor \frac{x}{m} \right\rfloor$ is quotient.

$$\text{mod} (29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = \color{red}5$$

Work in this system.

$a \equiv b \pmod{m}$.
Says two integers $a$ and $b$ are equivalent modulo $m$.

**Modulus** is $m$

$6 \equiv 3 + 3$
Notation

\( x \pmod{m} \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5
\]

Work in this system.
\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[
6 \equiv 3 + 3 \equiv 3 + 10
\]
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} \ (29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = 4 \equiv 5
\]

Work in this system.

\( a \equiv b \pmod{m} \).

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \).
Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of $x$ divided by $m$ in \{0, \ldots, m−1\}.

$$\mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m$$

\(\left\lfloor \frac{x}{m} \right\rfloor\) is quotient.

$$\mod(29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor\right) \times 12 = 29 - (2) \times 12 = 4$$

Work in this system.

\(a \equiv b \pmod{m}\).

Says two integers $a$ and $b$ are equivalent modulo $m$.

**Modulus** is $m$

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 =$
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 = \mathbf{5}
\]

Work in this system.

\( a \equiv b \pmod{m} \).

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]

\[ 6 = 3 + 3 \]
Notation

$x \pmod{m}$ or $\text{mod}(x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

$$\text{mod}(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m$$

$\left\lfloor \frac{x}{m} \right\rfloor$ is quotient.

$$\text{mod}(29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4$$

Work in this system.

$a \equiv b \pmod{m}$.
Says two integers $a$ and $b$ are equivalent modulo $m$.

**Modulus** is $m$

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10$
Notation

\( x \mod m \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5
\]

Work in this system.

\( a \equiv b \mod m \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[
6 \equiv 3 + 3 \equiv 3 + 10 \mod 7.
\]

\[
6 = 3 + 3 = 3 + 10 \mod 7.
\]
Notation

$x \pmod{m}$ or $\text{mod } (x, m)$
- remainder of $x$ divided by $m$ in $\{0, \ldots, m-1\}$.

$$\text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m$$

$\left\lfloor \frac{x}{m} \right\rfloor$ is quotient.

$$\text{mod } (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 \neq 5$$

Work in this system.

$a \equiv b \pmod{m}$.
Says two integers $a$ and $b$ are equivalent modulo $m$.

**Modulus** is $m$

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10 \pmod{7}$.

Generally, not $6 \pmod{7} = 13 \pmod{7}$. 
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod} (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 = 5 \]

Work in this system.
\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
\[ 6 = 3 + 3 = 3 + 10 \pmod{7}. \]

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
But probably won’t take off points,
Notation

\( x \mod m \) or \( \text{mod}(x,m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod}(x,m) = x - \lfloor \frac{x}{m} \rfloor m \]
\( \lfloor \frac{x}{m} \rfloor \) is quotient.

\[ \text{mod}(29,12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \text{\textcolor{red}{5}} \]

Work in this system.

\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \]
\[ 6 = 3 + 3 = 3 + 10 \pmod{7} \]

Generally, not \( 6 \equiv 13 \pmod{7} \).
But probably won’t take off points, still hard for us to read.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) is \( y \) where \( xy = 1 \);
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1; \)**

1 \( \text{is multiplicative identity element.} \)
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \);

1 is multiplicative identity element.

In modular arithmetic, 1 is the multiplicative identity element.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) is \( y \) where \( xy = 1 \);

1 is **multiplicative identity element**.

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) is \( y \) with \( xy = 1 \mod m \).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1; \) 
**1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) **is** \( y \) **with** \( xy = 1 \mod m).**

For 4 modulo 7 inverse is 2: 
\[ 2 \cdot 4 \equiv 8 \equiv 1 \mod 7. \]
Inverses and Factors.

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For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \) \( (\text{mod } 7) \).

Can solve \( 4x = 5 \) \( (\text{mod } 7) \).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \iff \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \iff x = \frac{3}{2} . \]

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For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \) **(mod 7).**

Can solve \( 4x = 5 \) **(mod 7).**

\[ 2 \cdot 4x = 2 \cdot 5 \text{ (mod 7)} \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

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**Multiplicative inverse of** \( x \) \text{ mod } m \text{ is } y \text{ with } xy = 1 \text{ (mod m)}.\)

For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \text{ (mod 7)}. \]

Can solve \( 4x = 5 \) \text{ (mod 7)}.
\[ 2 \cdot 4x = 2 \cdot 5 \text{ (mod 7)} \]
\[ 8x = 10 \text{ (mod 7)} \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

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For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}. \]

Can solve \( 4x = 5 \) (mod 7).
\[ 2 \cdot 4x = 2 \cdot 5 \pmod{7} \]
\[ 8x = 10 \pmod{7} \]
\[ x = 3 \pmod{7} \]
Inverses and Factors.

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For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \) **(mod 7).**

Can solve \( 4x = 5 \) **(mod 7).**

\[ 2 \cdot 4x = 2 \cdot 5 \] **(mod 7)**
\[ 8x = 10 \] **(mod 7)**
\[ x = 3 \] **(mod 7)**

Check!
Inverses and Factors.

Division: multiply by multiplicative inverse.

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For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \mod 7. \]

Can solve \( 4x = 5 \mod 7 \).

\begin{align*}
2 \cdot 4x &= 2 \cdot 5 \mod 7 \\
8x &= 10 \mod 7 \\
x &= 3 \mod 7
\end{align*}

Check! \( 4(3) = 12 = 5 \mod 7 \).
Inverses and Factors.

Division: multiply by multiplicative inverse.

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For 4 modulo 7 inverse is 2:
\[ 2 \cdot 4 \equiv 8 \equiv 1 \ (\text{mod} \ 7). \]

Can solve \( 4x = 5 \ (\text{mod} \ 7) \).
\( x = 3 \ (\text{mod} \ 7) \implies \text{Check!} \ 4(3) = 12 = 5 \ (\text{mod} \ 7). \)

For 8 modulo 12: no multiplicative inverse!
Inverses and Factors.

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For 4 modulo 7 inverse is 2: 
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Can solve \( 4x = 5 \mod 7 \).
\[ x = 3 \mod 7 \implies \text{Check! } 4(3) = 12 = 5 \mod 7. \]

For 8 modulo 12: no multiplicative inverse!

“Common factor of 4”
Inverses and Factors.

Division: multiply by multiplicative inverse.

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For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).

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\( x = 3 \mod 7 \) \(\because\) Check! \( 4(3) = 12 = 5 \mod 7 \).

For 8 modulo 12: no multiplicative inverse!

“Common factor of 4” \(\implies\)

\( 8k - 12\ell \) is a multiple of four for any \( \ell \) and \( k \) \(\implies\)
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

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**Multiplicative inverse of** \( x \) **mod** \( m \) **is** \( y \) **with** \( xy = 1 \mod m \).

For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \mod 7. \]

Can solve \( 4x = 5 \mod 7 \). \( x = 3 \mod 7 \): Check! \( 4(3) = 12 = 5 \mod 7 \).

For 8 modulo 12: no multiplicative inverse!

“Common factor of 4” \( \implies \)
\( 8k - 12\ell \) **is a multiple of four for any** \( \ell \) **and** \( k \) \( \implies \)
\( 8k \not\equiv 1 \mod 12 \) **for any** \( k \).
Mark true statements.
(A) Multiplicative inverse of 2 mod 5 is 3 mod 5.
(B) The multiplicative inverse of \((n - 1) \pmod{n} = ((n - 1) \pmod{n})\).
(C) Multiplicative inverse of 2 mod 5 is 0.5.
(D) Multiplicative inverse of 4 \(= -1 \pmod{5}\).
(E) \((-1) \times (-1) = 1\). Woohoo.
(F) Multiplicative inverse of 4 mod 5 is 4 mod 5.
Mark true statements.

(A) Multiplicative inverse of 2 mod 5 is 3 mod 5.
(B) The multiplicative inverse of \((n - 1) \mod n\) = \(((n - 1) \mod n)\).
(C) Multiplicative inverse of 2 mod 5 is 0.5.
(D) Multiplicative inverse of 4 = -1 (mod 5).
(E) \((-1) \times (-1) = 1\). Woohoo.
(F) Multiplicative inverse of 4 mod 5 is 4 mod 5.

(C) is false. 0.5 has no meaning in arithmetic modulo 5.
Greatest Common Divisor and Inverses.

Thm:
If greatest common divisor of $x$ and $m$, $\text{gcd}(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of $x$ and $m$, $gcd(x,m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

**Proof** $\implies$ :
**Claim:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$. 

\[ a - b \equiv 0 \pmod{m} \Rightarrow a - b = km \text{ for some integer } k. \]

Since $gcd(x,m) = 1$, the prime factorization of $m$ and $x$ do not contain common primes. Therefore, $(a - b)$'s factorization contains all primes in $m$'s factorization.

So $(a - b)$ has to be a multiple of $m$.

But $a, b \in \{0, 1, \ldots, m-1\}$. Contradiction.
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Each of \( m \) numbers in \( S \) correspond to one of \( m \) equivalence classes modulo \( m \).
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Proof of Claim:
Greatest Common Divisor and Inverses.

**Thm:** If greatest common divisor of \( x \) and \( m \), \( \text{gcd}(x, m) \), is 1, then \( x \) has a multiplicative inverse modulo \( m \).

**Proof \Rightarrow:**

**Claim:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

Each of \( m \) numbers in \( S \) correspond to one of \( m \) equivalence classes modulo \( m \).

\( \Rightarrow \) One must correspond to 1 modulo \( m \). **Inverse Exists!**

Proof of Claim: If not distinct, then \( \exists a, b \in \{0, \ldots, m-1\}, a \neq b \),
Greatest Common Divisor and Inverses.

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**Proof of Claim:** If not distinct, then $\exists a, b \in \{0, \ldots, m - 1\}$, $a \neq b$, where $(ax \equiv bx \mod m) \implies (a - b)x \equiv 0 \mod m$.
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$(ax \equiv bx \mod m) \implies (a − b)x \equiv 0 \mod m$ 

Or $(a − b)x = km$ for some integer $k$. 
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

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**Claim:** The set $S = \{0x, 1x, \ldots, (m - 1)x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

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$\gcd(x, m) = 1$
Greatest Common Divisor and Inverses.

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$\implies$ Prime factorization of $m$ and $x$ do not contain common primes.
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So $(a - b)$ has to be multiple of $m$. 
Greatest Common Divisor and Inverses.

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$\implies (a - b) \geq m.$
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

**Proof $\Rightarrow$:**

**Claim:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

Each of $m$ numbers in $S$ correspond to one of $m$ equivalence classes modulo $m$.

$\Rightarrow$ One must correspond to $1$ modulo $m$. **Inverse Exists!**

Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m-1\}, a \neq b$, where

$$(ax \equiv bx \pmod{m}) \Rightarrow (a - b)x \equiv 0 \pmod{m}$$

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$\Rightarrow (a - b)$ factorization contains all primes in $m$’s factorization.

So $(a - b)$ has to be multiple of $m$.

$\Rightarrow (a - b) \geq m$. But $a, b \in \{0, \ldots, m-1\}$. 
Greatest Common Divisor and Inverses.

Thm:
If greatest common divisor of $x$ and $m$, gcd$(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.

Proof $\implies$:
Claim: The set $S = \{0x, 1x, \ldots, (m - 1)x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

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So \( (a-b) \) has to be multiple of \( m \).
\( \implies (a-b) \geq m \). But \( a, b \in \{0, \ldots m-1\} \). Contradiction. \( \square \)
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).
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... For \( x = 4 \) and \( m = 6 \). All products of 4...
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For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \]
Thm: If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

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... For $x = 4$ and $m = 6$. All products of 4...

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\}$
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}\]
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

... 

For $x = 4$ and $m = 6$. All products of 4...

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}$

reducing $\pmod{6}$
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**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing \( \pmod{6} \)

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reducing (mod 6)

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reducing \( \mod 6 \)

\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct.
**Proof review. Consequence.**

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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reducing \((\mod 6)\)

\[ S = \{0, 4, 2, 0, 4, 2\} \]
Not distinct. Common factor 2.
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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reducing (mod 6)

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S = \{0, 4, 2, 0, 4, 2\}
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Not distinct. Common factor 2. Can’t be 1.
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Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} \]
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... For \(x = 4\) and \(m = 6\). All products of 4...
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\[S = \{0, 4, 2, 0, 4, 2\}\]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \(x = 5\) and \(m = 6\).
\[S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\}\] = \{0, 5, 4, 3, 2, 1\}
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For \( x = 5 \) and \( m = 6 \).

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All distinct,
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).  
\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]  
All distinct, contains 1!
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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reducing \( \mod 6 \)

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S = \{0, 4, 2, 0, 4, 2\}
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Not distinct. Common factor 2. Can't be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

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All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).
Proof review. Consequence.

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**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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All distinct, contains 1! 5 is multiplicative inverse of 5 \( (\mod 6) \).

(Hmm. What normal number is it own multiplicative inverse?)
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...
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(Hmm. What normal number is it own multiplicative inverse?) 1
**Proof review. Consequence.**

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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reducing (mod 6)

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Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...

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For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1! 5 is multiplicative inverse of 5 \((\mod 6)\).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

5\(x = 3 \mod 6\) What is \( x \)?
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing \( \mod 6 \)

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S = \{0, 4, 2, 0, 4, 2\}
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For \( x = 5 \) and \( m = 6 \).

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S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
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All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[5x = 3 \mod 6\] What is \( x \)? Multiply both sides by 5.
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing \( \mod 6 \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \] What is \( x \)? Multiply both sides by 5.

\[
x = 15
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]
reducing \( \mod 6 \)

\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]
All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \]

What is \( x \)? Multiply both sides by 5.

\[ x = 15 = 3 \mod 6 \]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...
   \[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]
   reducing \( \pmod{6} \)
   \[ S = \{0, 4, 2, 0, 4, 2\} \]
   Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
   \[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]
   All distinct, contains 1! 5 is multiplicative inverse of 5 \( \pmod{6} \).
   (Hmm. What normal number is it own multiplicative inverse?) 1 -1.

   \[ 5x = 3 \pmod{6} \] What is \( x \)? Multiply both sides by 5.
   \[ x = 15 = 3 \pmod{6} \]

   \[ 4x = 3 \pmod{6} \]
**Proof review. Consequence.**

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

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reducing \( \mod 6 \)

\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \] What is \( x \)? Multiply both sides by 5.

\[ x = 15 = 3 \mod 6 \]

\[ 4x = 3 \mod 6 \] No solutions.
**Proof review. Consequence.**

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

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S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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Reducing \( \mod 6 \)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
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All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \). (Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[
5x = 3 \mod 6 \quad \text{What is } x? \quad \text{Multiply both sides by 5.}
\]

\[
x = 15 = 3 \mod 6
\]

\[
4x = 3 \mod 6 \quad \text{No solutions. Can’t get an odd.}
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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\[
5x = 3 \mod 6 \quad \text{What is } x? \quad \text{Multiply both sides by 5.}
\]

\[
x = 15 = 3 \mod 6
\]

\[
4x = 3 \mod 6 \quad \text{No solutions. Can’t get an odd.}
\]

\[
4x = 2 \mod 6
\]
Proof review. Consequence.

Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

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reducing \( \pmod{6} \)

\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1! 5 is multiplicative inverse of 5 \( \pmod{6} \).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

5\( x = 3 \) (mod 6) What is \( x \)? Multiply both sides by 5.

\[ x = 15 = 3 \pmod{6} \]

4\( x = 3 \) (mod 6) No solutions. Can’t get an odd.

4\( x = 2 \) (mod 6) Two solutions!
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

For \( x = 4 \) and \( m = 6 \). All products of 4...
\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\} \]
reducing \( \mod 6 \)
\[ S = \{0, 4, 2, 0, 4, 2\} \]
Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).
\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]
All distinct, contains 1! 5 is multiplicative inverse of 5 \( \mod 6 \).
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \] What is \( x \)? Multiply both sides by 5.
\[ x = 15 = 3 \mod 6 \]
\[ 4x = 3 \mod 6 \] No solutions. Can’t get an odd.
\[ 4x = 2 \mod 6 \] Two solutions! \( x = 2, 5 \mod 6 \)
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

... 

For $x = 4$ and $m = 6$. All products of 4...

$S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}$

reducing (mod 6)

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct. Common factor 2. Can’t be 1. No inverse.

For $x = 5$ and $m = 6$.

$S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}$

All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

$5x = 3 \pmod{6}$ What is $x$? Multiply both sides by 5.

$x = 15 = 3 \pmod{6}$

$4x = 3 \pmod{6}$ No solutions. Can’t get an odd.

$4x = 2 \pmod{6}$ Two solutions! $x = 2, 5 \pmod{6}$

Very different for elements with inverses.
Proof Review 2: Bijections.

If $\gcd(x,m) = 1$. 

1. One to one: there is a unique pre-image (single $x$ where $y = f(x)$).
2. Onto: the sizes of the domain and co-domain are the same.

Example:

- $x = 3$, $m = 4$.
  - $f(1) = 3 \mod 4 = 3$.
  - $f(2) = 6 = 2 \mod 4$.
  - $f(3) = 1 \mod 3$.

Oh yeah.

- $f(0) = 0$.

Bijection $\equiv$ unique pre-image and same size.

All the images are distinct.

Example:

- $x = 2$, $m = 4$.
  - $f(1) = 2$.
  - $f(2) = 0$.
  - $f(3) = 2$.

Oh yeah.

- $f(0) = 0$.

Not a bijection.
Proof Review 2: Bijectons.

If $\gcd(x,m) = 1$.

Then the function $f(a) = xa \mod m$ is a bijection.
Proof Review 2: Bijections.

If $\gcd(x,m) = 1$.

Then the function $f(a) = xa \mod m$ is a bijection.

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Proof Review 2: Bijections.

If $\gcd(x,m) = 1$.
Then the function $f(a) = xa \mod m$ is a bijection.
One to one: there is a unique pre-image (single $x$ where $y = f(x)$.)
Onto: the sizes of the domain and co-domain are the same.

$x = 3, m = 4$. 

$\Rightarrow$ unique pre-image for any image.
Proof Review 2: Bijections.

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Then the function \( f(a) = xa \mod m \) is a bijection.
One to one: there is a unique pre-image (single \( x \) where \( y = f(x) \)).
Onto: the sizes of the domain and co-domain are the same.

\( x = 3, m = 4. \)
\( f(1) = 3(1) = 3 \mod 4, \)
Proof Review 2: Bijections.

If \( \gcd(x,m) = 1 \).

Then the function \( f(a) = xa \mod m \) is a bijection.

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Onto: the sizes of the domain and co-domain are the same.

\( x = 3, m = 4 \).
\[

def(1) = 3(1) = 3 \mod 4, \\
def(2) = 6 = 2 \mod 4, \\
def(3) = 1 \mod 3.
\]
Oh yeah. \( f(0) = 0 \mod 3 \).
Proof Review 2: Bijections.

If \( \gcd(x, m) = 1 \).
Then the function \( f(a) = xa \mod m \) is a bijection.
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Bijection
Proof Review 2: Bijections.

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Bijection $\equiv$ unique pre-image and same size.
Proof Review 2: Bijections.

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Bijection \( \equiv \) unique pre-image and same size.

All the images are distinct. \( \implies \) unique pre-image for any image.
Proof Review 2: Bijections.

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\( x = 3, m = 4 \).
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\( x = 2, m = 4 \).
Proof Review 2: Bijections.

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One to one: there is a unique pre-image (single \( x \) where \( y = f(x) \).)

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\( x = 2, m = 4 \).

\[ f(1) = 2, \]
\[ f(2) = 0, \]
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Proof Review 2: Bijections.

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\( x = 2, m = 4 \).

\( f(1) = 2 \),
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Oh yeah. \( f(0) = 0 \).
If \( \gcd(x,m) = 1 \).
Then the function \( f(a) = xa \mod m \) is a bijection.
One to one: there is a unique pre-image (single \( x \) where \( y = f(x) \)).
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\[ \begin{align*}
  x &= 3, \ m = 4. \\
  f(1) &= 3(1) = 3 \pmod{4}, \\
  f(2) &= 6 = 2 \pmod{4}, \\
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\end{align*} \]

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  x &= 2, \ m = 4. \\
  f(1) &= 2, \\
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  f(3) &= 2 \\
  \text{Oh yeah. } f(0) &= 0.
\end{align*} \]

Not a bijection.
Which is bijection?

(A) $f(x) = x$ for domain and range being $\mathbb{R}$
(B) $f(x) = ax \pmod{n}$ for $x \in \{0, \ldots, n-1\}$ and $\gcd(a, n) = 2$
(C) $f(x) = ax \pmod{n}$ for $x \in \{0, \ldots, n-1\}$ and $\gcd(a, n) = 1$
Which is bijection?
(A) $f(x) = x$ for domain and range being $\mathbb{R}$
(B) $f(x) = ax \pmod{n}$ for $x \in \{0,\ldots,n-1\}$ and $\gcd(a,n) = 2$
(C) $f(x) = ax \pmod{n}$ for $x \in \{0,\ldots,n-1\}$ and $\gcd(a,n) = 1$

(B) is not.
Only if

Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).
Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$. Assume the inverse of $a$ is $x^{-1}$, or $ax = 1 + km$. 
Only if

Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).

Assume the inverse of \( a \) is \( x^{-1} \), or \( ax = 1 + km \).

\[ x = nd \] and \( m = \ell d \) for \( d > 1 \).
Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume the inverse of $a$ is $x^{-1}$, or $ax = 1 + km$.

$x = nd$ and $m = \ell d$ for $d > 1$.

Thus,
Only if

Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.
Assume the inverse of $a$ is $x^{-1}$, or $ax = 1 + km$.
$x = nd$ and $m = \ell d$ for $d > 1$.
Thus,
\[a(nd) =\]
Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).
Assume the inverse of \( a \) is \( x^{-1} \), or \( ax = 1 + km \).

\[ x = nd \quad \text{and} \quad m = \ell d \quad \text{for} \quad d > 1. \]

Thus,

\[ a(nd) = 1 + k\ell d \]
Only if

Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume the inverse of $a$ is $x^{-1}$, or $ax = 1 + km$.

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Thus,

$$a(nd) = 1 + k\ell d$$ or
Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).
Assume the inverse of \( a \) is \( x^{-1} \), or \( ax = 1 + km \).
\[ x = nd \text{ and } m = \ell d \text{ for } d > 1. \]
Thus,
\[ a(nd) = 1 + k\ell d \text{ or } \]
\[ d(na - k\ell) = 1. \]
Thm: If $\gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume the inverse of $a$ is $x^{-1}$, or $ax = 1 + km$.

$x = nd$ and $m = \ell d$ for $d > 1$.

Thus,

$$a(nd) = 1 + k\ell d \text{ or }$$

$$d(na - k\ell) = 1.$$ 

But $d > 1$ and $z = (na - k\ell) \in \mathbb{Z}$. 


Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).

Assume the inverse of \( a \) is \( x^{-1} \), or \( ax = 1 + km \).

\[ x = nd \] and \( m = \ell d \) for \( d > 1 \).

Thus,

\[ a(nd) = 1 + k\ell d \text{ or} \]
\[ d(na - k\ell) = 1. \]

But \( d > 1 \) and \( z = (na - k\ell) \in \mathbb{Z} \).

so \( dz \neq 1 \) and \( dz = 1 \). Contradiction.
Thm: If $\gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume the inverse of $a$ is $x^{-1}$, or $ax = 1 + km$.

$x = nd$ and $m = \ell d$ for $d > 1$.

Thus,

$$a(nd) = 1 + k\ell d$$

or

$$d(na - k\ell) = 1.$$ 

But $d > 1$ and $z = (na - k\ell) \in \mathbb{Z}$.

so $dz \neq 1$ and $dz = 1$. Contradiction.
Finding inverses.

How to find the inverse?

Find $\gcd(x, m)$.

- Greater than 1? No multiplicative inverse.
- Equal to 1? Multiplicative inverse.

Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$.

Very slow.
Finding inverses.

How to find the inverse?

How to find if \( x \) has an inverse modulo \( m \)?
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?
Find $\gcd(x, m)$. 

Finding inverses.

How to find the inverse?

How to find if \( x \) has an inverse modulo \( m \)?

Find gcd \( (x, m) \).

Greater than 1?
Finding inverses.

How to find the inverse?

How to find if \( x \) has an inverse modulo \( m \)?

Find \( \text{gcd} \ (x, m) \).

  Greater than 1? No multiplicative inverse.
Finding inverses.

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How to find if $x$ has an inverse modulo $m$?

Find $\text{gcd}(x, m)$.
  - Greater than 1? No multiplicative inverse.
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Finding inverses.

How to find the inverse?
How to find if \( x \) has an inverse modulo \( m \)?

Find \( \gcd (x, m) \).

- Greater than 1? No multiplicative inverse.
- Equal to 1? Multiplicative inverse.

Algorithm:
How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find gcd $(x, m)$.
   - Greater than 1? No multiplicative inverse.
   - Equal to 1? Multiplicative inverse.

Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$. 
Finding inverses.

How to find the inverse?
How to find if \( x \) has an inverse modulo \( m \)?

Find gcd \((x, m)\).
  - Greater than 1? No multiplicative inverse.
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Algorithm: Try all numbers up to \( x \) to see if it divides both \( x \) and \( m \).
Very slow.
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\gcd(x, m)$.

- Greater than 1? No multiplicative inverse.
- Equal to 1? Multiplicative inverse.

Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$.

Very slow.
Inverses

Next up.
Inverses

Next up.
Inverses

Next up.

Euclid’s Algorithm.
Inverses

Next up.

Euclid’s Algorithm.
Runtime.
Inverses

Next up.

Euclid’s Algorithm.
Runtime.
Euclid’s Extended Algorithm.
Does 2 have an inverse mod 8?

No. Any multiple of 2 is 2 away from 0 + 8k for any k ∈ N.

Does 2 have an inverse mod 9?

Yes. 5 \cdot 2 = 10 = 1 \mod 9.

Does 6 have an inverse mod 9?

No. Any multiple of 6 is 3 away from 0 + 9k for any k ∈ N.

3 = \gcd(6, 9) ≠ 1.

x has an inverse modulo m if and only if \gcd(x, m) > 1?

Yes. \gcd(x, m) = 1?

Now what?: Compute gcd! Compute Inverse modulo m.
Does 2 have an inverse mod 8? No.
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$. 

Does 2 have an inverse mod 9? No.
Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9?
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.
Does 2 have an inverse mod 9? Yes.
Does 2 have an inverse mod 8? No.
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   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
   $2(5) = 10 = 1 \mod 9$. 
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   $2(5) = 10 = 1 \mod 9$.

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   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
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   Any multiple of 2 is 2 away from 0 + 8k for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
   $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from 0 + 9k for any $k \in \mathbb{N}$.
   $3 = gcd(6, 9)$!
Does 2 have an inverse mod 8? No.  
Any multiple of 2 is 2 away from \(0 + 8k\) for any \(k \in \mathbb{N}\).

Does 2 have an inverse mod 9? Yes. 5  
\[2(5) = 10 = 1 \mod 9.\]

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\[3 = \gcd(6, 9)\]

\(x\) has an inverse modulo \(m\) if and only if
Does 2 have an inverse mod 8? No.
    Any multiple of 2 is 2 away from 0 + 8k for any $k \in \mathbb{N}$.

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$x$ has an inverse modulo $m$ if and only if
    $\gcd(x, m) > 1$?
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5
   \[ 2(5) = 10 = 1 \mod 9. \]

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
   \[ 3 = \gcd(6, 9)! \]

$x$ has an inverse modulo $m$ if and only if
   \[ \gcd(x, m) > 1? \text{ No.} \]
   \[ \gcd(x, m) = 1? \]
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. 5

$2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.

Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.

$3 = \gcd(6, 9)!$

$x$ has an inverse modulo $m$ if and only if

$\gcd(x, m) > 1$? No.

$\gcd(x, m) = 1$? Yes.
Does 2 have an inverse mod 8? No.
Any multiple of 2 is 2 away from 0 + 8k for any k ∈ \mathbb{N}.

Does 2 have an inverse mod 9? Yes. 5
2(5) = 10 = 1 \mod 9.

Does 6 have an inverse mod 9? No.
Any multiple of 6 is 3 away from 0 + 9k for any k ∈ \mathbb{N}.
3 = gcd(6, 9)!

x has an inverse modulo m if and only if
gcd(x, m) > 1? No.
gcd(x, m) = 1? Yes.

Now what?:
Compute gcd!
Does 2 have an inverse mod 8? No.
   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

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   $2(5) = 10 = 1 \mod 9$.

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
   $3 = \gcd(6, 9)$!

$x$ has an inverse modulo $m$ if and only if
   $\gcd(x, m) > 1$? No.
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Now what?:
   Compute gcd!
   Compute Inverse modulo $m$. 
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   Any multiple of 2 is 2 away from $0 + 8k$ for any $k \in \mathbb{N}$.

Does 2 have an inverse mod 9? Yes. $5$
   \[ 2(5) = 10 = 1 \mod 9. \]

Does 6 have an inverse mod 9? No.
   Any multiple of 6 is 3 away from $0 + 9k$ for any $k \in \mathbb{N}$.
   \[ 3 = \gcd(6, 9)! \]

$x$ has an inverse modulo $m$ if and only if
   \[ \gcd(x, m) > 1? \text{ No.} \]
   \[ \gcd(x, m) = 1? \text{ Yes.} \]

Now what?:
   Compute \( \gcd! \)
   Compute Inverse modulo $m$.  

**Notation:** $d|x$ means “$d$ divides $x$” or
Divisibility...

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 
Divisibility...

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Fact:** If $d | x$ and $d | y$ then $d | (x + y)$ and $d | (x - y)$.
Notation: $d \mid x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d \mid x$ and $d \mid y$ then $d \mid (x + y)$ and $d \mid (x - y)$.

Is it a fact?
Divisibility...

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Is it a fact? Yes?
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x - y)$.

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Is it a fact? Yes? No?

**Proof:** \(d | x\) and \(d | y\) or...
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

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Proof: $d|x$ and $d|y$ or $x = \ell d$ and $y = kd$
Divisibility...

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**Proof:** $d | x$ and $d | y$ or $x = \ell d$ and $y = kd$

$$\implies x - y = kd - \ell d$$
Notation: \(d \mid x\) means “\(d\) divides \(x\)” or \(x = kd\) for some integer \(k\).

Fact: If \(d \mid x\) and \(d \mid y\) then \(d \mid (x + y)\) and \(d \mid (x - y)\).

Is it a fact? Yes? No?

Proof: \(d \mid x\) and \(d \mid y\) or
\[
x = \ell d \quad \text{and} \quad y = kd
\]

\[
\implies x - y = kd - \ell d = (k - \ell)d
\]
Notation: \(d|x\) means “\(d\) divides \(x\)” or 
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Proof: \(d|x\) and \(d|y\) or 
\[x = \ell d\] and \(y = kd\)

\[
\implies x - y = kd - \ell d = (k - \ell)d \implies d|(x - y)
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Divisibility...

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x = \ell d \quad \text{and} \quad y = kd
\]
\[
\implies x - y = kd - \ell d = (k - \ell)d \implies d \mid (x - y)
\]
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or

---

**Lemma 1:**
If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x, y)$.

**Proof:**

\[
\text{mod}(x, y) = x - \lfloor \frac{x}{y} \rfloor \cdot y = x - s \cdot y
\]

for integer $s = kd - s \ell d$ for integers $k, \ell$ where $x = kd$ and $y = \ell d$.

Therefore $d|\text{mod}(x, y)$.

And $d|y$ since it is in condition.

**Lemma 2:**
If $d|y$ and $d|\text{mod}(x, y)$ then $d|y$ and $d|x$.

**Proof:**

Similar.

---

**GCD Mod Corollary:**

\[
\gcd(x, y) = \gcd(y, \text{mod}(x, y))
\]

**Proof:**

$x$ and $y$ have the same set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma 1 and 2.

Same common divisors $\Rightarrow$ largest is the same.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 

Lemma 1: If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod}(x, y)$.

Proof: 

\[
\text{mod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y = x - s \cdot y
\]

where $x = kd$ and $y = \ell d$ for integers $k$, $\ell$. 

Therefore $d | \text{mod}(x, y)$.

And $d | y$ since it is in condition.

Lemma 2: If $d | y$ and $d | \text{mod}(x, y)$ then $d | y$ and $d | x$.

Proof: Similar.

GCD Mod Corollary: $\gcd(x, y) = \gcd(y, \text{mod}(x, y))$.

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$$\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y$$
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\[
\text{mod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y
\]
\[
= x - s \cdot y \quad \text{for integer } s
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= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
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More divisibility

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\[
\text{mod } (x,y) = x - \lfloor x/y \rfloor \cdot y \\
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= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \).

**Proof:**

\[
\text{mod} \ (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - s \cdot y \quad \text{for integer } s
\]
\[
= kd - sld \quad \text{for integers } k, l \text{ where } x = kd \text{ and } y = ld
\]
\[
= (k - sl)d
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Therefore \( d \mid \text{mod} \ (x, y) \).
More divisibility

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= (k - s\ell)d
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Therefore $d|\text{mod}(x,y)$. And $d|y$ since it is in condition.
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Therefore \(d \mid \text{mod} \ (x, y)\). And \(d \mid y\) since it is in condition. 
\[\square\]
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**Proof:**

\[
\begin{align*}
\text{mod} \ (x, y) &= x - \lfloor x/y \rfloor \cdot y \\
&= x - s \cdot y \quad \text{for integer } s \\
&= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
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\end{align*}
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Therefore \( d \mid \text{mod} \ (x, y) \). And \( d \mid y \) since it is in condition. \( \square \)

**Lemma 2:** If \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \) then \( d \mid y \) and \( d \mid x \).

**Proof...:** Similar.
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= kd - s \ell d \quad \text{for integers} \ k, \ell \text{ where } x = kd \text{ and } y = \ell d
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= (k - s\ell)d
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Therefore $d|\text{mod}(x, y)$. And $d|y$ since it is in condition.

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**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \text{mod}(x, y))$. 
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Therefore $d| \text{mod} \ (x, y)$. And $d|y$ since it is in condition. \hfill \square

**Lemma 2:** If $d|y$ and $d| \text{mod} \ (x, y)$ then $d|y$ and $d|x$.

**Proof...:** Similar. Try this at home. \hfill \squareish.

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Same common divisors $\implies$ largest is the same.
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$. 
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**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$?
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)?  7
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0.
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Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?
Euclid’s algorithm.

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Hey, what’s \( \gcd(7, 0) \)?  7  since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?  \( x \)
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Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

```scheme
(define (euclid x y)
    (if (= y 0)
        x
        (euclid y (mod x y)))) ***
```
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \text{mod } (x, y)) \).

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**Theorem:** \( (\text{euclid } x y) = \gcd(x, y) \) if \( x \geq y \).
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod(x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)? \( x \)

\[
\text{(define (euclid } x y) \\
(\text{if } (= y 0) \\
\quad x \\
\quad (euclid \ y (\text{mod } x y)))) \] ***

**Theorem:** \( (\text{euclid } x y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \mod(x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)? \( x \)

\[
(\text{define} \ (\text{euclid} \ x \ y) \\
\quad (\text{if} \ (= \ y \ 0) \\
\quad \quad x \\
\quad \quad (\text{euclid} \ y \ (\mod \ x \ y)))) \quad ***
\]

**Theorem:** \( (\text{euclid} \ x \ y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.
**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

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**Theorem:** $(\text{euclid } x y) = \gcd(x, y)$ if $x \geq y$.

**Proof:** Use Strong Induction.

**Base Case:** $y = 0$, “$x$ divides $y$ and $x$”

$\implies$ “$x$ is common divisor and clearly largest.”
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod (x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))
```

**Theorem:** \( (\text{euclid } x \ y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
\( \implies \) “\( x \) is common divisor and clearly largest.”

**Induction Step:** \( \mod (x, y) < y \leq x \) when \( x \geq y \)
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

```scheme
(define (euclid x y)
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```

**Theorem:** $(\text{euclid } x \ y) = \gcd(x, y)$ if $x \geq y$.

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  $\implies$ “$x$ is common divisor and clearly largest.”

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call in line (***)) meets conditions plus arguments “smaller”
Euclid’s algorithm.

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Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

(define (euclid x y)
  (if (= y 0)
    x
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**Theorem:** \((\text{euclid } x \ y) = \gcd(x, y)\) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
\[ \implies \text{“} x \text{ is common divisor and clearly largest.”} \]

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)

call in line (***)) meets conditions plus arguments “smaller”
and by strong induction hypothesis
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y)))))  ***

**Theorem:** \( (\text{euclid } x \ y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \implies \text{“} x \text{ is common divisor and clearly largest.”} \]

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)

call in line (***)) meets conditions plus arguments “smaller”
and by strong induction hypothesis
computes \( \gcd(y, \mod(x, y)) \)
Euclid's algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what's \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What's \( \gcd(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))) ***
```

**Theorem:** \( \text{(euclid } x y \text{)} = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[\implies \text{“} x \text{ is common divisor and clearly largest.”}\]

**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)

call in line (***), meets conditions plus arguments “smaller”
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computes \( \gcd(y, \mod(x, y)) \)
which is \( \gcd(x, y) \) by GCD Mod Corollary.
Euclid’s algorithm.

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What’s \(\gcd(x, 0)\)?  

\[
\text{(define (euclid x y)} \\
\text{ (if (= y 0) } \\
\text{ x} \\
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\]
Excursion: Value and Size.

Before discussing running time of gcd procedure...
Excursion: Value and Size.

Before discussing running time of gcd procedure...
What is the value of 1,000,000?
Excursion: Value and Size.

Before discussing running time of gcd procedure...
What is the value of 1,000,000?
one million or 1,000,000!
Excursion: Value and Size.

Before discussing running time of gcd procedure...

What is the value of 1,000,000?

One million or 1,000,000!

What is the “size” of 1,000,000?
Excursion: Value and Size.

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Number of digits in base 10: 7.
Excursion: Value and Size.

Before discussing running time of gcd procedure...

What is the value of 1,000,000?
One million or 1,000,000!

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Number of digits in base 10: 7.
Number of bits (a digit in base 2): 21.
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For a number $x$, what is its size in bits?
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$$ n = b(x) \approx \log_2 x $$
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$$n = b(x) \approx \log_2 x$$
Euclid procedure is fast.

**Theorem:** $(\text{euclid } x \ y)$ uses $2n$ ”divisions” where $n = b(x) \approx \log_2 x$. 

Is this good? Better than trying all numbers in \( \{2, \ldots, y/2\} \)?

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

If $y \approx x$ roughly $y$ uses $n$ bits...

$2^n - 1$ divisions! Exponential dependence on size!

200 bit number. $2^{100} \approx 10^{30} =$ ”million, trillion, trillion” divisions!

$2^n$ is much faster! .. roughly 200 divisions.
Euclid procedure is fast.

**Theorem:** \( (\text{euclid} \ x \ y) \) uses \( 2n \) ”divisions” where \( n = b(x) \approx \log_2 x \).

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Check 2,
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) "divisions" where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots, y/2\}\)?

Check 2, check 3,
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) ”divisions” where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots \ y/2\}\)?

Check 2, check 3, check 4,
Euclid procedure is fast.

**Theorem:** (euclid x y) uses $2n$ ”divisions” where $n = b(x) \approx \log_2 x$.

Is this good? Better than trying all numbers in \{2, \ldots y/2\}?

Check 2, check 3, check 4, check 5 \ldots, check $y/2$. 


Euclid procedure is fast.

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If \(y \approx x\)
Euclid procedure is fast.

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Check 2, check 3, check 4, check 5 . . . , check \(y/2\).

If \(y \approx x\) roughly \(y\) uses \(n\) bits . . .

\(2^{n-1}\) divisions! Exponential dependence on size!
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) "divisions" where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots y/2\}\)?

Check 2, check 3, check 4, check 5 \ldots, check \(y/2\).

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101 bit number.
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Check 2, check 3, check 4, check 5 . . . , check \(y/2\).

If \(y \approx x\) roughly \(y\) uses \(n\) bits ...

\(2^{n-1}\) divisions! Exponential dependence on size!

101 bit number. \(2^{100} \approx 10^{30} = \) “million, trillion, trillion” divisions!

\(2n\) is much faster! .. roughly 200 divisions.
Assume $\log_2 1,000,000$ is 20 to the nearest integer. Mark what’s true.
Assume $\log_2 1,000,000$ is 20 to the nearest integer. Mark what’s true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.
Assume \( \log_2 1,000,000 \) is 20 to the nearest integer. Mark what’s true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

(A) and (C).
Poll

Which are correct?

(A) $\gcd(700, 568) = \gcd(568, 132)$
(B) $\gcd(8, 3) = \gcd(3, 2)$
(C) $\gcd(8, 3) = 1$
(D) $\gcd(4, 0) = 4$
Which are correct?

(A) gcd(700,568) = gcd (568,132)
(B) gcd(8,3) = gcd(3,2)
(C) gcd(8,3) = 1
(D) gcd(4,0) = 4
Algorithms at work.

Trying everything

Try gcd(700, 568)

Try gcd(568, 132)

Try gcd(132, 40)

Try gcd(40, 12)

Try gcd(12, 4)

Try gcd(4, 0)

Notice: The first argument decreases rapidly. At least a factor of 2 in two recursive calls. (The second is less than the first.)
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( \frac{y}{2} \).
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd x y)” at work.

\[
euclid(700, 568)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( \frac{y}{2} \).
“(gcd x y)” at work.

\[
euclid(700, 568) \\
euclid(568, 132)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

euclid(700, 568)
  euclid(568, 132)
    euclid(132, 40)
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 ..., check $y/2$.
“(gcd x y)” at work.

\[
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

\[
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12) \\
euclid(12, 4)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

```
euclid(700, 568)
euclid(568, 132)
euclid(132, 40)
euclid(40, 12)
euclid(12, 4)
euclid(4, 0)
```
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 \ldots, check \( y/2 \).

“(gcd x y)” at work.

\[
\begin{align*}
\text{euclid}(700, 568) \\
\text{euclid}(568, 132) \\
\text{euclid}(132, 40) \\
\text{euclid}(40, 12) \\
\text{euclid}(12, 4) \\
\text{euclid}(4, 0) \\
4
\end{align*}
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y / 2$.
“(gcd x y)” at work.

\[
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12) \\
euclid(12, 4) \\
euclid(4, 0) = 4
\]

Notice: The first argument decreases rapidly.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

\[
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12) \\
euclid(12, 4) \\
euclid(4, 0) \\
4
\]

Notice: The first argument decreases rapidly. At least a factor of 2 in two recursive calls.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

\[
\text{euclid}(700, 568) \\
\quad \text{euclid}(568, 132) \\
\quad \quad \text{euclid}(132, 40) \\
\quad \quad \quad \text{euclid}(40, 12) \\
\quad \quad \quad \quad \text{euclid}(12, 4) \\
\quad \quad \quad \quad \quad \text{euclid}(4, 0) \\
\quad \quad \quad \quad \quad \quad 4
\]

Notice: The first argument decreases rapidly.
   At least a factor of 2 in two recursive calls.

(The second is less than the first.)
Runtime Proof.

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))
```

**Theorem:** \((\text{euclid } x \ y)\) uses \(O(n)\) ”divisions” where \(n = b(x)\).
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After $2\log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
Runtime Proof.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After $2\log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.

One more recursive call to finish.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Theorem: (euclid x y) uses \(O(n)\) ”divisions” where \(n = b(x)\).

Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

After \(2\log_2 x = O(n)\) recursive calls, argument \(x\) is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After $2\log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
$O(n)$ divisions.
(define (euclid x y)
 (if (= y 0)
  x
  (euclid y (mod x y)))))

**Fact:**
First arg decreases by at least factor of two in two recursive calls.
Runtime Proof (continued.)

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))
```

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

**Proof of Fact:** Recall that first argument decreases every call.
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
  \( \implies \) true in one recursive call;
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.
Case 1: \(y < \frac{x}{2}\), first argument is \(y\)
   \(\implies\) true in one recursive call;
Case 2: Will show “\(y \geq \frac{x}{2}\)” \(\implies\) “\(\text{mod}(x, y) \leq \frac{x}{2}\).”
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
  \( \Rightarrow \) true in one recursive call;

Case 2: Will show “\( y \geq \frac{x}{2} \) \( \Rightarrow \) “\( \text{mod}(x, y) \leq \frac{x}{2} \).”
  \( \text{mod}(x, y) \) is second argument in next recursive call,
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

**Proof of Fact:** Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
  \( \implies \) true in one recursive call;

Case 2: Will show \( y \geq x/2 \) \( \implies \) \( mod(x, y) \leq x/2. \)

\( mod(x, y) \) is second argument in next recursive call, and becomes the first argument in the next one.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

**Proof of Fact:** Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $$\implies \text{true in one recursive call;}$$

Case 2: Will show “$y \geq x/2$” $\implies \text{“mod}(x, y) \leq x/2.”$
  \[
  \text{mod (x, y) is second argument in next recursive call,}
  \]
  \[
  \text{and becomes the first argument in the next one.}
  \]

When $y \geq x/2$, then


Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
  \( \Rightarrow \) true in one recursive call;

Case 2: Will show "\( y \geq x/2 \) \( \Rightarrow \) "mod\((x, y) \leq x/2." 

\( \text{mod} \,(x, y) \) is second argument in next recursive call, and becomes the first argument in the next one.
When \( y \geq x/2 \), then
\[
\left\lfloor \frac{x}{y} \right\rfloor = 1,
\]
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show “$y \geq x/2$” $\implies$ “$\text{mod}(x, y) \leq x/2$.”

  $\text{mod}(x, y)$ is second argument in next recursive call, and becomes the first argument in the next one.

When $y \geq x/2$, then

  $\left\lfloor \frac{x}{y} \right\rfloor = 1$,
  
  $\text{mod}(x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor =$
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
    \( \implies \) true in one recursive call;

Case 2: Will show \( y \geq \frac{x}{2} \) \( \implies \) \( \text{mod}(x, y) \leq \frac{x}{2} \).

\( \text{mod} (x, y) \) is second argument in next recursive call, and becomes the first argument in the next one.

When \( y \geq \frac{x}{2} \), then
\[
\left\lfloor \frac{x}{y} \right\rfloor = 1,
\]
\[
\text{mod} (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - \frac{x}{2}
\]
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.
Case 1: $y < \frac{x}{2}$, first argument is $y$
  $\implies$ true in one recursive call;
Case 2: Will show “$y \geq \frac{x}{2}$” $\implies$ “$\text{mod}(x, y) \leq \frac{x}{2}$.”
  $\text{mod} (x, y)$ is second argument in next recursive call, and becomes the first argument in the next one.
When $y \geq \frac{x}{2}$, then
  \[
  \left\lfloor \frac{x}{y} \right\rfloor = 1,
  \]
  $\text{mod} (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - \frac{x}{2} = \frac{x}{2}$
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
  \( \implies \) true in one recursive call;

Case 2: Will show "\( y \geq x/2 \) \( \implies \) "mod(x, y) \( \leq \) x/2."
  mod \( x, y \) is second argument in next recursive call, and becomes the first argument in the next one.
  When \( y \geq x/2 \), then
  \[ \left\lfloor \frac{x}{y} \right\rfloor = 1, \]
  \[
  \text{mod} \ (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - x/2 = x/2
  \]
Remark

(define (euclid x y) (if (= y 0) x (euclid y (- x y))))
Remark

(define (euclid x y) (if (= y 0) x (euclid y (- x y))))

Didn’t necessarily need to do gcd.
Remark

(define (euclid x y) (if (= y 0) x (euclid y (- x y))))

Didn’t necessarily need to do gcd.
Runtime proof still works.
Finding an inverse?

We showed how to efficiently tell if there is an inverse.
Finding an inverse?

We showed how to efficiently tell if there is an inverse.
Extend euclid to find inverse.
Euclid’s GCD algorithm.

\[
\begin{align*}
\text{(define (euclid x y)} & \\
 & \begin{cases} 
\text{x} & \text{if } (= y 0) \\
\text{(euclid y (mod x y))} & \text{otherwise}
\end{cases}
\end{align*}
\]
Euclid’s GCD algorithm.

\[
\text{(define (euclid x y)} \\
\text{ (if (= y 0)} \\
\text{ \quad x)} \\
\text{ (euclid y (mod x y))})
\]

Computes the gcd\((x, y)\) in \(O(n)\) divisions.
Euclid’s GCD algorithm.

{(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))
)

Computes the gcd($x, y$) in $O(n)$ divisions.

For $x$ and $m$, if gcd($x, m$) = 1 then $x$ has an inverse modulo $m$. 
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse.
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse. How do we find a multiplicative inverse?
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: \( x \equiv y \pmod{N} \) if \( x = y + kN \) for some integer \( k \).
Modular Arithmetic Lecture in a minute.

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For $a \equiv b \pmod{N}$, and $c \equiv d \pmod{N}$,
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Division?
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Division? Multiply by multiplicative inverse.
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\( a \pmod{N} \) has multiplicative inverse, \( a^{-1} \pmod{N} \).
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If and only if \( \gcd(a, N) = 1 \).
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$a \pmod{N}$ has multiplicative inverse, $a^{-1} \pmod{N}$.

If and only if $gcd(a, N) = 1$.

Why?
Modular Arithmetic Lecture in a minute.

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Why? If: $f(x) = ax \pmod{N}$ is a bijection on $\{1, \ldots, N-1\}$. 
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$ax - ay = 0 \pmod{N} \implies a(x - y)$ is a multiple of $N$. 
Modular Arithmetic Lecture in a minute.

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Modular Arithmetic Lecture in a minute.

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Euclid’s Alg: \( \gcd(x, y) = \gcd(y \mod x, x) \)
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Fast cuz value drops by a factor of two every two recursive calls.
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Know if there is an inverse, but how do we find it?
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Know if there is an inverse, but how do we find it? On Tuesday!
Euclid’s Extended GCD Theorem:
For any $x, y$ there are integers $a, b$ where

$$ax + by = \text{gcd}(x, y)$$

"Make $d$ out of sum of multiples of $x$ and $y$.

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\text{gcd}(x, m) = 1$.

$$ax + bm = 1$$

$$ax \equiv 1 \pmod{m}$$

So $a$ is a multiplicative inverse of $x$ (mod $m$)!!

Example: For $x = 12$ and $y = 35$, $\text{gcd}(12, 35) = 1$.

$$(3)12 + (-1)35 = 1$$

$a = 3$ and $b = -1$.

The multiplicative inverse of 12 (mod 35) is 3.
Euclid’s Extended GCD Theorem:
For any $x, y$ there are integers $a, b$ where
$$ax + by$$
Extended GCD

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For any \(x, y\) there are integers \(a, b\) where
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What is multiplicative inverse of \( x \) modulo \( m \)?
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**Extended GCD**

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What is multiplicative inverse of $x$ modulo $m$?

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$$ax + bm = 1$$

$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$
Euclid’s Extended GCD Theorem:
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ax + bm = 1
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So \( a \) multiplicative inverse of \( x \) \pmod m!!
Extended GCD

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The multiplicative inverse of 12 (mod 35) is 3.
Make $d$ out of $x$ and $y$..?

$$\text{gcd}(35, 12)$$
Make $d$ out of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \% 12)
\]
Make \( d \) out of \( x \) and \( y \)...

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35%12) \\
gcd(11, 1) ;; gcd(11, 12%11)
\]
Make $d$ out of $x$ and $y$..?

gcd(35, 12)
gcd(12, 11) ;; gcd(12, \% 12)
gcd(11, 1) ;; gcd(11, 12 \% 11)
gcd(1, 0)
1
Make $d$ out of $x$ and $y$..?

```plaintext
gcd(35, 12)
gcd(12, 11) ;; gcd(12, 35%12)
gcd(11, 1) ;; gcd(11, 12%11)
gcd(1, 0)
  1
```

How did gcd get 11 from 35 and 12?
Make $d$ out of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \% 12) \\
gcd(11, 1) ;; gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?

\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]
Make \( d \) out of \( x \) and \( y \)...

\[
gcd(35, 12) \\
gcd(12, 11) \ ; ; \ gcd(12, 35 \% 12) \\
gcd(11, 1) \ ; ; \ gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\]

How did \( \text{gcd} \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( \text{gcd} \) get 1 from 12 and 11?

12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
Make $d$ out of $x$ and $y$..?

\[
\text{gcd}(35, 12) \\
\text{gcd}(12, 11) ;; \text{gcd}(12, 35 \mod 12) \\
\text{gcd}(11, 1) ;; \text{gcd}(11, 12 \mod 11) \\
\text{gcd}(1, 0) \\
1
\]

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\[
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\[
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\]
Make $d$ out of $x$ and $y$...?

\[
gcd(35, 12) \\
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gcd(11, 1) ;; gcd(11, 12 \mod 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor \times 12 = 35 - (2 \times 12) = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor \times 11 = 12 - (1 \times 11) = 1
\]

Algorithm finally returns 1.
Make $d$ out of $x$ and $y$..?

$$\text{gcd}(35, 12)$$
$$\text{gcd}(12, 11) ;; \text{gcd}(12, 35 \% 12)$$
$$\text{gcd}(11, 1) ;; \text{gcd}(11, 12 \% 11)$$
$$\text{gcd}(1, 0)$$
$$1$$

How did gcd get 11 from 35 and 12?

$$35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11$$

How does gcd get 1 from 12 and 11?

$$12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1$$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Make $d$ out of $x$ and $y$..?

$$\text{gcd}(35, 12)$$
$$\text{gcd}(12, 11) ;; \text{gcd}(12, 35 \mod 12)$$
$$\text{gcd}(11, 1) ;; \text{gcd}(11, 12 \mod 11)$$
$$\text{gcd}(1, 0)$$
$$1$$

How did gcd get 11 from 35 and 12?
$$35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11$$

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$$12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1$$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Get 1 from 12 and 11.
Make \( d \) out of \( x \) and \( y \)?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;;& \; \gcd(12, 35 \% 12) \\
gcd(11, 1) ;;& \; \gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did \( \gcd \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( \gcd \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11
\]
Make \( d \) out of \( x \) and \( y \)?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) \quad ;; \quad gcd(12, 35 \% 12) \\
gcd(11, 1) \quad ;; \quad gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did \( \text{gcd} \) get 11 from 35 and 12?

\[
35 - \lfloor \frac{35}{12} \rfloor \times 12 = 35 - (2) \times 12 = 11
\]

How does \( \text{gcd} \) get 1 from 12 and 11?

\[
12 - \lfloor \frac{12}{11} \rfloor \times 11 = 12 - (1) \times 11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12)
\]

Get 11 from 35 and 12 and plugin....
Make \( d \) out of \( x \) and \( y \)...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) \;; \; gcd(12, 35 \% 12) \\
gcd(11, 1) \;; \; gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did \( gcd \) get 11 from 35 and 12?
\[35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11\]

How does \( gcd \) get 1 from 12 and 11?
\[12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35\]

Get 11 from 35 and 12 and plugin.... Simplify.
Make $d$ out of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) \quad ;; \quad gcd(12, 35 \% 12) \\
gcd(11, 1) \quad ;; \quad gcd(11, 12 \% 11) \\
gcd(1, 0) \\
\]

1

How did \( gcd \) get 11 from 35 and 12?
\[
35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11
\]

How does \( gcd \) get 1 from 12 and 11?
\[
12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin.... Simplify.
Make \( d \) out of \( x \) and \( y \)...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) \ ;; \ gcd(12, 35 \mod 12) \\
gcd(11, 1) \ ;; \ gcd(11, 12 \mod 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did \( \gcd \) get 11 from 35 and 12?

\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( \gcd \) get 1 from 12 and 11?

\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin.... Simplify. \( a = 3 \) and \( b = -1 \).

\[
a = 3 \quad \text{and} \quad b = -1.
\]
Extended GCD Algorithm.

\textbf{ext-gcd}(x, y)
\begin{align*}
\text{if } y &= 0 \text{ then return } (x, 1, 0) \\
\text{else} & \\
(d, a, b) & := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{align*}

Claim: Returns \((d, a, b)\) such that
\(d = \text{gcd}(a, b)\) and \(d = ax + by\).

Example:
\[
\begin{align*}
\text{ext-gcd}(35, 12) & \quad \text{ext-gcd}(12, 11) \\
& \quad \text{ext-gcd}(11, 1) \\
& \quad \text{ext-gcd}(1, 0) \\
& \quad \text{return } (1,1,0) \\
& \quad \text{return } (1,0,1) \\
& \quad \text{return } (1,1,-1) \\
& \quad \text{return } (1,-1,3)
\end{align*}
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

if \( y = 0 \) then return \((x, 1, 0)\)
else

\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]

return \((d, b, a - \text{floor}(x/y) \times b)\)

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \quad \begin{cases} 
\text{if } y = 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
\quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \\
\quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{cases}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\text{ext-gcd}(35, 12)
\]
Extended GCD Algorithm.

\[
\operatorname{ext-gcd}(x, y)
\]

\[
\begin{align*}
\text{if } y & = 0 \text{ then return}(x, 1, 0) \\
\text{else} & \\
(d, a, b) & := \operatorname{ext-gcd}(y, \operatorname{mod}(x,y)) \\
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{align*}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\begin{align*}
\operatorname{ext-gcd}(35, 12) \\
\operatorname{ext-gcd}(12, 11)
\end{align*}
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]
\[
\begin{align*}
\text{if } y &= 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
&\quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \\
&\quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{align*}
\]

Claim: Returns \((d, a, b)\): \(d = \text{gcd}(a, b)\) and \(d = ax + by\).
Example:

\[
\text{ext-gcd}(35, 12) \\
\text{ext-gcd}(12, 11) \\
\text{ext-gcd}(11, 1)
\]

\[
\frac{47}{52}
\]
Extended GCD Algorithm.

\[\text{ext-gcd}(x, y)\]
\[
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[\text{ext-gcd}(35, 12)\]
\[\quad \text{ext-gcd}(12, 11)\]
\[\quad \quad \text{ext-gcd}(11, 1)\]
\[\quad \quad \quad \text{ext-gcd}(1, 0)\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

\[
\begin{align*}
\text{if } y &= 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
&\quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
&\quad \text{return } (d, b, a - \text{floor}(x/y) \cdot b)
\end{align*}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - [x/y] \cdot b =
\]

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
&\quad \text{ext-gcd}(12, 11) \\
&\quad \quad \text{ext-gcd}(11, 1) \\
&\quad \quad \quad \text{ext-gcd}(1, 0) \\
&\quad \text{return } (1, 1, 0) ;; 1 = (1)1 + (0)0
\end{align*}
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x,y)
\]
\[
\begin{align*}
&\text{if } y = 0 \text{ then return } (x, 1, 0) \\
&\text{else} \\
&\quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \\
&\quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{align*}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1\)

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
\text{ext-gcd}(12, 11) \\
&\quad \text{ext-gcd}(11, 1) \\
&\quad \text{ext-gcd}(1, 0) \\
&\quad \text{return } (1, 1, 0) \ ;; 1 = (1)1 + (0)0 \\
&\quad \text{return } (1, 0, 1) \ ;; 1 = (0)11 + (1)1
\end{align*}
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

if \(y = 0\) then return \((x, 1, 0)\)
else

\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]

return \((d, b, a - \text{floor}(x/y) \times b)\)

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1\)

\[
\text{ext-gcd}(35, 12)
\]

\[
\text{ext-gcd}(12, 11)
\]

\[
\text{ext-gcd}(11, 1)
\]

\[
\text{ext-gcd}(1, 0)
\]

return \((1, 1, 0)\); \(1 = (1)1 + (0)0\)

return \((1, 0, 1)\); \(1 = (0)11 + (1)1\)

return \((1, 1, -1)\); \(1 = (1)12 + (-1)11\)
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

\[
\begin{align*}
\text{if } y = 0 & \text{ then return }(x, 1, 0) \\
\text{else} & \\
& \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
& \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{align*}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 35/12 \rfloor \cdot (-1) = 3\)

\[
\begin{align*}
\text{ext-gcd}(35, 12) & \\
\text{ext-gcd}(12, 11) & \\
& \quad \text{ext-gcd}(11, 1) \\
& \quad \text{ext-gcd}(1, 0) \\
& \quad \text{return } (1, 1, 0) \quad ;; 1 = (1)1 + (0)0 \\
& \quad \text{return } (1, 0, 1) \quad ;; 1 = (0)11 + (1)1 \\
& \quad \text{return } (1, 1, -1) \quad ;; 1 = (1)12 + (-1)11 \\
& \quad \text{return } (1, -1, 3) \quad ;; 1 = (-1)35 + (3)12
\end{align*}
\]
Extended GCD Algorithm.

ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
  else
    (d, a, b) := ext-gcd(y, mod(x,y))
    return (d, b, a - floor(x/y) * b)

Claim: Returns (d,a,b): d = gcd(a,b) and d = ax + by.
Example:

ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
        return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1) ;; 1 = (0)11 + (1)1
        return (1,1,-1) ;; 1 = (1)12 + (-1)11
        return (1,-1, 3) ;; 1 = (-1)35 +(3)12
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then } \text{return}(x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \\
\quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]
Extended GCD Algorithm.

ext-gcd(x, y)
    if y = 0 then return (x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)

**Theorem:** Returns \((d, a, b)\), where \(d = \text{gcd}(a, b)\) and

\[ d = ax + by. \]
Correctness.

**Proof:** Strong Induction.\(^1\)

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.¹
Base: ext-gcd(x, 0) returns (d = x, 1, 0) with x = (1)x + (0)y.

Induction Step: Returns (d, A, B) with d = Ax + By
Ind hyp: ext-gcd(y, mod (x, y)) returns (d, a, b) with
  d = ay + b( mod (x, y))

¹Assume d is gcd(x, y) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** \texttt{ext-gcd}(x, 0) returns \( (d = x, 1, 0) \) with \( x = (1)x + (0)y \).

**Induction Step:** Returns \( (d, A, B) \) with \( d = Ax + By \)

Ind hyp: \texttt{ext-gcd}(y, \texttt{mod}(x, y)) returns \( (d, a, b) \) with \[ d = ay + b(\texttt{mod}(x, y)) \]

\texttt{ext-gcd}(x, y) calls \texttt{ext-gcd}(y, \texttt{mod}(x, y)) so

\(^1\)Assume \( d \) is \( \texttt{gcd}(x, y) \) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** ext-gcd\((x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: ext-gcd\((y, \mod(x, y))\) returns \((d, a, b)\) with \(d = ay + b(\mod(x, y))\)

ext-gcd\((x, y)\) calls ext-gcd\((y, \mod(x, y))\) so

\[
d = ay + b \cdot (\mod(x, y))
\]

---

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \mod(x, y))\) returns \((d, a, b)\) with
\[
d = ay + b(\mod(x, y))
\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod(x, y))\) so
\[
d = ay + b\cdot(\mod(x, y))
\]
\[
= ay + b\cdot(x - \left\lfloor \frac{x}{y} \right\rfloor y)
\]

---

\(^1\)Assume \(d\) is \(\gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.¹

Base: \( \text{ext-gcd}(x, 0) \) returns \((d = x, 1, 0)\) with \( x = (1)x + (0)y \).

Induction Step: Returns \((d, A, B)\) with \( d = Ax + By \)

Ind hyp: \( \text{ext-gcd}(y, \text{mod}(x, y)) \) returns \((d, a, b)\) with \( d = ay + b(\text{mod}(x, y)) \)

\( \text{ext-gcd}(x, y) \) calls \( \text{ext-gcd}(y, \text{mod}(x, y)) \) so

\[
\begin{align*}
  d &= ay + b \cdot (\text{mod}(x, y)) \\
  &= ay + b \cdot (x - \left\lfloor \frac{x}{y} \right\rfloor y) \\
  &= bx + (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b)y
\end{align*}
\]

¹Assume \( d \) is \( \text{gcd}(x, y) \) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** `ext-gcd(x, 0)` returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

**Ind hyp:** `ext-gcd(y, \mod(x, y))` returns \((d, a, b)\) with \(d = ay + b(\mod(x, y))\)

`ext-gcd(x, y)` calls `ext-gcd(y, \mod(x, y))` so

\[
\begin{align*}
d &= ay + b \cdot (\mod(x, y)) \\
&= ay + b \cdot (x - \left\lceil \frac{x}{y} \right\rceil y) \\
&= bx + (a - \left\lceil \frac{x}{y} \right\rceil \cdot b)y
\end{align*}
\]

And `ext-gcd` returns \((d, b, (a - \left\lceil \frac{x}{y} \right\rceil \cdot b))\) so theorem holds!

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.¹

Base: \( \text{ext-gcd}(x, 0) \) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

Induction Step: Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \( \text{ext-gcd}(y, \mod(x, y)) \) returns \((d, a, b)\) with
\[
d = ay + b(\mod(x, y))
\]

\( \text{ext-gcd}(x, y) \) calls \( \text{ext-gcd}(y, \mod(x, y)) \) so
\[
d = ay + b \cdot (\mod(x, y))
\]
\[
= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y)
\]
\[
= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y
\]

And \( \text{ext-gcd} \) returns \((d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))\) so theorem holds!  

¹Assume \(d\) is \(gcd(x, y)\) by previous proof.
Prove: returns \((d, A, B)\) where \(d = Ax + By\).

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \mod(x, y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

50 / 52
Prove: returns \((d, A, B)\) where \(d = Ax + By\).

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Recursively: \(d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)\)

Prove: returns \((d, A, B)\) where \(d = Ax + By\).

\[
\text{ext-gcd}(x, y)
\]
\[
\text{if } y = 0 \text{ then return } (x, 1, 0)
\]
\[
\text{else}
\]
\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]
\[
\text{return } (d, b, a - \text{floor}(x/y) \cdot b)
\]

Recursively: \(d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y\)
Prove: returns \((d, A, B)\) where \(d = Ax + By\).

ext-gcd\((x, y)\)

if \(y = 0\) then return\((x, 1, 0)\)

else

\((d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))\)

return \((d, b, a - \text{floor}(x/y) \ast b)\)

Recursively: \(d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y\)

Returns \((d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))\).
Hand Calculation Method for Inverses.

Example: gcd(7, 60) = 1.
Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.

$\text{egcd}(7, 60)$. 
Hand Calculation Method for Inverses.

Example: \( \text{gcd}(7, 60) = 1 \).

\[ \text{egcd}(7, 60). \]

\[
7(0) + 60(1) = 60
\]
Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$. 
$\text{egcd}(7,60)$.

\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7
\end{align*}
Hand Calculation Method for Inverses.

Example: \( \text{gcd}(7, 60) = 1 \).
\[ \text{egcd}(7, 60). \]

\[
7(0) + 60(1) = 60 \\
7(1) + 60(0) = 7 \\
7(-8) + 60(1) = 4
\]
Hand Calculation Method for Inverses.

Example: \( \gcd(7, 60) = 1 \).
\[ \text{egcd}(7, 60). \]

\[
7(0) + 60(1) = 60 \\
7(1) + 60(0) = 7 \\
7(-8) + 60(1) = 4 \\
7(9) + 60(-1) = 3
\]
Hand Calculation Method for Inverses.

Example: \( \gcd(7, 60) = 1 \).
\[
e\gcd(7, 60).
\]

\[
\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7 \\
7(-8) + 60(1) &= 4 \\
7(9) + 60(-1) &= 3 \\
7(-17) + 60(2) &= 1
\end{align*}
\]
Example: \( \text{gcd}(7, 60) = 1 \).

\( \text{egcd}(7, 60) \).

\[
\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7 \\
7(-8) + 60(1) &= 4 \\
7(9) + 60(-1) &= 3 \\
7(-17) + 60(2) &= 1
\end{align*}
\]
Hand Calculation Method for Inverses.

Example: $\text{gcd}(7, 60) = 1$.
$\text{egcd}(7, 60)$.

\[
\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7 \\
7(-8) + 60(1) &= 4 \\
7(9) + 60(-1) &= 3 \\
7(-17) + 60(2) &= 1
\end{align*}
\]

Confirm:
Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.
$\text{egcd}(7, 60)$.

\[
\begin{align*}
7(0) + 60(1) & = 60 \\
7(1) + 60(0) & = 7 \\
7(-8) + 60(1) & = 4 \\
7(9) + 60(-1) & = 3 \\
7(-17) + 60(2) & = 1 \\
\end{align*}
\]

Confirm: $-119 + 120 = 1$
Conclusion: Can find multiplicative inverses in $O(n)$ time!
Wrap-up

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Very different from elementary school: try 1, try 2, try 3...
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Inverse of 500,000,357 modulo 1,000,000,000,000?
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$\leq 80$ divisions.
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versus 1,000,000
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Internet Security.
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$(100000000000000000000000000000000000000000000)^5$ divisions.
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$\leq 80$ divisions.
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512 divisions vs.
$(\underbrace{1000000000000000000000000000000000}_{5\text{ digits}})\text{ divisions}$. 

Internet Security:
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Inverse of 500,000,357 modulo 1,000,000,000,000?

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Internet Security: Next Week.