Lecture 7. Outline.

1. Isoperimetric inequality for hypercube.
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2. Modular Arithmetic.
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   Clock Math!!!
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3. Inverses for Modular Arithmetic: Greatest Common Divisor.
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   Division!!!
Lecture 7. Outline.

1. Isoperimetric inequality for hypercube.

2. Modular Arithmetic.
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4. Euclid’s GCD Algorithm.
Lecture 7. Outline.

1. Isoperimetric inequality for hypercube.
2. Modular Arithmetic.  
   Clock Math!!!
3. Inverses for Modular Arithmetic: Greatest Common Divisor.  
   Division!!!
4. Euclid’s GCD Algorithm.  
   A little tricky here!
Isoperimetry.

For 3-space:

Surface Area: $4\pi r^2$, Volume: $\frac{4}{3}\pi r^3$. Ratio: $\frac{1}{3r} = \Theta\left(\frac{V - \frac{1}{3}}{V}\right)$. Graphical Analog: Cut into two pieces and find ratio of edges/vertices on small side. Tree: $\Theta\left(\frac{1}{|V|}\right)$. Hypercube: $\Theta\left(1\right)$. Surface Area is roughly at least the volume!
Isoperimetry.

For 3-space:

The sphere minimizes surface area to volume.
Isoperimetry.

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Ratio: $1/3r = \Theta(V^{-1/3})$. 
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Tree: $\Theta(1/|V|)$.

Hypercube: $\Theta(1)$. 
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Surface Area is roughly at least the volume!
Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.
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An $n$-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$. 

\begin{tabular}{l}
(A) Lower left forward node name is 0000 \\
(B) Lower left back node name is 0001 \\
(C) Upper right forward node is 1011 \\
(D) Upper right back node name is 1111 \\
\end{tabular}
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Thm: Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$.

Terminology: $(S, V - S)$ is a cut. $(E \cap S \times (V - S))$ cut edges.

Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.
Hypercube: Can’t cut me!

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**Terminology:**
$(S, V - S)$ is cut.
**Thm:** Any subset $S$ of the hypercube where $|S| \leq |V|/2$ has $\geq |S|$ edges connecting it to $V - S$; $|E \cap S \times (V - S)| \geq |S|$

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.
Cuts in graphs.

$S$ is red, $V - S$ is blue.

What is size of cut? Number of edges between red and blue.

Hypercube: any cut that cuts off $x$ nodes has $\geq x$ edges.
Cuts in graphs.

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Number of edges between red and blue.
Cuts in graphs.

$S$ is red, $V - S$ is blue.

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Number of edges between red and blue. 4.
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S is red, $V - S$ is blue.

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Number of edges between red and blue. 4.

Hypercube: any cut that cuts off $x$ nodes has $\geq x$ edges.
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Base Case: \(n = 1\)
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Proof:
Base Case: \(n = 1\) \(V = \{0, 1\}\).
Proof of Large Cuts.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

**Proof:**
Base Case: \(n = 1\) \(V = \{0, 1\}\).
- \(S = \{0\}\) has one edge leaving.
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Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.
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Use recursive definition into two subcubes.
Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

- Two cubes connected by edges.
Induction Step Idea

**Thm:** For any cut \((S, V − S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.
Induction Step Idea

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

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Two cubes connected by edges.

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Two cubes connected by edges.

Case 1: Count edges inside subcube inductively.

Case 2: Count inside and across.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side.

Use recursive definition into two subcubes.

Two cubes connected by edges.

**Case 1:** Count edges inside subcube inductively.

**Case 2:** Count inside and across.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).
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**Proof:** Induction Step.
**Induction Step**

**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

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Recursive definition:
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

Recursive definition:

\[ H_0 = (V_0, E_0), H_1 = (V_1, E_1), \] edges \(E_x\) that connect them.
Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

Proof: Induction Step.
Recursive definition:
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H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.}
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H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)
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**Induction Step**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**
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- \(H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)\)
- \(S = S_0 \cup S_1\) where \(S_0\) in first, and \(S_1\) in other.
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)

Both \(S_0\) and \(S_1\) are small sides.
Induction Step

Thm: For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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Case 1: \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)
Both \(S_0\) and \(S_1\) are small sides. So by induction.
Induction Step

**Thm:** For any cut \( (S, V - S) \) in the hypercube, the number of cut edges is at least the size of the small side, \( |S| \).

**Proof: Induction Step.**

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H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.}
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**Case 1:** \( |S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2 \)

Both \( S_0 \) and \( S_1 \) are small sides. So by induction.

Edges cut in \( H_0 \geq |S_0| \).
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

Recursive definition:

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H_0 = (V_0, E_0), H_1 = (V_1, E_1), \text{ edges } E_x \text{ that connect them.}
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H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)
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S = S_0 \cup S_1 \text{ where } S_0 \text{ in first, and } S_1 \text{ in other.}
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**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)

Both \(S_0\) and \(S_1\) are small sides. So by induction.

- Edges cut in \(H_0 \geq |S_0|\).
- Edges cut in \(H_1 \geq |S_1|\).
**Induction Step**

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step.**

Recursive definition:

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Both \(S_0\) and \(S_1\) are small sides. So by induction.

- Edges cut in \(H_0 \geq |S_0|\).
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Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** *Induction Step.*

Recursive definition:

\[H_0 = (V_0, E_0), H_1 = (V_1, E_1),\] edges \(E_x\) that connect them.

\[H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)\]

\(S = S_0 \cup S_1\) where \(S_0\) in first, and \(S_1\) in other.

**Case 1:** \(|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2\)

Both \(S_0\) and \(S_1\) are small sides. So by induction.

- Edges cut in \(H_0 \geq |S_0|\).
- Edges cut in \(H_1 \geq |S_1|\).

Total cut edges \(\geq |S_0| + |S_1| = |S|\).
Induction Step

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step.

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Both \(S_0\) and \(S_1\) are small sides. So by induction.

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- Edges cut in \(H_1 \geq |S_1|\).

Total cut edges \(\geq |S_0| + |S_1| = |S|\).
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[|S_0| \geq |V_0|/2.\]

Edges in \(E_x\) connect corresponding nodes.
Induction Step. Case 2.

**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[|S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2.\]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2 \]

\[|S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2.\]

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]
Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.


Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.

$\implies S_1$ edges cut in $E_1$.

$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$

Edges in $E_x$ connect corresponding nodes.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]

\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]

\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]

\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[|S_1| \leq |V_1|/2\] since \(|S| \leq |V|/2.\)

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]

\[|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\]

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Edges in \(E_x\) connect corresponding nodes.
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof:** Induction Step. Case 2.

\[|S_0| \geq |V_0|/2.\]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)

\[|S_1| \leq |V_1|/2\] since \(|S| \leq |V|/2.\)

\[\implies \geq |S_1| \text{ edges cut in } E_1.\]

\[|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2\]

\[\implies \geq |V_0| - |S_0| \text{ edges cut in } E_0.\]

Edges in \(E_x\) connect corresponding nodes.

\[\implies = |S_0| - |S_1| \text{ edges cut in } E_x.\]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

**Proof: Induction Step. Case 2.**

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)
\[ |S_1| \leq |V_1|/2 \quad \text{since} \quad |S| \leq |V|/2. \]
\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]
\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]
\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in \(E_x\) connect corresponding nodes.
\[ \implies = |S_0| - |S_1| \text{ edges cut in } E_x. \]
**Thm:** For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

**Proof:** Induction Step. Case 2.

\[ |S_0| \geq |V_0|/2. \]

Recall Case 1: $|S_0|, |S_1| \leq |V|/2$

\[ |S_1| \leq |V_1|/2 \text{ since } |S| \leq |V|/2. \]

\[ \implies \geq |S_1| \text{ edges cut in } E_1. \]

\[ |S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2 \]

\[ \implies \geq |V_0| - |S_0| \text{ edges cut in } E_0. \]

Edges in $E_x$ connect corresponding nodes.

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Total edges cut:
Induction Step. Case 2.

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Total edges cut:  
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**Proof:** Induction Step. Case 2.

1. \(|S_0| \geq |V_0|/2\).
2. Recall Case 1: \(|S_0|, |S_1| \leq |V|/2\)
   - \(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2\).
   - \(\Rightarrow \geq |S_1|\) edges cut in \(E_1\).
3. \(|S_0| \geq |V_0|/2\) \(\Rightarrow |V_0 - S| \leq |V_0|/2\)
   - \(\Rightarrow \geq |V_0| - |S_0|\) edges cut in \(E_0\).

Edges in \(E_x\) connect corresponding nodes.
- \(\Rightarrow = |S_0| - |S_1|\) edges cut in \(E_x\).

Total edges cut:
- \(\geq |S_1|\)
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Edges in \(E_x\) connect corresponding nodes.

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Total edges cut:

\[\geq |S_1| + |V_0| - |S_0|\]
**Thm:** For any cut \((S, V - S)\) in the hypercube, the number of cut edges is at least the size of the small side, \(|S|\).

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\(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2.\)

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Edges in \(E_x\) connect corresponding nodes.

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Total edges cut:

\[\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1|\]
Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.  

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Recall Case 1: $|S_0|, |S_1| \leq |V|/2$  
$|S_1| \leq |V_1|/2$ since $|S| \leq |V|/2$.  
$\implies \geq |S_1|$ edges cut in $E_1$.  
$|S_0| \geq |V_0|/2 \implies |V_0 - S| \leq |V_0|/2$  
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Total edges cut:  
$\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|$
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\[ \geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0| \]

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1. **Induction Step. Case 2.**
   
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   Total edges cut:
   
   \[\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|\]
   
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**Proof: Induction Step. Case 2.**

\(|S_0| \geq |V_0|/2.\)

Recall Case 1: \(|S_0|, |S_1| \leq |V|/2 \Rightarrow \)

\(|S_1| \leq |V_1|/2\) since \(|S| \leq |V|/2.\)

\(|S_0| \geq |V_0|/2 \Rightarrow |V_0 - S| \leq |V_0|/2 \Rightarrow \)

\(\geq |V_0| - |S_0|\) edges cut in \(E_0.\)

Edges in \(E_x\) connect corresponding nodes.
\(\Rightarrow = |S_0| - |S_1|\) edges cut in \(E_x.\)

Total edges cut:
\(\geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0|\)

\(|V_0| = |V|/2 \geq |S|\).

Also, case 3 where \(|S_1| \geq |V|/2\) is symmetric.
Hypercube proof: poll

Hypercube has large cuts proof uses these ideas:
(A) If cuts are same size on two sides it works by induction.
(B) Uses the fact that it is planar.
(C) Recursive definition of hypercube.
(D) If different size, can count edges between to subcubes.
(E) Applies Euler’s formula.
Hypercube proof: poll

**Hypercube has large cuts proof uses these ideas:**
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(C) Recursive definition of hypercube.
(D) If different size, can count edges between to subcubes.
(E) Applies Euler’s formula.

(A), (D), and (E).
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0, 1\}^n$. 
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \(\{0, 1\}^n\).

Central area of study in computer science!
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \( \{0, 1\}^n \).

Central area of study in computer science!

Yes/No Computer Programs \( \equiv \) Boolean function on \( \{0, 1\}^n \)
The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on \( \{0, 1\}^n \).

Central area of study in computer science!

Yes/No Computer Programs \( \equiv \) Boolean function on \( \{0, 1\}^n \)

Central object of study.
Modular Arithmetic.

Applications: cryptography, error correction.
Key ideas for modular arithmetic.

Theorem: If \( d \mid x \) and \( d \mid y \), then \( d \mid (y - x) \).
Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y−x)$.
Proof:
Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y - x)$.

Proof:

$x = ad$, $y = bd$, 

$$(x - y) = (ad - bd) = d(a - b).$$
Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y - x)$.

Proof:
$x = ad, y = bd,$
$(x - y) = (ad - bd) = d(a - b) \implies d|(x - y).$
Key ideas for modular arithmetic.

Theorem: If \( d \mid x \) and \( d \mid y \), then \( d \mid (y - x) \).

Proof:
\[
x = ad, \quad y = bd,
\]
\[
(x - y) = (ad - bd) = d(a - b) \implies d \mid (x - y).
\]

Theorem: Every number \( n \geq 2 \) can be represented as a product of primes.
Theorem: If $d | x$ and $d | y$, then $d | (y - x)$.

Proof:

$x = ad, y = bd$,

$(x - y) = (ad - bd) = d(a - b) \implies d | (x - y)$.

Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Proof: Either prime, or $n = a \times b$, and use strong induction.

(Uniqueness? Later.)
What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler’s formula.
(C) The definition of a prime number.
(D) Euclid’s Lemma.
What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler’s formula.
(C) The definition of a prime number.
(D) Euclid’s Lemma.

(A) and (C)
Next Up.

Modular Arithmetic.
Clock Math

If it is 1:00 now.
Clock Math

If it is 1:00 now.
What time is it in 2 hours?
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
If it is 1:00 now.
   What time is it in 2 hours? 3:00!
   What time is it in 5 hours?
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!

16 / 34
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours?

16 / 34
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!

Actually 4:00.
16 is the "same as 4" with respect to a 12 hour clock system.

Clock time equivalent up to addition/subtraction of 12.

What time is it in 100 hours?
101:00!
or 5:00.
101 = 12 \times 8 + 5.
5 is the same as 101 for a 12 hour clock system.

Clock time equivalent up to addition of any integer multiple of 12.
Custom is only to use the representative in \{12, 1,..., 11\}.

(Asserted remainder, except for 12 and 0 are equivalent.)
Clock Math

If it is 1:00 now.
   What time is it in 2 hours? 3:00!
   What time is it in 5 hours? 6:00!
   What time is it in 15 hours? 16:00!
      Actually 4:00.
Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.
If it is 1:00 now.  
What time is it in 2 hours? 3:00!  
What time is it in 5 hours? 6:00!  
What time is it in 15 hours? 16:00!  
Actually 4:00.  

16 is the “same as 4” with respect to a 12 hour clock system. Clock time equivalent up to to addition/subtraction of 12.
Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
   Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours?
If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
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   Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00!
Clock Math

If it is 1:00 now.
- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
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Clock Math

If it is 1:00 now.
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5 is the same as 101 for a 12 hour clock system.
Clock Math

If it is 1:00 now.
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Clock Math

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Custom is only to use the representative in \{12, 1, \ldots, 11\}
(April remainder, except for 12 and 0 are equivalent.)
Day of the week.

This is Thursday is September 16, 2021.
This is Thursday is September 16, 2021. What day is it a year from then?
Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Day of the week.

This is Thursday is September 16, 2021. What day is it a year from then? on September 16, 2022?
Number days.
Day of the week.

This is Thursday is September 16, 2021. What day is it a year from then? on September 16, 2022? Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.
This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.
This is Thursday is September 16, 2021. What day is it a year from then? on September 16, 2022?
Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation: subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5.

369 = 7*(52) + 5 or September 16, 2022 is a Friday.
Day of the week.

This is Thursday is September 16, 2021. What day is it a year from then? on September 16, 2022? Number days. 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4. 5 days from then.
Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, . . ., 6 for Saturday.

Today: day 4.
5 days from then. day 9
Day of the week.

This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2
This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
Day of the week.

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What day is it a year from then? on September 16, 2022?
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  5 days from then. day 9 or day 2 or Tuesday.
  25 days from then.
This is Thursday is September 16, 2021. What day is it a year from then? on September 16, 2022? Number days.

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5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29

11 days from then is day 1 which is Monday!
Day of the week.

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What day is it a year from then? on September 16, 2022?
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Day of the week.

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What day will it be a year from then? on September 16, 2022?
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Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
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two days are equivalent up to addition/subtraction of multiple of 7.
Day of the week.

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- 11 days from then is day 1 which is Monday!

What day is it a year from then?
This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year.
This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
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What day is it a year from then?
Next year is not a leap year. So 365 days from then.
This is Thursday is September 16, 2021.

What day is it a year from then? on September 16, 2022?

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11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.
Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
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What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
  0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
  5 days from then. day 9 or day 2 or Tuesday.
  25 days from then. day 29 or day 1. \(29 = (7 \times 4 + 1\)
  two days are equivalent up to addition/subtraction of multiple of 7.
  11 days from then is day 1 which is Monday!

What day is it a year from then?
  Next year is not a leap year. So 365 days from then.
  Day 4+365 or day 369.
Smallest representation:
  subtract 7 until smaller than 7.
Day of the week.

This is Thursday is September 16, 2021.  
What day is it a year from then? on September 16, 2022?  
Number days.  
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.  
5 days from then. day 9 or day 2 or Tuesday.  
25 days from then. day 29 or day 1. $29 = (7)4 + 1$  
two days are equivalent up to addition/subtraction of multiple of 7.  
11 days from then is day 1 which is Monday!

What day is it a year from then?  
Next year is not a leap year. So 365 days from then.  
Day 4+365 or day 369.  
Smallest representation:  
subtract 7 until smaller than 7.  
divide and get remainder.
Day of the week.

This is Thursday is September 16, 2021.
What day is it a year from then? on September 16, 2022?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
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25 days from then. day 29 or day 1. 29 = (7)4 + 1
two days are equivalent up to addition/subtraction of multiple of 7.
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What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7
Day of the week.

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Number days.
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two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7 leaves quotient of 52 and remainder 5.
Day of the week.

This is Thursday is September 16, 2021.
   What day is it a year from then? on September 16, 2022?
   Number days.
      0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
   5 days from then. day 9 or day 2 or Tuesday.
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   Next year is not a leap year. So 365 days from then.
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Smallest representation:
   subtract 7 until smaller than 7.
   divide and get remainder.
   369/7 leaves quotient of 52 and remainder 5. \(369 = 7(52) + 5\)
Day of the week.

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What day is it a year from then? on September 16, 2022?

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Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5. 369 = 7(52) + 5

or September 16, 2022 is a Friday.
Day of the week.

This is Thursday is September 16, 2021.
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369/7 leaves quotient of 52 and remainder 5. 369 = 7(52) + 5
or September 16, 2022 is a Friday.
Years and years...

80 years?
Years and years...

80 years? 20 leap years.
Years and years...

80 years?  20 leap years.  $366 \times 20$ days
Years and years...

80 years? 20 leap years. $366 \times 20$ days
60 regular years.
Years and years...

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Today is day 4.
Years and years...

80 years? 20 leap years. $366 \times 20$ days
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Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. 
Years and years...

80 years? 20 leap years. $366 \times 20$ days
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Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?
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Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7?

What is remainder of 365 when dividing by 7?

Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$.
Remainder when dividing by 7?

$104 = 14 \times 7 + 6$.
Or February 11, 2101 is Saturday!
Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.
Or Day 6.
September 16, 2101 is Saturday.
"Reduce" at any time in calculation!
Years and years...

80 years? 20 leap years. $366 \times 20$ days
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“Reduce” at any time in calculation!
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or \( "x \equiv y \ (\text{mod} \ m)" \)
if and only if \( (x - y) \) is divisible by \( m \).
Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.
...or $x$ and $y$ have the same remainder w.r.t. $m$. 

Useful Fact:

- Addition, subtraction, multiplication can be done with any equivalent $x$ and $y$.
- \[a \equiv c \pmod{m} \quad \text{and} \quad b \equiv d \pmod{m} \implies a + b \equiv c + d \pmod{m} \quad \text{and} \quad a \cdot b \equiv c \cdot d \pmod{m} \]

Proof:

- If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$.
- If $b \equiv d \pmod{m}$, then $b = d + jm$ for some integer $j$.
- Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.
- \[a + b \equiv c + d \pmod{m} \]

Can calculate with representative in \{0, ..., $m-1$\}.  

\[19 / 34\]
Modular Arithmetic: refresher.

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if and only if \( (x - y) \) is divisible by \( m \).
...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
...or \( x = y + km \) for some integer \( k \).
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Mod 7 equivalence classes:
Modular Arithmetic: refresher.

$x$ is congruent to $y$ modulo $m$ or “$x \equiv y \pmod{m}$” if and only if $(x - y)$ is divisible by $m$.
...or $x$ and $y$ have the same remainder w.r.t. $m$.
...or $x = y + km$ for some integer $k$.

Mod 7 equivalence classes:
$\{\ldots, -7, 0, 7, 14, \ldots\}$
Modular Arithmetic: refresher.

*x is congruent to y modulo m* or “*x ≡ y (mod m)*” if and only if *(x − y)* is divisible by *m*.
...or *x* and *y* have the same remainder w.r.t. *m*.
...or *x = y + km* for some integer *k*.

Mod 7 equivalence classes:
{..., −7, 0, 7, 14, ...}  {..., −6, 1, 8, 15, ...}
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \pmod{m} \)” if and only if \((x - y)\) is divisible by \( m \).
...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
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Mod 7 equivalence classes:
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots
Modular Arithmetic: refresher.

\( x \text{ is congruent to } y \mod m \) or “\( x \equiv y \mod m \)”
if and only if \( (x - y) \) is divisible by \( m \).
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Mod 7 equivalence classes:
\{\ldots, -7, 0, 7, 14, \ldots\}  \{\ldots, -6, 1, 8, 15, \ldots\} ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).
Modular Arithmetic: refresher.

\[ x \text{ is congruent to } y \bmod m \text{ or } x \equiv y \pmod{m} \]
if and only if \((x - y)\) is divisible by \(m\).
...or \(x\) and \(y\) have the same remainder w.r.t. \(m\).
...or \(x = y + km\) for some integer \(k\).

Mod 7 equivalence classes:
\[ \{ \ldots, -7, 0, 7, 14, \ldots \} \quad \{ \ldots, -6, 1, 8, 15, \ldots \} \ldots \]

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \(x\) and \(y\).

or \(a \equiv c \pmod{m}\) and \(b \equiv d \pmod{m}\)
Modular Arithmetic: refresher.

\[ x \text{ is congruent to } y \text{ modulo } m \text{ or } “x \equiv y \pmod{m}” \]
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\[ \implies a + b \equiv c + d \pmod{m} \text{ and } a \cdot b \equiv c \cdot d \pmod{m} \]"
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Mod 7 equivalence classes:
\{..., -7, 0, 7, 14, ...\} \quad \{..., -6, 1, 8, 15, ...\} ...

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$\implies a + b \equiv c + d \pmod{m}$ and $a \cdot b \equiv c \cdot d \pmod{m}$”

**Proof:** If $a \equiv c \pmod{m}$, then $a = c + km$ for some integer $k$. 
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...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
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\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \quad \ldots
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**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)

\[ \Rightarrow a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m} \)"

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
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or “ \( a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)
\[ \implies a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m} \)”

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore,
Modular Arithmetic: refresher.

\textbf{x is congruent to y modulo m} or “\(x \equiv y \pmod{m}\)” if and only if \((x - y)\) is divisible by \(m\).

...or \(x\) and \(y\) have the same remainder w.r.t. \(m\).

...or \(x = y + km\) for some integer \(k\).

Mod 7 equivalence classes:
\[
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots
\]

\textbf{Useful Fact:} Addition, subtraction, multiplication can be done with any equivalent \(x\) and \(y\).

or “\(a \equiv c \pmod{m}\) and \(b \equiv d \pmod{m}\)
\[\implies a + b \equiv c + d \pmod{m}\) and \(a \cdot b \equiv c \cdot d \pmod{m}\)”

\textbf{Proof:} If \(a \equiv c \pmod{m}\), then \(a = c + km\) for some integer \(k\).
If \(b \equiv d \pmod{m}\), then \(b = d + jm\) for some integer \(j\).
Therefore, \(a + b = c + d + (k + j)m\)
Modular Arithmetic: refresher.

*x is congruent to y modulo m* or “*x ≡ y (mod m)*”
if and only if (*x − y*) is divisible by *m*.
...or *x* and *y* have the same remainder w.r.t. *m*.
...or *x = y + km* for some integer *k*.

Mod 7 equivalence classes:
  \{ ..., −7, 0, 7, 14, ... \} \{ ..., −6, 1, 8, 15, ... \} ...

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent *x* and *y*.

or “*a ≡ c (mod m)* and *b ≡ d (mod m)*
  \implies \ a + b ≡ c + d (mod m) and \ a \cdot b ≡ c \cdot d (mod m)”

**Proof:** If *a ≡ c (mod m)*, then *a = c + km* for some integer *k*.
If *b ≡ d (mod m)*, then *b = d + jm* for some integer *j*.
Therefore, \ a + b = c + d + (k + j)m \ and since *k + j* is integer.
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or \( "x \equiv y \pmod{m}" \)
if and only if \( (x - y) \) is divisible by \( m \).
...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or \( " \ a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)
\[ \implies \ a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m} \)"

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore, \( \ a + b = c + d + (k + j)m \) \quad and since \( k + j \) is integer.
\[ \implies \ a + b \equiv c + d \pmod{m}. \]
Modular Arithmetic: refresher.

\( x \text{ is congruent to } y \text{ modulo } m \) or \( x \equiv y \pmod{m} \)
if and only if \( (x - y) \) is divisible by \( m \).
...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).
...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\{ \ldots , -7, 0, 7, 14, \ldots \} \quad \{ \ldots , -6, 1, 8, 15, \ldots \} \ldots

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or \( " a \equiv c \pmod{m} \) and \( b \equiv d \pmod{m} \)
\[ \implies a + b \equiv c + d \pmod{m} \) and \( a \cdot b \equiv c \cdot d \pmod{m} " \]

**Proof:** If \( a \equiv c \pmod{m} \), then \( a = c + km \) for some integer \( k \).
If \( b \equiv d \pmod{m} \), then \( b = d + jm \) for some integer \( j \).
Therefore, \( a + b = c + d + (k + j)m \) and since \( k + j \) is integer.
\[ \implies a + b \equiv c + d \pmod{m} \).
Modular Arithmetic: refresher.

\( x \) is congruent to \( y \) modulo \( m \) or “\( x \equiv y \) (mod \( m \))” if and only if \( (x - y) \) is divisible by \( m \).

...or \( x \) and \( y \) have the same remainder w.r.t. \( m \).

...or \( x = y + km \) for some integer \( k \).

Mod 7 equivalence classes:
\[
\{\ldots, -7, 0, 7, 14, \ldots\} \quad \{\ldots, -6, 1, 8, 15, \ldots\} \quad \ldots
\]

**Useful Fact:** Addition, subtraction, multiplication can be done with any equivalent \( x \) and \( y \).

or “\( a \equiv c \) (mod \( m \)) and \( b \equiv d \) (mod \( m \))

\[ \implies a + b \equiv c + d \) (mod \( m \)) \) and \( a \cdot b = c \cdot d \) (mod \( m \))”

**Proof:** If \( a \equiv c \) (mod \( m \)), then \( a = c + km \) for some integer \( k \). If \( b \equiv d \) (mod \( m \)), then \( b = d + jm \) for some integer \( j \). Therefore, \( a + b = c + d + (k + j)m \) and since \( k + j \) is integer.

\[ \implies a + b \equiv c + d \) (mod \( m \)). \]

Can calculate with representative in \( \{0, \ldots, m - 1\} \).
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
Notation

\[ x \, (\text{mod} \ m) \text{ or } \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).
Notation

\[ x \quad (\text{mod} \quad m) \quad \text{or} \quad \text{mod} \quad (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \text{mod} \quad (x, m) = x - \lfloor \frac{x}{m} \rfloor m \]
Notation

\[ x \pmod{m} \text{ or } \mod(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\mod(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[
\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}
\]
Notation

\( x \pmod{m} \) or \( \text{mod } (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod } (29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12
\]

Work in this system.

\( a \equiv b \pmod{m} \).
- Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

\[
6 \equiv 3 + 3 \pmod{7}.
\]
- Generally, not \( 6 \equiv 13 \pmod{7} \).

But probably won't take off points, still hard for us to read.
Notation

\[ x \pmod{m} \text{ or } \text{mod } (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod } (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12
\]
Notation

\( x \pmod{m} \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[
\text{mod} (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4
\]
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5
\]
Notation

\[ x \pmod m \] or \[ \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \] is quotient.

\[ \text{mod} \ (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5 \]

Work in this system.
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5
\]

Work in this system.
\( a \equiv b \pmod{m} \).
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = \boxed{5}
\]

Work in this system.
\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).
**Notation**

\[ x \ (\text{mod} \ m) \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \boxed{4} = 5 \]

Work in this system.

\( a \equiv b \ (\text{mod} \ m) \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)
**Notation**

\[ x \pmod{m} \text{ or } \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \mod (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \mod (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5 \]

Work in this system.

\( a \equiv b \pmod{m} \).

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv \)
Notation

\[ x \pmod{m} \text{ or } \text{mod } (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[
\text{mod } (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod } (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \not{4} = 5
\]

Work in this system.

\[ a \equiv b \pmod{m} \]

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \]
Notation

\[ x \pmod{m} \text{ or } \text{mod (} x, m \text{)} \]
- remainder of \( x \) divided by \( m \) in \{0, \ldots, m - 1\}.

\[
\text{mod (} x, m \text{)} = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[
\left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.}
\]

\[
\text{mod (} 29, 12 \text{)} = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4 \neq 5
\]

Work in this system.
\[
a \equiv b \pmod{m}.
\]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[
6 \equiv 3 + 3 \equiv 3 + 10
\]
Notation

\[ x \pmod{m} \text{ or } \text{mod} (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[
\text{mod} (29, 12) = 29 - \left\lfloor \frac{29}{12} \right\rfloor \times 12 = 29 - (2) \times 12 = \boxed{5}
\]

Work in this system.

\[ a \equiv b \pmod{m} \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \text{X} = 5 \]

Work in this system.
\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \).

6 = 
Notation

\[ x \pmod{m} \text{ or } \text{mod} \ (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m - 1\} \).

\[ \text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \text{mod} \ (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 5 \]

Work in this system.
\[ a \equiv b \pmod{m}. \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}. \]
\[ 6 = 3 + 3 \]
Notation

\( x \pmod{m} \) or \( \text{mod} (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} (29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 = 4 \quad \Rightarrow \quad X = 5
\]

Work in this system.
\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\( 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \).

\( 6 = 3 + 3 = 3 + 10 \)
Notation

\[ x \ (\mod \ m) \text{ or } \mod (x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \mod (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\[ \left\lfloor \frac{x}{m} \right\rfloor \text{ is quotient.} \]

\[ \mod (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = \red{5} \]

Work in this system.
\[ a \equiv b \ (\mod \ m) . \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \ (\mod \ 7) . \]
\[ 6 = 3 + 3 = 3 + 10 \ (\mod \ 7) . \]
Notation

\( x \pmod{m} \) or \( \text{mod} \ (x, m) \)
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[
\text{mod} \ (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m
\]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[
\text{mod} \ (29, 12) = 29 - \left( \left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 4
\]

Work in this system.
\( a \equiv b \pmod{m} \).
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[
6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}.
\]

\[
6 = 3 + 3 = 3 + 10 \pmod{7}.
\]

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
Notation

\[ x \equiv b \pmod{m} \]

- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod} (x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]

\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod} (29, 12) = 29 - \left(\left\lfloor \frac{29}{12} \right\rfloor \right) \times 12 = 29 - (2) \times 12 = 5 \]

Work in this system.

\[ a \equiv b \pmod{m} \]

Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \]

\[ 6 = 3 + 3 = 3 + 10 \pmod{7} \]

Generally, not \( 6 \equiv 13 \pmod{7} \).

But probably won’t take off points,
Notation

\[ x \pmod{m} \text{ or } \text{mod}(x, m) \]
- remainder of \( x \) divided by \( m \) in \( \{0, \ldots, m-1\} \).

\[ \text{mod}(x, m) = x - \left\lfloor \frac{x}{m} \right\rfloor m \]
\( \left\lfloor \frac{x}{m} \right\rfloor \) is quotient.

\[ \text{mod}(29, 12) = 29 - (\left\lfloor \frac{29}{12} \right\rfloor) \times 12 = 29 - (2) \times 12 \]
\[ X = 5 \]

Work in this system.

\[ a \equiv b \pmod{m} \]
Says two integers \( a \) and \( b \) are equivalent modulo \( m \).

**Modulus** is \( m \)

\[ 6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7} \]

\[ 6 = 3 + 3 = 3 + 10 \pmod{7} \]

Generally, not \( 6 \pmod{7} = 13 \pmod{7} \).
But probably won’t take off points, still hard for us to read.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

Multiplicative inverse of \( x \) is \( y \) where \( xy = 1 \);
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \); **1** **is multiplicative identity element.**
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \); **1** **is multiplicative identity element.**

In modular arithmetic, **1** is the multiplicative identity element.
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[
2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}.
\]

**Multiplicative inverse of** \(x\) is \(y\) where \(xy = 1\); \(1\) is multiplicative identity element.

In modular arithmetic, \(1\) is the multiplicative identity element.

**Multiplicative inverse of** \(x \mod m\) is \(y\) with \(xy = 1 \pmod{m}\).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) is \( y \) where \( xy = 1 \); \( 1 \) is multiplicative identity element.

In modular arithmetic, \( 1 \) is the multiplicative identity element.  

**Multiplicative inverse of** \( x \mod m \) is \( y \) with \( xy = 1 \) \((\mod m)\).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}.\]

**Multiplicative inverse of** \(x\) **is** \(y\) **where** \(xy = 1\); **1 is multiplicative identity element.**

In modular arithmetic, \(1\) is the multiplicative identity element.

**Multiplicative inverse of** \(x \mod m\) **is** \(y\) **with** \(xy = 1 \mod m\).

For 4 modulo 7 inverse is 2: \(2 \cdot 4 \equiv 8 \equiv 1 \mod 7\).

Can solve \(4x = 5 \mod 7\).
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) **is** \( y \) **where** \( xy = 1 \);

**1 is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) **is** \( y \) **with** \( xy = 1 \mod m \).

For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).

Can solve \( 4x = 5 \mod 7 \).

\[ 2 \cdot 4x = 2 \cdot 5 \mod 7 \]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[
2x = 3 \implies \left( \frac{1}{2} \right) \cdot 2x = \left( \frac{1}{2} \right) \cdot 3 \implies x = \frac{3}{2}.
\]

**Multiplicative inverse of** \(x\) **is** \(y\) **where** \(xy = 1\); 1 **is multiplicative identity element.**

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \(x \mod m\) **is** \(y\) **with** \(xy = 1 \pmod{m}\).

For 4 modulo 7 inverse is 2: 2 \(\cdot\) 4 \(\equiv\) 8 \(\equiv\) 1 (mod 7).

Can solve 4\(x\) = 5 (mod 7).

\[
2 \cdot 4x = 2 \cdot 5 \pmod{7} \\
8x = 10 \pmod{7}
\]
Inverses and Factors.

Division: multiply by multiplicative inverse.

\[ 2x = 3 \implies \left(\frac{1}{2}\right) \cdot 2x = \left(\frac{1}{2}\right) \cdot 3 \implies x = \frac{3}{2}. \]

**Multiplicative inverse of** \( x \) is \( y \) where \( xy = 1 \); \( 1 \) is multiplicative identity element.

In modular arithmetic, 1 is the multiplicative identity element.

**Multiplicative inverse of** \( x \mod m \) is \( y \) with \( xy = 1 \) (mod \( m \)).

For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \pmod{7}. \]

Can solve \( 4x = 5 \pmod{7} \).

\[ 2 \cdot 4x = 2 \cdot 5 \pmod{7} \]

\[ 8x = 10 \pmod{7} \]

\[ x = 3 \pmod{7} \]
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For 4 modulo 7 inverse is 2: \[ 2 \cdot 4 \equiv 8 \equiv 1 \mod 7. \]

Can solve \( 4x = 5 \mod 7. \)

\[ 2 \cdot 4x = 2 \cdot 5 \mod 7 \]
\[ 8x = 10 \mod 7 \]
\[ x = 3 \mod 7 \]

Check!
Inverses and Factors.

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For 4 modulo 7 inverse is 2: \( 2 \cdot 4 \equiv 8 \equiv 1 \mod 7 \).

Can solve \( 4x = 5 \mod 7 \).

\[
\begin{align*}
2 \cdot 4x &= 2 \cdot 5 \mod 7 \\
8x &= 10 \mod 7 \\
x &= 3 \mod 7 \\
\end{align*}
\]

Check! \( 4(3) = 12 = 5 \mod 7 \).
Inverses and Factors.

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Can solve \( 4x = 5 \) (mod 7).
\( x = 3 \) (mod 7) :: Check! \( 4(3) = 12 = 5 \) (mod 7).
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$$2 \cdot 4 \equiv 8 \equiv 1 \pmod 7.$$  
Can solve $4x = 5 \pmod 7$.  
$x = 3 \pmod 7$ :::: Check! $4(3) = 12 = 5 \pmod 7$.

For 8 modulo 12: no multiplicative inverse!
Inverses and Factors.

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“Common factor of 4”
Inverses and Factors.

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“Common factor of 4” \( \implies \)
\( 8k - 12\ell \) **is a multiple of four for any** \( \ell \) **and** \( k \).
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\( x \equiv 3 \mod 7 \) :::: Check! \( 4(3) = 12 = 5 \mod 7 \).

For 8 modulo 12: no multiplicative inverse!

“Common factor of 4” \( \implies \)

\( 8k - 12\ell \) **is a multiple of four** **for any** \( \ell \) **and** \( k \) \( \implies \)

\( 8k \not\equiv 1 \mod 12 \) **for any** \( k \).
Mark true statements.
(A) Multiplicative inverse of 2 mod 5 is 3 mod 5.
(B) The multiplicative inverse of \((n - 1) \mod n\) equals \((n - 1) \mod n\).
(C) Multiplicative inverse of 2 mod 5 is 0.5.
(D) Multiplicative inverse of 4 \(-1\) \(\mod 5\).
(E) \((-1) \times (-1) = 1\). Woohoo.
(F) Multiplicative inverse of 4 mod 5 is 4 mod 5.
Mark true statements.
(A) Multiplicative inverse of 2 mod 5 is 3 mod 5.
(B) The multiplicative inverse of \((n - 1) \pmod{n} = ((n - 1) \pmod{n})\).
(C) Multiplicative inverse of 2 mod 5 is 0.5.
(D) Multiplicative inverse of 4 = −1 (mod 5).
(E) \((-1) \times (-1) = 1\). Woohoo.
(F) Multiplicative inverse of 4 mod 5 is 4 mod 5.

(C) is false. 0.5 has no meaning in arithmetic modulo 5.
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$.
Greatest Common Divisor and Inverses.

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If greatest common divisor of \( x \) and \( m \), \( \text{gcd}(x, m) \), is 1, then \( x \) has a multiplicative inverse modulo \( m \).

**Proof \( \implies \):**

**Claim:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).
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**Claim:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

Each of $m$ numbers in $S$ correspond to one of $m$ equivalence classes modulo $m$. 
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Proof $\Rightarrow$:
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Proof of Claim:
Greatest Common Divisor and Inverses.

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Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m-1\}, a \neq b$,
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$\gcd(x, m) = 1$
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$\implies$ Prime factorization of $m$ and $x$ do not contain common primes.
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So $(a-b)$ has to be multiple of $m$. 
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Or $(a - b)x = km$ for some integer $k$.

$\gcd(x, m) = 1$

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$\implies (a - b)$ factorization contains all primes in $m$’s factorization.

So $(a - b)$ has to be multiple of $m$.

$\implies (a - b) \geq m.$
Greatest Common Divisor and Inverses.

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Proof $\implies$:
Claim: The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

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$\implies$ One must correspond to 1 modulo $m$. Inverse Exists!

Proof of Claim: If not distinct, then $\exists a, b \in \{0, \ldots, m-1\}, a \neq b$, where

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$\implies (a - b) \geq m$. But $a, b \in \{0, \ldots m-1\}$.
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**Proof \implies:**

**Claim:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

Each of \( m \) numbers in \( S \) correspond to one of \( m \) equivalence classes modulo \( m \).

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Proof of Claim: If not distinct, then \( \exists a, b \in \{0, \ldots, m-1\}, a \neq b \), where

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Or \( (a - b)x = km \) for some integer \( k \).

\[ \gcd(x, m) = 1 \]

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So \( (a - b) \) has to be multiple of \( m \).

\[ \implies (a - b) \geq m. \text{ But } a, b \in \{0, \ldots m-1\}. \text{ Contradiction.} \]
**Greatest Common Divisor and Inverses.**

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If greatest common divisor of \( x \) and \( m \), \( \gcd(x, m) \), is 1, then \( x \) has a multiplicative inverse modulo \( m \).

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Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$. 

---

**Proof Sketch:**

The set $S = \{0 \cdot x, 1 \cdot x, ..., (m-1) \cdot x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

---

For $x = 4$ and $m = 6$. All products of 4...

$S = \{0 \cdot 4, 1 \cdot 4, 2 \cdot 4, 3 \cdot 4, 4 \cdot 4, 5 \cdot 4\} = \{0, 4, 8, 12, 16, 20\}$

Reducing (mod 6)

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct.

Common factor 2.

Can't be 1.

No inverse.

For $x = 5$ and $m = 6$.

$S = \{0 \cdot 5, 1 \cdot 5, 2 \cdot 5, 3 \cdot 5, 4 \cdot 5, 5 \cdot 5\} = \{0, 5, 4, 3, 2, 1\}$

All distinct,

contains 1!

$5$ is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?)

1 - 1.

$5 \cdot x \equiv 3 \pmod{6}$

What is $x$?

Multiply both sides by 5.

$x = 15 \equiv 3 \pmod{6}$

No solutions.

Can't get an odd.

$4 \cdot x \equiv 2 \pmod{6}$

Two solutions!

$x = 2, 5 \pmod{6}$

Very different for elements with inverses.
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).
Proof review. Consequence.

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... For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \]

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For \( x = 4 \) and \( m = 6 \). All products of 4...

\[ S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} \]
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

... For \( x = 4 \) and \( m = 6 \). All products of 4...

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Proof review. Consequence.

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... For \( x = 4 \) and \( m = 6 \). All products of 4...

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reducing \((\text{mod } 6)\)

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\( 5 \times 5 \equiv 3 \pmod{6} \)

What is \( x \)?

Multiply both sides by 5.

\( x = 15 = 3 \pmod{6} \)

No solutions.

Can't get an odd.

\( 4 \times 2 \equiv 2 \pmod{6} \)

Two solutions!

\( x = 2, 5 \pmod{6} \)

Very different for elements with inverses.
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**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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... For \( x = 4 \) and \( m = 6 \). All products of 4...

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\[
S = \{0, 4, 2, 0, 4, 2\}
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Proof review. Consequence.

**Thm:** If \(\gcd(x, m) = 1\), then \(x\) has a multiplicative inverse modulo \(m\).

**Proof Sketch:** The set \(S = \{0x, 1x, \ldots, (m - 1)x\}\) contains \(y \equiv 1 \mod m\) if all distinct modulo \(m\).

For \(x = 4\) and \(m = 6\). All products of 4...
\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
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reducing $(\pmod{6})$

$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct.
Proof review. Consequence.

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reducing \( \text{mod } 6 \)

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Not distinct. Common factor 2.
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Not distinct. Common factor 2. Can’t be 1.
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\[
\ldots
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All distinct,
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For \( x = 5 \) and \( m = 6 \).

\[ S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\} \]

All distinct, contains 1!
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

**Proof Sketch:** The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all distinct modulo \( m \).

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All distinct, contains 1! 5 is multiplicative inverse of 5 \( \pmod{6} \).
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

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(Hmm. What normal number is it own multiplicative inverse?)
Proof review. Consequence.

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(Hmm. What normal number is it own multiplicative inverse?) 1
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Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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\[ 5x \equiv 3 \pmod{6} \]
Proof review. Consequence.

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All distinct, contains 1! 5 is multiplicative inverse of 5 \((\mod 6)\).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

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5x \equiv 3 \pmod{6} \text{ What is } x? \text{ Multiply both sides by 5.}
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Proof review. Consequence.

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All distinct, contains 1! 5 is multiplicative inverse of 5 \( \pmod{6} \). (Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \pmod{6} \] What is \( x \)? Multiply both sides by 5.
\[ x = 15 \]
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

**Proof Sketch:** The set $S = \{0x, 1x, \ldots, (m - 1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

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$S = \{0, 4, 2, 0, 4, 2\}$

Not distinct. Common factor 2. Can’t be 1. No inverse.

For $x = 5$ and $m = 6$.

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All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).

(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

$5x \equiv 3 \mod 6$ What is $x$? Multiply both sides by 5.

$x = 15 \equiv 3 \mod 6$
Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

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reducing (mod 6)

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Not distinct. Common factor 2. Can’t be 1. No inverse.

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$5x = 3 \pmod{6}$ What is $x$? Multiply both sides by 5.

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\[ S = \{0, 4, 2, 0, 4, 2\} \]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

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5x = 3 \pmod 6 \text{ What is } x? \text{ Multiply both sides by 5.}
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\[
x = 15 = 3 \pmod 6
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\[
4x = 3 \pmod 6 \text{ No solutions. Can't get an odd.}
\]
\[
4x = 2 \pmod 6 \text{ Two solutions!}
\]
Proof review. Consequence.

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Reducing \((\mod 6)\)

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Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

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(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \] What is \( x \)? Multiply both sides by 5.

\[ x = 15 = 3 \mod 6 \]

\[ 4x = 3 \mod 6 \] No solutions. Can’t get an odd.

\[ 4x = 2 \mod 6 \] Two solutions! \( x = 2, 5 \mod 6 \)
Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m - 1)x\} \) contains \( y \equiv 1 \mod m \) if all distinct modulo \( m \).

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All distinct, contains 1! 5 is multiplicative inverse of 5 (mod 6).
(Hmm. What normal number is it own multiplicative inverse?) 1 -1.

\[ 5x = 3 \mod 6 \] What is \( x \)? Multiply both sides by 5.
\[ x = 15 = 3 \mod 6 \]

\[ 4x = 3 \mod 6 \] No solutions. Can’t get an odd.
\[ 4x = 2 \mod 6 \] Two solutions! \( x = 2, 5 \mod 6 \)

Very different for elements with inverses.
Proof Review 2: Bijections.

If \( \gcd(x,m) = 1 \).
Proof Review 2: Bijections.

If \( \gcd(x,m) = 1 \).
Then the function \( f(a) = xa \mod m \) is a bijection.
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$x = 3, m = 4$.

$f(1) = 3(1) = 3 \pmod{4}, f(2) = 6 = 2 \pmod{4}, f(3) = 1 \pmod{3}$. 

Oh yeah.

$f(0) = 0$.

Bijection $\equiv$ unique pre-image and same size.
All the images are distinct.
$x = 2, m = 4$.

$f(1) = 2, f(2) = 0, f(3) = 2$

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Not a bijection.
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Not a bijection.
Poll

Which is bijection?
(A) $f(x) = x$ for domain and range being $\mathbb{R}$
(B) $f(x) = ax \pmod{n}$ for $x \in \{0, ..., n-1\}$ and $\gcd(a, n) = 2$
(C) $f(x) = ax \pmod{n}$ for $x \in \{0, ..., n-1\}$ and $\gcd(a, n) = 1$
Which is bijection?
(A) $f(x) = x$ for domain and range being $\mathbb{R}$
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(C) $f(x) = ax \pmod{n} \text{ for } x \in \{0, \ldots, n - 1\} \text{ and } \gcd(a, n) = 1$

(B) is not.
Only if

Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).
Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume $a$ is $x^{-1}$, or $ax = 1 + km$. 
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Assume \( a \) is \( x^{-1} \), or \( ax = 1 + km \).

\[ x = nd \quad \text{and} \quad m = \ell d \quad \text{for} \quad d > 1. \]
Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).
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Thus,
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Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).

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Thus,

\[ a(nd) = 1 + k\ell d \text{ or } d(na - k\ell) = 1. \]
Only if

**Thm:** If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

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$x = nd$ and $m = \ell d$ for $d > 1$.

Thus,

$$a(nd) = 1 + k\ell d \text{ or } d(na - k\ell) = 1.$$  

But $d > 1$ and $n = (na - k\ell) \in \mathbb{Z}$. 
Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume $a$ is $x^{-1}$, or $ax = 1 + km$.

$x = nd$ and $m = ld$ for $d > 1$.

Thus,

$$a(nd) = 1 + kld \text{ or } d(na - k\ell) = 1.$$ 

But $d > 1$ and $n = (na - k\ell) \in \mathbb{Z}$.

so $dn \neq 1$ and $dn = 1$. Contradiction.
Thm: If $\gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$. Assume $a$ is $x^{-1}$, or $ax = 1 + km$. 

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Finding inverses.

How to find the inverse?

Find \( \gcd(x, m) \).

Greater than 1?

No multiplicative inverse.

Equal to 1?

Multiplicative inverse.

Algorithm: Try all numbers up to \( x \) to see if it divides both \( x \) and \( m \).

Very slow.
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?
Find gcd $(x, m)$.
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\gcd(x, m)$.
   Greater than 1?
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Finding inverses.

How to find the inverse?
How to find if \( x \) has an inverse modulo \( m \)?

Find \( \gcd(x, m) \).
    Greater than 1? No multiplicative inverse.
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Algorithm: Try all numbers up to \( x \) to see if it divides both \( x \) and \( m \).
Very slow.
Inverses

Next up.
Inverses

Next up.
Inverses

Next up.

Euclid’s Algorithm.
Inverses

Next up.

Euclid’s Algorithm.
Runtime.
Inverses

Next up.
Euclid’s Algorithm.
Runtime.
Euclid’s Extended Algorithm.
Does 2 have an inverse mod 8?

No. Any multiple of 2 is 2 away from 0 + 8k for any k ∈ N.

Does 2 have an inverse mod 9?

Yes. 5
2(5) = 10 = 1 mod 9.

Does 6 have an inverse mod 9?

No. Any multiple of 6 is 3 away from 0 + 9k for any k ∈ N.

3 = \gcd(6, 9) ≠ 1

x has an inverse modulo m if and only if \gcd(x, m) > 1?

No.

\gcd(x, m) = 1?

Yes.

Now what?:

Compute gcd!

Compute Inverse modulo m.
Does 2 have an inverse mod 8? No.
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Compute $\text{gcd}(6, 9)$!  
Compute Inverse modulo $m$. 

$3 = \gcd(6, 9)$!
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   $2(5) = 10 = 1 \pmod{9}$. 

$\gcd(x, m) = 1$? Yes.

Now what? Compute $\gcd$! Compute Inverse modulo $m$. 

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Does 2 have an inverse mod 8? No.
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\[
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\[
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\(x\) has an inverse modulo \(m\) if and only if
\[
gcd(x, m) > 1? \text{ No.}
\]
\[
gcd(x, m) = 1?
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   \[ 3 = gcd(6, 9)! \]

$x$ has an inverse modulo $m$ if and only if
   
   $gcd(x, m) > 1$? No.
   
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Compute $\gcd$!
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   $gcd(x, m) > 1$? No.
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Divisibility...

**Notation:** $d | x$ means “$d$ divides $x$” or
Divisibility...

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).
Divisibility...

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Fact:** If $d | x$ and $d | y$ then $d | (x + y)$ and $d | (x - y)$. 
Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d|x$ and $d|y$ then $d|(x + y)$ and $d|(x − y)$.

Is it a fact?
Divisibility...

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Is it a fact? Yes? No?

**Proof:** \( d \mid x \) and \( d \mid y \) or
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Is it a fact? Yes? No?

**Proof:** $d|x$ and $d|y$ or $x = \ell d$ and $y = kd$

$\implies x - y = kd - \ell d$
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$x = \ell d$ and $y = kd$

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Notation: $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Fact: If $d | x$ and $d | y$ then $d | (x + y)$ and $d | (x - y)$.

Is it a fact? Yes? No?

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More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x, y)$.

**Proof:**

\[
\text{mod}(x, y) = x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y = x - s \cdot y
\]

for integer $s = kd - s \ell d$ for integers $k, \ell$ where $x = kd$ and $y = \ell d$.

Therefore $d|\text{mod}(x, y)$.

And $d|y$ since it is in condition.

**Lemma 2:** If $d|y$ and $d|\text{mod}(x, y)$ then $d|y$ and $d|x$.

**Proof:** Similar.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \text{mod}(x, y))$.

**Proof:** $x$ and $y$ have the same set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma 1 and 2.

Same common divisors $\Rightarrow$ largest is the same.
More divisibility

**Notation:** $d \mid x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d \mid x$ and $d \mid y$ then $d \mid y$ and $d \mid \text{mod}(x, y)$.

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**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d| \text{mod} \ (x, y)$.

**Proof:**
\[
\text{mod} \ (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{ mod } (x, y)$.

**Proof:**

$$\text{mod } (x, y) = x - \lfloor x/y \rfloor \cdot y$$

$$= x - [s] \cdot y \quad \text{for integer } s$$
More divisibility

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\text{mod } (x, y) &= x - \lfloor x/y \rfloor \cdot y \\
&= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
&= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
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**Proof...:** Similar.
More divisibility

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**Lemma 1:** If \(d \mid x\) and \(d \mid y\) then \(d \mid y\) and \(d \mid \text{mod}(x, y)\).

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\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer} \ s \\
= kd - s\ell d \quad \text{for integers} \ k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell) d
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Therefore \(d \mid \text{mod}(x, y)\). And \(d \mid y\) since it is in condition.

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**Proof...:** Similar. Try this at home.
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\[\square\text{ish.}\]
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= x - [s] \cdot y \quad \text{for integer } s
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= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
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= (k - s \ell )d
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Same common divisors \(\implies\) largest is the same.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).
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Hey, what’s \( \gcd(7, 0) \)?
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \ mod \ (x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?
Euclid’s algorithm.

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Hey, what’s $\text{gcd}(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\text{gcd}(x, 0)$? $x$
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Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?  \( x \)

(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y))))) ***
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```
(define (euclid x y)
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```

**Theorem:** \((\text{euclid } x \ y) = \gcd(x, y)\) if \( x \geq y \).
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
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(define (euclid x y)
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**Theorem:** \( (\text{euclid} \ x \ y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? $7$ since $7$ divides $7$ and $7$ divides $0$
What’s $\gcd(x, 0)$? $x$

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))
```

*Theorem:* $(\text{euclid } x y) = \gcd(x, y)$ if $x \geq y$.

*Proof:* Use Strong Induction.

*Base Case:* $y = 0$, “$x$ divides $y$ and $x$”
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)?  \( x \)

```
(define (euclid x y)
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**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \implies \text{ “} x \text{ is common divisor and clearly largest.”} \]
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Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

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**Induction Step:** \( \mod(x, y) < y \leq x \) when \( x \geq y \)
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))  ***

**Theorem:** $(\text{euclid } x \ y) = \gcd(x, y)$ if $x \geq y$.

**Proof:** Use Strong Induction.

**Base Case:** $y = 0$, “$x$ divides $y$ and $x$”
  $\implies$ “$x$ is common divisor and clearly largest.”

**Induction Step:** $\mod(x, y) < y \leq x$ when $x \geq y$

call in line (***)) meets conditions plus arguments “smaller”
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \ mod \ (x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

\[
\text{(define (euclid x y)}
\begin{align*}
  &\quad \text{(if (= y 0)} \\
  &\quad \quad \text{x} \\
  &\quad \quad \quad \text{(euclid y (mod x y)))} \quad ***
\end{align*}
\]

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call in line (***), meets conditions plus arguments “smaller”
and by strong induction hypothesis
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \, \text{mod} \,(x, y)) \).

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**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
  \( \implies \) “\( x \) is common divisor and clearly largest.”

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**Base Case:** $y = 0$, "$x$ divides $y$ and $x$" 
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computes $\gcd(y, \mod(x, y))$
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Modular Arithmetic Lecture in a minute.

Modular Arithmetic: \( x \equiv y \pmod{N} \) if \( x = y + kN \) for some integer \( k \).
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer $k$.

For $a \equiv b \pmod{N}$, and $c \equiv d \pmod{N}$,
\[ ac = bd \pmod{N} \] and \[ a + b = c + d \pmod{N}. \]
Modular Arithmetic Lecture in a minute.

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For $a \equiv b \pmod{N}$, and $c \equiv d \pmod{N}$,

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Division?

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Division? Multiply by multiplicative inverse.
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Division? Multiply by multiplicative inverse.
\( a \pmod{N} \) has multiplicative inverse, \( a^{-1} \pmod{N} \).
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\( a \pmod{N} \) has multiplicative inverse, \( a^{-1} \pmod{N} \).
If and only if \( \gcd(a, N) = 1 \).
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For \( a \equiv b \pmod{N} \), and \( c \equiv d \pmod{N} \),
\[
ac = bd \pmod{N} \quad \text{and} \quad a + b = c + d \pmod{N}.
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Division? Multiply by multiplicative inverse.
- \( a \pmod{N} \) has multiplicative inverse, \( a^{-1} \pmod{N} \).
  - If and only if \( \gcd(a, N) = 1 \).

Why?
Modular Arithmetic Lecture in a minute.

Modular Arithmetic: $x \equiv y \pmod{N}$ if $x = y + kN$ for some integer $k$.

For $a \equiv b \pmod{N}$, and $c \equiv d \pmod{N}$,
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Division? Multiply by multiplicative inverse.

$a \pmod{N}$ has multiplicative inverse, $a^{-1} \pmod{N}$.

If and only if $\gcd(a, N) = 1$.

Why? If: $f(x) = ax \pmod{N}$ is a bijection on $\{1, \ldots, N - 1\}$. 

Euclid's Alg: \[ \gcd(x, y) = \gcd(y \mod x, x) \]
Fast cuz value drops by a factor of two every two recursive calls.

Know if there is an inverse, but how do we find it?

On Tuesday!
Modular Arithmetic Lecture in a minute.

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For $a \equiv b \pmod{N}$, and $c \equiv d \pmod{N}$,

$ac = bd \pmod{N}$ and $a + b = c + d \pmod{N}$.

Division? Multiply by multiplicative inverse.

$a \pmod{N}$ has multiplicative inverse, $a^{-1} \pmod{N}$.

If and only if $\gcd(a, N) = 1$.

Why? If: $f(x) = ax \pmod{N}$ is a bijection on $\{1, \ldots, N-1\}$.

$ax - ay = 0 \pmod{N} \implies a(x - y)$ is a multiple of $N$. 
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