Today

Finish Euclid.

Bijection/CRT/Isomorphism.

Fermat's Little Theorem.
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Finish Euclid.
Bijection/CRT/Isomorphism.
Fermat’s Little Theorem.
Greatest Common Divisor and Inverses.

**Thm:**
If greatest common divisor of $x$ and $m$, $\gcd(x, m)$, is 1, then $x$ has a multiplicative inverse modulo $m$. 

Proof:
Claim:
The set $S = \{0 \times x, 1 \times x, \ldots, (m - 1) \times x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$.

Each of $m$ numbers in $S$ correspond to one of $m$ equivalence classes modulo $m$.

$\Rightarrow$ One must correspond to 1 modulo $m$.

Inverse Exists!
Proof of Claim:
If not distinct, then $\exists a, b \in \{0, \ldots, m - 1\}, a \neq b$, where $(a \times x) \equiv (b \times x) \pmod{m}$.

Or $(a - b) \times x \equiv 0 \pmod{m}$.

$\gcd(x, m) = 1 \Rightarrow$ Prime factorization of $m$ and $x$ do not contain common primes.

$\Rightarrow (a - b)$ factorization contains all primes in $m$'s factorization.

So $(a - b)$ has to be multiple of $m$.

$\Rightarrow (a - b) \geq m$.

But $a, b \in \{0, \ldots, m - 1\}$.

Contradiction.
Greatest Common Divisor and Inverses.

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**Claim:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \pmod{m}$ if all distinct modulo $m$. 
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Or $(a - b)x = km$ for some integer $k$. 
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Or $(a - b)x = km$ for some integer $k$.

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**Proof $\Rightarrow$ :**

**Claim:** The set $S = \{0x, 1x, \ldots, (m-1)x\}$ contains $y \equiv 1 \mod m$ if all distinct modulo $m$.

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Proof of Claim: If not distinct, then \( \exists a, b \in \{0, \ldots, m−1\}, a \neq b \), where
\[ (ax \equiv bx \mod m) \implies (a−b)x \equiv 0 \mod m \]
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\[ \gcd(x, m) = 1 \]
\[ \implies \text{Prime factorization of } m \text{ and } x \text{ do not contain common primes.} \]
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So \( (a−b) \) has to be multiple of \( m \).
\[ \implies (a−b) \geq m. \text{ But } a, b \in \{0, \ldots m−1\}. \text{ Contradiction.} \]
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Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$. 
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... For \( x = 4 \) and \( m = 6 \). All products of 4...
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reducing \( \pmod{6} \)

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Not distinct. Common factor 2.
**Proof review. Consequence.**

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Not distinct. Common factor 2. Can’t be 1.
Proof review. Consequence.

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Not distinct. Common factor 2. Can’t be 1. No inverse.
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For \( x = 5 \) and \( m = 6 \).
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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\( S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} \)
Proof: Let \( x \) be a positive integer. We say that \( x \) has a multiplicative inverse modulo \( m \) if there exists an integer \( y \) such that \( xy \equiv 1 \pmod{m} \). This integer \( y \) is called the multiplicative inverse of \( x \) modulo \( m \).

Theorem: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

Proof Sketch: The set \( S = \{0x, 1x, \ldots, (m-1)x\} \) contains \( y \equiv 1 \pmod{m} \) if all elements are distinct modulo \( m \).

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Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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... For \( x = 4 \) and \( m = 6 \). All products of 4...

\[
S = \{0(4), 1(4), 2(4), 3(4), 4(4), 5(4)\} = \{0, 4, 8, 12, 16, 20\}
\]

reducing (mod 6)

\[
S = \{0, 4, 2, 0, 4, 2\}
\]

Not distinct. Common factor 2. Can’t be 1. No inverse.

For \( x = 5 \) and \( m = 6 \).

\[
S = \{0(5), 1(5), 2(5), 3(5), 4(5), 5(5)\} = \{0, 5, 4, 3, 2, 1\}
\]

All distinct, contains 1!
Thm: If \( \gcd(x,m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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Proof review. Consequence.

**Thm:** If $\gcd(x, m) = 1$, then $x$ has a multiplicative inverse modulo $m$.

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(Hmm. What normal number is it own multiplicative inverse?)
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Proof review. Consequence.

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\( 5x = 3 \pmod{6} \)
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What is \( x \)?
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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5x = 3 \mod 6 \text{ What is } x? \text{ Multiply both sides by 5.}
\]

\[
x = 15
\]
Proof review. Consequence.

**Thm:** If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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reducing (mod 6)

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\[ x = 15 = 3 \mod 6 \]
Proof review. Consequence.

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\[
4x = 3 \mod 6
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Thm: If gcd\((x, m) = 1\), then \(x\) has a multiplicative inverse modulo \(m\).

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\[ 5x = 3 \mod 6 \]
What is \(x\)? Multiply both sides by 5.
\[ x = 15 = 3 \mod 6 \]

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x = 15 = 3 \mod 6
\]

4\( x = 3 \mod 6 \) No solutions. Can’t get an odd.
Thm: If gcd\((x, m) = 1\), then \(x\) has a multiplicative inverse modulo \(m\).

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Thm: If \( \gcd(x, m) = 1 \), then \( x \) has a multiplicative inverse modulo \( m \).

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5x = 3 \pmod{6} \text{ What is } x? \text{ Multiply both sides by 5.}
\]

\[
x = 15 = 3 \pmod{6}
\]

\[
4x = 3 \pmod{6} \text{ No solutions. Can’t get an odd.}
\]

\[
4x = 2 \pmod{6} \text{ Two solutions!}
\]
Proof review. Consequence.

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5x = 3 \pmod 6 \text{ What is } x? \text{ Multiply both sides by 5.}
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4x = 3 \pmod 6 \text{ No solutions. Can’t get an odd.}
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4x = 2 \pmod 6 \text{ Two solutions! } x = 2, 5 \pmod 6
\]

Very different for elements with inverses.
Proof Review 2: Bijections.

If \( \gcd(x,m) = 1 \).

Bijection \( \equiv \) unique pre-image and same size.

All the images are distinct. = \( \Rightarrow \) unique pre-image for any image.
Proof Review 2: Bijections.

If \( \gcd(x, m) = 1 \).
Then the function \( f(a) = xa \mod m \) is a bijection.
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One to one: there is a unique pre-image.
Proof Review 2: Bijectons.

If $\gcd(x,m) = 1$.

- Then the function $f(a) = xa \mod m$ is a bijection.
- One to one: there is a unique pre-image.
- Onto: the sizes of the domain and co-domain are the same.

$x = 3, m = 4.$
Proof Review 2: Bijectons.

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\( x = 3, m = 4 \).

\( f(1) = 3(1) = 3 \mod 4 \),
Proof Review 2: Bijections.

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\( x = 3, m = 4 \).

\( f(1) = 3(1) = 3 \pmod{4}, f(2) = 6 = 2 \pmod{4}, \)
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$f(1) = 3(1) = 3 \mod 4, f(2) = 6 = 2 \mod 4, f(3) = 1 \mod 3$. 

Oh yeah.
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Oh yeah. $f(0) = 0$. 
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Bijection $\equiv$ unique pre-image and same size.
Proof Review 2: Bijectons.

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Bijection \( \equiv \) unique pre-image and same size.

All the images are distinct. \( \implies \) unique pre-image for any image.
If \( \gcd(x,m) = 1 \).
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One to one: there is a unique pre-image.
Onto: the sizes of the domain and co-domain are the same.

\[ x = 3, \ m = 4. \]
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Bijection \equiv \text{unique pre-image and same size.}
All the images are distinct. \( \implies \text{unique pre-image for any image.} \)

\[ x = 2, \ m = 4. \]
Proof Review 2: Bijectons.

If \( \gcd(x,m) = 1 \).

Then the function \( f(a) = xa \mod m \) is a bijection.

One to one: there is a unique pre-image.

Onto: the sizes of the domain and co-domain are the same.

\( x = 3, m = 4 \).

\( f(1) = 3(1) = 3 \mod 4, f(2) = 6 = 2 \mod 4, f(3) = 1 \mod 3 \).

Oh yeah. \( f(0) = 0 \).

Bijection \( \equiv \) unique pre-image and same size.

All the images are distinct. \( \implies \) unique pre-image for any image.

\( x = 2, m = 4 \).

\( f(1) = 2, f(2) = 0, f(3) = 2 \)
Proof Review 2: Bijections.

If \( \gcd(x, m) = 1 \).
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One to one: there is a unique pre-image.
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Bijection \( \equiv \) unique pre-image and same size.

All the images are distinct. \( \implies \) unique pre-image for any image.

\( x = 2, m = 4 \).

\( f(1) = 2, f(2) = 0, f(3) = 2 \)

Oh yeah. \( f(0) = 0 \).
Proof Review 2: Bijections.

If \( \gcd(x,m) = 1 \).

Then the function \( f(a) = xa \mod m \) is a bijection.

One to one: there is a unique pre-image.

Onto: the sizes of the domain and co-domain are the same.

\( x = 3, m = 4 \).

\( f(1) = 3(1) = 3 \mod 4, f(2) = 6 = 2 \mod 4, f(3) = 1 \mod 3 \).

Oh yeah. \( f(0) = 0 \).

Bijection \( \equiv \) unique pre-image and same size.

All the images are distinct. \( \implies \) unique pre-image for any image.

\( x = 2, m = 4 \).

\( f(1) = 2, f(2) = 0, f(3) = 2 \)

Oh yeah. \( f(0) = 0 \).

Not a bijection.
Which is bijection?
(A) $f(x) = x$ for domain and range being $\mathbb{R}$
(B) $f(x) = ax \pmod{n}$ for $x \in \{0, \ldots, n-1\}$ and $\gcd(a, n) = 2$
(C) $f(x) = ax \pmod{n}$ for $x \in \{0, \ldots, n-1\}$ and $\gcd(a, n) = 1$
Which is bijection?
(A) $f(x) = x$ for domain and range being $\mathbb{R}$
(B) $f(x) = ax \mod (n)$ for $x \in \{0, ..., n-1\}$ and $\gcd(a, n) = 2$
(C) $f(x) = ax \mod n$ for $x \in \{0, ..., n-1\}$ and $\gcd(a, n) = 1$

(B) is not.
Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$. 
Thm: If $\gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$. Assume $a$ is $x^{-1}$, or $ax = 1 + km$. 
Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).

Assume \( a \) is \( x^{-1} \), or \( ax = 1 + km \).

\[ x = nd \text{ and } m = \ell d \text{ for } d > 1. \]
Thm: If $\gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume $a$ is $x^{-1}$, or $ax = 1 + km$.

$x = nd$ and $m = \ell d$ for $d > 1$.

Thus,
Thm: If $gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume $a$ is $x^{-1}$, or $ax = 1 + km$.

$x = nd$ and $m = \ell d$ for $d > 1$.

Thus,

$$a(nd) = 1 + k\ell d$$

or

$$d(na - k\ell) = 1.$$
Thm: If $\gcd(x, m) \neq 1$ then $x$ has no multiplicative inverse modulo $m$.

Assume $a$ is $x^{-1}$, or $ax = 1 + km$.

$x = nd$ and $m = \ell d$ for $d > 1$.

Thus,

$$a(nd) = 1 + k\ell d$$

or

$$d(na - k\ell) = 1.$$

But $d > 1$ and $n = (na - k\ell) \in \mathbb{Z}$.  
Only if

Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).
Assume \( a \) is \( x^{-1} \), or \( ax = 1 + km \).

\[ x = nd \text{ and } m = \ell d \text{ for } d > 1. \]

Thus,

\[ a(nd) = 1 + k\ell d \text{ or } d(na - k\ell) = 1. \]

But \( d > 1 \) and \( n = (na - k\ell) \in \mathbb{Z}. \)
so \( dn \neq 1 \) and \( dn = 1 \). Contradiction.
Thm: If \( \gcd(x, m) \neq 1 \) then \( x \) has no multiplicative inverse modulo \( m \).

Assume \( a \) is \( x^{-1} \), or \( ax = 1 + km \).

\( x = nd \) and \( m = \ell d \) for \( d > 1 \).

Thus,

\[ a(nd) = 1 + k\ell d \text{ or } d(na - k\ell) = 1. \]

But \( d > 1 \) and \( n = (na - k\ell) \in \mathbb{Z} \).

so \( dn \neq 1 \) and \( dn = 1 \). Contradiction.
Finding inverses.

How to find the inverse?

Algorithm:
Try all numbers up to \( x \) to see if it divides both \( x \) and \( m \).

Very slow.
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?
Finding inverses.

How to find the inverse?
How to find if $x$ has an inverse modulo $m$?
Find $\gcd(x, m)$. 
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\gcd(x, m)$.
  
  Greater than 1?
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find gcd $(x, m)$.
  
  Greater than 1? No multiplicative inverse.
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\text{gcd} (x, m)$.
   Greater than 1? No multiplicative inverse.
   Equal to 1?
Finding inverses.

How to find the inverse?

How to find if \( x \) has an inverse modulo \( m \)?

Find \( \gcd(x, m) \).
  Greater than 1? No multiplicative inverse.
  Equal to 1? Multiplicative inverse.
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\gcd(x, m)$.
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Algorithm:
Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\gcd(x, m)$.
  
  Greater than 1? No multiplicative inverse.
  
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Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$. 

Finding inverses.

How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\gcd(x, m)$.
  
  Greater than 1? No multiplicative inverse.
  
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Very slow.
How to find the inverse?

How to find if $x$ has an inverse modulo $m$?

Find $\gcd(x, m)$.
- Greater than 1? No multiplicative inverse.
- Equal to 1? Multiplicative inverse.

Algorithm: Try all numbers up to $x$ to see if it divides both $x$ and $m$.

Very slow.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:**
If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod}(x, y)$.

**Proof:**
$\text{mod}(x, y) = x - \lfloor x / y \rfloor \cdot y = x - s \cdot y$ for integer $s = kd - s \ell d$ for integers $k, \ell$ where $x = kd$ and $y = \ell d$.

Therefore $d | \text{mod}(x, y)$.

And $d | y$ since it is in condition.

**Lemma 2:**
If $d | y$ and $d | \text{mod}(x, y)$ then $d | y$ and $d | x$.

**Proof:** Similar.

GCD Mod Corollary:
$\gcd(x, y) = \gcd(y, \text{mod}(x, y))$.

**Proof:** $x$ and $y$ have the same set of common divisors as $x$ and $\text{mod}(x, y)$ by Lemma 1 and 2.

Same common divisors $\Rightarrow$ largest is the same.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$. 

Lemma 1: If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod}(x, y)$.

**Proof:**

\[
\text{mod}(x, y) = x - \lfloor \frac{x}{y} \rfloor \cdot y = x - \lfloor s \rfloor \cdot y = kd - s \ell d \text{ for integers } k, \ell
\]

Therefore $d | \text{mod}(x, y)$.

And $d | y$ since it is in condition.

Lemma 2: If $d | y$ and $d | \text{mod}(x, y)$ then $d | y$ and $d | x$.

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More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or 
\( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod}(x, y) \).
More divisibility

Notation: $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

Lemma 1: If $d|x$ and $d|y$ then $d|y$ and $d|\text{mod}(x, y)$.

Proof:
\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y
\]
More divisibility

**Notation:** $d|\ x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|\ x$ and $d|\ y$ then $d|\ y$ and $d|\ \text{mod}(x, y)$.

**Proof:**

\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
More divisibility

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**Proof:**

$$\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y$$
$$= x - s \cdot y \quad \text{for integer } s$$
$$= kd - sld \quad \text{for integers } k, l \text{ where } x = kd \text{ and } y = ld$$
More divisibility

**Notation:** $d|x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d|x$ and $d|y$ then $d|y$ and $d|\mod(x, y)$.

**Proof:**
\[
\mod(x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or
$x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s
\]
\[
= kd - sld \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]
Therefore $d | \text{mod} (x, y)$. 

**GCD Mod Corollary:**
\[
\gcd (x, y) = \gcd (y, \text{mod} (x, y))
\]

**Proof:** $x$ and $y$ have same set of common divisors as $x$ and $\text{mod} (x, y)$ by Lemma 1 and 2.

Same common divisors $\Rightarrow$ largest is the same.
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or 
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**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \).

**Proof:**
\[
\begin{align*}
\text{mod} \ (x, y) &= x - \lfloor x/y \rfloor \cdot y \\
&= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
&= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
&= (k - s\ell)d
\end{align*}
\]
Therefore \( d \mid \text{mod} \ (x, y) \). And \( d \mid y \) since it is in condition.
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod} \ (x, y) \).

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\text{mod} \ (x, y) = x - \lfloor x/y \rfloor \cdot y \\
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= (k - s \ell)d
\]

Therefore $d | \text{mod} \ (x, y)$. And $d | y$ since it is in condition.

**Lemma 2:** If $d | y$ and $d | \text{mod} \ (x, y)$ then $d | y$ and $d | x$.

**Proof...:** Similar.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.

**Proof:**
\[
\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y
\]
\[
= x - s \cdot y \quad \text{for integer } s
\]
\[
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d
\]
\[
= (k - s\ell)d
\]
Therefore $d | \text{mod} (x, y)$. And $d | y$ since it is in condition. \(\square\)

**Lemma 2:** If $d | y$ and $d | \text{mod} (x, y)$ then $d | y$ and $d | x$.

**Proof...:** Similar. Try this at home.
More divisibility

**Notation:** \(d|x\) means “\(d\) divides \(x\)” or \(x = kd\) for some integer \(k\).

**Lemma 1:** If \(d|x\) and \(d|y\) then \(d|y\) and \(d|\text{mod}(x, y)\).

**Proof:**
\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]
Therefore \(d|\text{mod}(x, y)\). And \(d|y\) since it is in condition. \(\square\)

**Lemma 2:** If \(d|y\) and \(d|\text{mod}(x, y)\) then \(d|y\) and \(d|x\).

**Proof...:** Similar. Try this at home. \(\square\)ish.

GCD Mod Corollary: \(\gcd(x, y) = \gcd(y, \text{mod}(x, y))\).

**Proof:** \(x\) and \(y\) have same set of common divisors as \(x\) and \(\text{mod}(x, y)\) by Lemma 1 and 2. Same common divisors \(\Rightarrow\) largest is the same.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} (x, y)$.

**Proof:**

$$\text{mod} (x, y) = x - \lfloor x/y \rfloor \cdot y$$

$$= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s$$

$$= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d$$

$$= (k - s\ell)d$$

Therefore $d | \text{mod} (x, y)$. And $d | y$ since it is in condition.

**Lemma 2:** If $d | y$ and $d | \text{mod} (x, y)$ then $d | y$ and $d | x$.

**Proof...:** Similar. Try this at home.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \ \text{mod} (x, y))$. 
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \mod(x, y) \).

**Proof:**
\[
\mod(x, y) = x - \lfloor x/y \rfloor \cdot y = x - \lfloor s \rfloor \cdot y \quad \text{for integer} \ s
\]
\[
= kd - s \ell d \quad \text{for integers} \ k, \ell \ \text{where} \ x = kd \ \text{and} \ y = \ell d
\]
\[
= (k - s \ell) d
\]

Therefore \( d \mid \mod(x, y) \). And \( d \mid y \) since it is in condition. \( \square \)

**Lemma 2:** If \( d \mid y \) and \( d \mid \mod(x, y) \) then \( d \mid y \) and \( d \mid x \).

**Proof...:** Similar. Try this at home. \( \square \)ish.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

**Proof:** \( x \) and \( y \) have same set of common divisors as \( x \) and \( \mod(x, y) \) by Lemma 1 and 2.
More divisibility

**Notation:** \( d \mid x \) means “\( d \) divides \( x \)” or \( x = kd \) for some integer \( k \).

**Lemma 1:** If \( d \mid x \) and \( d \mid y \) then \( d \mid y \) and \( d \mid \text{mod}(x, y) \).

**Proof:**
\[
\text{mod}(x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s \ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s \ell)d
\]
Therefore \( d \mid \text{mod}(x, y) \). And \( d \mid y \) since it is in condition.  \( \square \)

**Lemma 2:** If \( d \mid y \) and \( d \mid \text{mod}(x, y) \) then \( d \mid y \) and \( d \mid x \).

**Proof...:** Similar. Try this at home.  \( \square \)ish.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \text{mod}(x, y)) \).

**Proof:** \( x \) and \( y \) have **same** set of common divisors as \( x \) and \( \text{mod}(x, y) \) by Lemma 1 and 2.

Same common divisors \( \implies \) largest is the same.
More divisibility

**Notation:** $d | x$ means “$d$ divides $x$” or $x = kd$ for some integer $k$.

**Lemma 1:** If $d | x$ and $d | y$ then $d | y$ and $d | \text{mod} \,(x, y)$.

**Proof:**

\[
\text{mod} \,(x, y) = x - \lfloor x/y \rfloor \cdot y \\
= x - \lfloor s \rfloor \cdot y \quad \text{for integer } s \\
= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\
= (k - s\ell)d
\]

Therefore $d | \text{mod} \,(x, y)$. And $d | y$ since it is in condition. □

**Lemma 2:** If $d | y$ and $d | \text{mod} \,(x, y)$ then $d | y$ and $d | x$.

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**Proof:** $x$ and $y$ have **same** set of common divisors as $x$ and $\text{mod} \,(x, y)$ by Lemma 1 and 2.

Same common divisors $\implies$ largest is the same. □
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)?
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \ mod \ (x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
Euclid’s algorithm.

**GCD Mod Corollary:** \( \text{gcd}(x, y) = \text{gcd}(y, \text{mod}(x, y)) \).

Hey, what’s \( \text{gcd}(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \text{gcd}(x, 0) \)?

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

***Theorem:**

\( (\text{euclid } x \ y) = \text{gcd}(x, y) \) if \( x \geq y \).

**Proof:**

Use Strong Induction.

**Base Case:**

\( y = 0, \) "x divides y and x" \( \Rightarrow \) "x is common divisor and clearly largest."

**Induction Step:**

\( \text{mod}(x, y) < y \leq x \) when \( x \geq y \) call in line (***) meets conditions plus arguments "smaller" and by strong induction hypothesis computes \( \text{gcd}(y, \text{mod}(x, y)) \) which is \( \text{gcd}(x, y) \) by GCD Mod Corollary.
Euclid’s algorithm.

GCD Mod Corollary: \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)
Euclid’s algorithm.

**GCD Mod Corollary:** $\text{gcd}(x, y) = \text{gcd}(y, \mod (x, y))$.

Hey, what’s gcd(7, 0)? 7 since 7 divides 7 and 7 divides 0
What’s gcd(x, 0)? x

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))) ) ***
```
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod (x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))) ) ***
```

**Theorem:** \( (\text{euclid } x \ y) = \gcd(x, y) \) if \( x \geq y \).
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? $7$ since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

\[
\begin{align*}
\text{(define (euclid x y)} \hfill \star \star \star \\
\text{(if (= y 0)} \hfill \star \star \star \\
\text{x) \hfill \star \star \star \\
\text{(euclid y (mod x y))])))} \quad \star \star \star \\
\text{Theorem:} \quad (\text{euclid x y}) = \gcd(x, y) \text{ if } x \geq y.
\]

**Proof:** Use Strong Induction.
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0
What’s \( \gcd(x, 0) \)? \( x \)

```scheme
(define (euclid x y)
    (if (= y 0)
        x
        (euclid y (mod x y))))  ***
```

**Theorem:** \( (\text{euclid } x \ y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.
**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0

What’s \( \gcd(x, 0) \)? \( x \)

(define (euclid x y)
  (if (= y 0)
    x
    (euclid y (mod x y))))  ***

**Theorem:** \( (\text{euclid } x \ y) = \gcd(x, y) \) if \( x \geq y \).

**Proof:** Use Strong Induction.

**Base Case:** \( y = 0 \), “\( x \) divides \( y \) and \( x \)”

\[ \implies \text{“} x \text{ is common divisor and clearly largest.”} \]
Euclid’s algorithm.

**GCD Mod Corollary:** $\gcd(x, y) = \gcd(y, \mod(x, y))$.

Hey, what’s $\gcd(7, 0)$? 7 since 7 divides 7 and 7 divides 0
What’s $\gcd(x, 0)$? $x$

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))) ***
```

**Theorem:** $(\text{euclid } x y) = \gcd(x, y)$ if $x \geq y$.

**Proof:** Use Strong Induction.
**Base Case:** $y = 0$, “$x$ divides $y$ and $x$”
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**Induction Step:** $\mod(x, y) < y \leq x$ when $x \geq y$
Euclid’s algorithm.

**GCD Mod Corollary:** \( \gcd(x, y) = \gcd(y, \mod(x, y)) \).

Hey, what’s \( \gcd(7, 0) \)? 7 since 7 divides 7 and 7 divides 0

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call in line (***) meets conditions plus arguments “smaller”
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Excursion: Value and Size.

Before discussing running time of gcd procedure...
Excursion: Value and Size.

Before discussing running time of gcd procedure...

What is the value of 1,000,000?
Excursion: Value and Size.

Before discussing running time of gcd procedure...
What is the value of 1,000,000?
one million or 1,000,000!
Excursion: Value and Size.

Before discussing running time of gcd procedure...
What is the value of 1,000,000?
one million or 1,000,000!
What is the “size” of 1,000,000?
Excursion: Value and Size.

Before discussing running time of gcd procedure...

What is the value of 1,000,000?
one million or 1,000,000!

What is the “size” of 1,000,000?
Number of digits in base 10: 7.
Before discussing running time of gcd procedure...

What is the value of 1,000,000?

One million or 1,000,000!

What is the “size” of 1,000,000?

Number of digits in base 10: 7.

Number of bits (a digit in base 2): 21.
Excursion: Value and Size.

Before discussing running time of gcd procedure...

What is the value of 1,000,000?  
One million or 1,000,000!

What is the “size” of 1,000,000?  
Number of digits in base 10: 7.

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For a number $x$, what is its size in bits?
Excursion: Value and Size.

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For a number $x$, what is its size in bits?

\[ n = b(x) \approx \log_2 x \]
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\[ n = b(x) \approx \log_2 x \]
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) ”divisions” where \(n = b(x) \approx \log_2 x\).
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**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) "divisions" where \(n = b(x) \approx \log_2 x\).

Is this good?
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) “divisions” where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots y/2\}\)?
Euclid procedure is fast.

**Theorem:** (euclid x y) uses $2n$ ”divisions” where $n = b(x) \approx \log_2 x$.

Is this good? Better than trying all numbers in \{2, \ldots y/2\}? Check 2,
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) "divisions" where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots y/2\}\)?

Check 2, check 3,
Euclid procedure is fast.

**Theorem:** (euclid $x$ $y$) uses $2n$ ”divisions” where $n = b(x) \approx \log_2 x$.

Is this good? Better than trying all numbers in \{2, \ldots \ y/2\}?

Check 2, check 3, check 4,
Euclid procedure is fast.

**Theorem:** $(\text{euclid } x \ y)$ uses $2n$ ”divisions” where $n = b(x) \approx \log_2 x$.

Is this good? Better than trying all numbers in $\{2, \ldots y/2\}$? Check $2$, check $3$, check $4$, check $5$ \ldots , check $y/2$. 

$2^n$ is much faster! Roughly $2^{100} \approx 10^{30} =$ ”million, trillion, trillion” divisions!
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) ”divisions” where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots y/2\}\)?

Check 2, check 3, check 4, check 5 . . . , check \(y/2\).
Euclid procedure is fast.

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Is this good? Better than trying all numbers in \(\{2, \ldots, y/2\}\)?

Check 2, check 3, check 4, check 5 . . . , check \(y/2\).

If \(y \approx x\)
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) ”divisions” where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots, y/2\}\)?

Check 2, check 3, check 4, check 5 . . . , check \(y/2\).

If \(y \approx x\) roughly \(y\) uses \(n\) bits
Euclid procedure is fast.

**Theorem:** \((euclid \ x \ y)\) uses \(2n\) "divisions" where \(n = b(x) \approx \log_2 x\).

Is this good? Better than trying all numbers in \(\{2, \ldots, y/2\}\)?

Check 2, check 3, check 4, check 5 . . . , check \(y/2\).

If \(y \approx x\) roughly \(y\) uses \(n\) bits ...
   \(2^{n-1}\) divisions! Exponential dependence on size!
Euclid procedure is fast.

Theorem: (euclid x y) uses $2n$ "divisions" where $n = b(x) \approx \log_2 x$.

Is this good? Better than trying all numbers in $\{2, \ldots, y/2\}$?

Check 2, check 3, check 4, check 5 \ldots, check $y/2$.

If $y \approx x$ roughly $y$ uses $n$ bits ... 

$2^{n-1}$ divisions! Exponential dependence on size!

101 bit number.
Euclid procedure is fast.

**Theorem:** \((\text{euclid } x \ y)\) uses \(2n\) ”divisions” where \(n = b(x) \approx \log_2 x\). Is this good? Better than trying all numbers in \(\{2, \ldots y/2\}\)?

Check 2, check 3, check 4, check 5 \ldots check \(y/2\).

If \(y \approx x\) roughly \(y\) uses \(n\) bits ...

\[2^{n-1}\] divisions! Exponential dependence on size!

101 bit number. \(2^{100} \approx 10^{30} = \text{“million, trillion, trillion” divisions!}\)
Euclid procedure is fast.

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Is this good? Better than trying all numbers in $\{2, \ldots y/2\}$?

Check 2, check 3, check 4, check 5 . . . , check $y/2$.

If $y \approx x$ roughly $y$ uses $n$ bits . . .

$2^{n-1}$ divisions! Exponential dependence on size!

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$2n$ is much faster!
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If \(y \approx x\) roughly \(y\) uses \(n\) bits ... 

\(2^{n-1}\) divisions! Exponential dependence on size!

101 bit number. \(2^{100} \approx 10^{30} = \text{“million, trillion, trillion” divisions!}\)

\(2n\) is much faster! .. roughly 200 divisions.
Assume \( \log_2 1,000,000 \) is 20 to the nearest integer. Mark what’s true.
Assume $\log_2 1,000,000$ is 20 to the nearest integer. Mark what’s true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.
Assume $\log_2 1,000,000$ is 20 to the nearest integer. Mark what’s true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

(A) and (C).
Poll

Which are correct?

(A) gcd(700, 568) = gcd(568, 132)
(B) gcd(8, 3) = gcd(3, 2)
(C) gcd(8, 3) = 1
(D) gcd(4, 0) = 4
Poll

Which are correct?

(A) \( \gcd(700,568) = \gcd(568,132) \)
(B) \( \gcd(8,3) = \gcd(3,2) \)
(C) \( \gcd(8,3) = 1 \)
(D) \( \gcd(4,0) = 4 \)
Algorithms at work.

Trying everything

\( \text{euclid}(700, 568) \)
\( \text{euclid}(568, 132) \)
\( \text{euclid}(132, 40) \)
\( \text{euclid}(40, 12) \)
\( \text{euclid}(12, 4) \)
\( \text{euclid}(4, 0) \)

Notice: The first argument decreases rapidly. At least a factor of 2 in two recursive calls. (The second is less than the first.)
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 \ldots, check \( y/2 \).
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).

“(gcd x y)” at work.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd x y)” at work.

\[
\text{euclid}(700, 568)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 ..., check \(y/2\).
“(gcd \ x \ y)” at work.

\[
euclid(700, 568) \\
euclid(568, 132)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check y/2.
“(gcd x y)” at work.

\[
\begin{align*}
\text{euclid}(700, 568) \\
\text{euclid}(568, 132) \\
\text{euclid}(132, 40)
\end{align*}
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
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\[
\begin{align*}
euclid(700,568) \\
euclid(568,132) \\
euclid(132,40) \\
euclid(40,12)
\end{align*}
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd x y)” at work.

\[
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12) \\
euclid(12, 4)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd x y)” at work.

\[
\text{euclid}(700, 568) \\
\text{euclid}(568, 132) \\
\text{euclid}(132, 40) \\
\text{euclid}(40, 12) \\
\text{euclid}(12, 4) \\
\text{euclid}(4, 0)
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check $y/2$.
“(gcd $x$ $y$)” at work.

\[
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12) \\
euclid(12, 4) \\
euclid(4, 0) \\
4
\]
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . ., check $y/2$.

“(gcd x y)” at work.

\[
\begin{align*}
\text{euclid}(700, 568) \\
\text{euclid}(568, 132) \\
\text{euclid}(132, 40) \\
\text{euclid}(40, 12) \\
\text{euclid}(12, 4) \\
\text{euclid}(4, 0) \\
&= 4
\end{align*}
\]

Notice: The first argument decreases rapidly.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 . . . , check \( y/2 \).
“(gcd x y)” at work.

\[
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12) \\
euclid(12, 4) \\
euclid(4, 0) \\
4
\]

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.
Algorithms at work.

Trying everything
Check 2, check 3, check 4, check 5 ... , check \( y/2 \).
“(gcd x y)” at work.

\[
\begin{align*}
euclid(700, 568) \\
euclid(568, 132) \\
euclid(132, 40) \\
euclid(40, 12) \\
euclid(12, 4) \\
euclid(4, 0) \\
4
\end{align*}
\]

Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.
(The second is less than the first.)
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$. 
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Theorem: (euclid x y) uses \(O(n)\) ”divisions” where \(n = b(x)\).
Proof:

Fact: First arg decreases by at least factor of two in two recursive calls.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls. After $2\log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** \((\text{euclid } x \ y)\) uses \(O(n)\) "divisions" where \(n = b(x)\).

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After \(2\log_2 x = O(n)\) recursive calls, argument \(x\) is 1 bit number.
One more recursive call to finish.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After $2\log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
Runtime Proof.

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))
```

**Theorem:** \((euclid x y)\) uses \(O(n)\) "divisions" where \(n = b(x)\).

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After \(2^{\log_2 x} = O(n)\) recursive calls, argument \(x\) is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
\(O(n)\) divisions.
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\Rightarrow$ true in one recursive call;
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < \frac{x}{2}$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show “$y \geq \frac{x}{2}$” $\implies$ “$\text{mod}(x, y) \leq \frac{x}{2}$.”
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact: 
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show "$y \geq x/2$ $\implies$ \textquote{mod}(x,y) \leq x/2.$" 
  \textquote{mod}(x,y) is second argument in next recursive call,
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \(y < \frac{x}{2}\), first argument is \(y\)
  \(\implies\) true in one recursive call;

Case 2: Will show “\(y \geq \frac{x}{2}\) ” \(\implies\) “\(\text{mod}(x, y) \leq \frac{x}{2}\).”
  \(\text{mod} (x,y)\) is second argument in next recursive call,
  and becomes the first argument in the next one.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show “$y \geq x/2$” $\implies$ “$mod(x,y) \leq x/2$.”
  $mod(x,y)$ is second argument in next recursive call,
  and becomes the first argument in the next one.
When $y \geq x/2$, then
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \(y < \frac{x}{2}\), first argument is \(y\)
  \(\Rightarrow\) true in one recursive call;

Case 2: Will show \(y \geq \frac{x}{2}\) \(\Rightarrow\) \(\text{mod}(x, y) \leq \frac{x}{2}\).
  \(\text{mod}(x, y)\) is second argument in next recursive call,
  and becomes the first argument in the next one.
When \(y \geq \frac{x}{2}\), then
  \(\left\lfloor \frac{x}{y} \right\rfloor = 1,\)
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.
Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
\[ \implies \text{true in one recursive call} \]

Case 2: Will show \( y \geq \frac{x}{2} \) \( \implies \) \( \text{mod}(x, y) \leq \frac{x}{2} \).
\( \text{mod} (x, y) \) is second argument in next recursive call, and becomes the first argument in the next one.

When \( y \geq \frac{x}{2} \), then
\[ \left\lfloor \frac{x}{y} \right\rfloor = 1, \]
\[ \text{mod} (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = \]
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
  $\implies$ true in one recursive call;

Case 2: Will show “$y \geq x/2$” $\implies$ “$\text{mod}(x, y) \leq x/2$.”
  $\text{mod } (x, y)$ is second argument in next recursive call, and becomes the first argument in the next one.

When $y \geq x/2$, then

$$\left\lfloor \frac{x}{y} \right\rfloor = 1,$$

$$\text{mod } (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - x/2$$
Runtime Proof (continued.)

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  \]
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Mark correct answers.
Note: Mod(x,y) is the remainder of x divided by y.
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Note: Mod(x,y) is the remainder of x divided by y.

(A) \( \text{mod} (x, y) < y \)
(B) If euclid(x,y) calls euclid(u,v) calls euclid (a,b) then \( a \leq x/2 \).
(C) euclid(x,y) calls euclid (u,v) means \( u = y \).
(D) if \( y > x/2 \), \( \text{mod}(x, y) < y/2 \)
(E) if \( y > x/2 \), \( \text{mod}(x, y) = (y - x) \)
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(D) is not always true.
Finding an inverse?

We showed how to efficiently tell if there is an inverse.
Finding an inverse?

We showed how to efficiently tell if there is an inverse.
Extend euclid to find inverse.
Euclid’s GCD algorithm.

\[
\text{(define (euclid } x \ y) \\
\text{ (if (= y 0)} \\
\quad x \\
\text{ (euclid } y \ (\text{mod } x \ y))))
\]
Euclid’s GCD algorithm.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

 Computes the gcd(x, y) in $O(n)$ divisions. (Remember $n = \log_2 x$.)
Euclid’s GCD algorithm.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Computes the gcd($x, y$) in $O(n)$ divisions. (Remember $n = \log_2 x$.)

For $x$ and $m$, if gcd($x, m$) = 1 then $x$ has an inverse modulo $m$. 
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse.
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse.

How do we **find** a multiplicative inverse?
Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

$$ax + by$$
Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

$$ax + by = d$$

where $d = \gcd(x, y)$. 

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3) \cdot 12 + (-1) \cdot 35 = 1.$$

$a = 3$ and $b = -1$.

The multiplicative inverse of 12 (mod 35) is 3.

Check: $3 \cdot (12) = 36 = 1 \pmod{35}$.
Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that
$$ax + by = d$$
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“Make $d$ out of sum of multiples of $x$ and $y$.”
Extended GCD

Euclid’s Extended GCD Theorem: For any \( x, y \) there are integers \( a, b \) such that
\[
ax + by = d \\
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\]

“Make \( d \) out of sum of multiples of \( x \) and \( y \).”

What is multiplicative inverse of \( x \) modulo \( m \)?
Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

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“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\text{gcd}(x, m) = 1$. 

Example: For $x = 12$ and $y = 35$, $\text{gcd}(12, 35) = 1$.

$$(3) \cdot 12 + (-1) \cdot 35 = 1.$$

$a = 3$ and $b = -1$.

The multiplicative inverse of 12 (mod 35) is 3.

Check: $3 \cdot (12) = 36 = 1 \text{ (mod 35)}$. 


Extended GCD

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So $a$ multiplicative inverse of $x$ ($\pmod{m}$)!!
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Check: $3 \cdot 12 = 36 \equiv 1 \pmod{35}$. 
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$$ax + by = d$$ where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”

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“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

\[ ax + bm = 1 \]
\[ ax \equiv 1 - bm \equiv 1 \pmod{m}. \]

So $a$ multiplicative inverse of $x$ (mod $m$)!!

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

\[ (3)12 + (-1)35 = 1. \]
\[ a = 3 \text{ and } b = -1. \]
Extended GCD

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$$ax + bm = 1$$
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Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$  

$a = 3$ and $b = -1$.

The multiplicative inverse of 12 (mod 35) is 3.

Check: $3(12)$
Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that
\[ ax + by = d \]
where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

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$a = 3$ and $b = -1$.

The multiplicative inverse of $12 \pmod{35}$ is $3$.

Check: $3(12) = 36 = 1 \pmod{35}$. 
Make \( d \) out of multiples of \( x \) and \( y \)..?

\[
\text{gcd}(35, 12)
\]
Make \( d \) out of multiples of \( x \) and \( y \)..?

\[
\text{gcd}(35, 12) \\
\text{gcd}(12, 11) ;; \text{gcd}(12, 35 \% 12)
\]
Make $d$ out of multiples of $x$ and $y$..?

\[
\begin{align*}
gcd(35,12) \\
gcd(12, 11) ;;& \quad gcd(12, 35\%12) \\
gcd(11, 1) ;;& \quad gcd(11, 12\%11)
\end{align*}
\]
Make $d$ out of multiples of $x$ and $y$..?

\[
\text{gcd}(35, 12) \\
\text{gcd}(12, 11) ;; \text{gcd}(12, 35 \% 12) \\
\text{gcd}(11, 1) ;; \text{gcd}(11, 12 \% 11) \\
\text{gcd}(1, 0) \\
1
\]
Make $d$ out of multiples of $x$ and $y$..?

```
gcd(35,12)
gcd(12, 11) ;; gcd(12, 35%12)
gcd(11, 1) ;; gcd(11, 12%11)
gcd(1,0)
1
```

How did gcd get 11 from 35 and 12?
Make $d$ out of multiples of $x$ and $y$? 

\[
gcd(35, 12) \\
gcd(12, 11) \quad ;; \quad gcd(12, 35 \mod 12) \\
gcd(11, 1) \quad ;; \quad gcd(11, 12 \mod 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2) 12 = 11
\]
Make $d$ out of multiples of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \% 12) \\
gcd(11, 1) ;; gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
Make $d$ out of multiples of $x$ and $y$ ..?

\[
gcd(35,12) \\
gcd(12, 11) \quad ;; \quad gcd(12, 35\%12) \\
gcd(11, 1) \quad ;; \quad gcd(11, 12\%11) \\
gcd(1,0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \lfloor \frac{35}{12} \rfloor \cdot 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \lfloor \frac{12}{11} \rfloor \cdot 11 = 12 - (1)11 = 1
\]
Make $d$ out of multiples of $x$ and $y$?

\[
gcd(35, 12) \\
gcd(12, 11) ;; \ gcd(12, 35 \mod 12) \\
gcd(11, 1) ;; \ gcd(11, 12 \mod 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.
Make $d$ out of multiples of $x$ and $y$?

\begin{align*}
gcd(35, 12) \\
gcd(12, 11) \;; \quad gcd(12, 35 \% 12) \\
gcd(11, 1) \;; \quad gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}

How did $gcd$ get 11 from 35 and 12?

$35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11$

How does $gcd$ get 1 from 12 and 11?

$12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Make $d$ out of multiples of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) \quad ;; \quad gcd(12, 35\%12) \\
gcd(11, 1) \quad ;; \quad gcd(11, 12\%11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11\]

How does gcd get 1 from 12 and 11?
\[12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Get 1 from 12 and 11.
Make $d$ out of multiples of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \% 12) \\
gcd(11, 1) ;; gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]
Make \( d \) out of multiples of \( x \) and \( y \)..<br>

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) & \quad ;; \quad gcd(12, 35 \% 12) \\
gcd(11, 1) & \quad ;; \quad gcd(11, 12 \% 11) \\
gcd(1, 0) & = 1
\end{align*}
\]

How did \( \text{gcd} \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor \times 12 = 35 - (2)12 = 11
\]

How does \( \text{gcd} \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor \times 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin....
Make $d$ out of multiples of $x$ and $y$..?

```
gcd(35, 12)
gcd(12, 11) ;; gcd(12, 35%12)
gcd(11, 1) ;; gcd(11, 12%11)
gcd(1, 0)
  1
```

How did gcd get 11 from 35 and 12?

$35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

$12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$

Get 11 from 35 and 12 and plugin.... Simplify.
Make \( d \) out of multiples of \( x \) and \( y \) ..?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35 \% 12) \\
gcd(11, 1) ;; gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did \( \text{gcd} \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( \text{gcd} \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin.... Simplify.
Make \( d \) out of multiples of \( x \) and \( y \)...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;; \gcd(12, 35 \% 12) \\
gcd(11, 1) ;; \gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did \( \gcd \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor \times 12 = 35 - (2) \times 12 = 11
\]

How does \( \gcd \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor \times 11 = 12 - (1) \times 11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1) \times 11 = 12 - (1)(35 - (2) \times 12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin.... Simplify. \( a = 3 \) and \( b = -1 \).
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

\[
\text{if } y = 0 \text{ then return}(x, 1, 0)
\]

\[
\text{else}
\]

\[
(d, a, b) \text{ := ext-gcd}(y, \text{mod}(x, y))
\]

\[
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return }(x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \quad \text{return } (d, b, \ a - \text{floor}(x/y) \ * \ b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Extended GCD Algorithm.

```plaintext
ext-gcd(x,y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y) * b)
```

Claim: Returns \((d, a, b)\): \(d = \text{gcd}(a, b)\) and \(d = ax + by\).

Example:

```plaintext
ext-gcd(35,12)
```
Extended GCD Algorithm.

\[\text{ext-gcd}(x, y)\]

\[
\begin{align*}
\text{if } y &= 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
(d, a, b) &= \text{ext-gcd}(y, \text{mod}(x,y)) \\
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{align*}
\]

Claim: Returns \((d, a, b)\): \(d = \text{gcd}(a, b)\) and \(d = ax + by\).

Example:

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
\text{ext-gcd}(12, 11)
\end{align*}
\]
Extended GCD Algorithm.

```
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y) * b)
```

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

- \(\text{ext-gcd}(35, 12)\)
- \(\text{ext-gcd}(12, 11)\)
- \(\text{ext-gcd}(11, 1)\)
Extended GCD Algorithm.

ext-gcd(x,y)
   if y = 0 then return(x, 1, 0)
   else
      (d, a, b) := ext-gcd(y, mod(x,y))
      return (d, b, a - floor(x/y) * b)

Claim: Returns (d, a, b): d = gcd(a,b) and d = ax + by.
Example:

   ext-gcd(35,12)
      ext-gcd(12, 11)
         ext-gcd(11, 1)
            ext-gcd(1,0)
Extended GCD Algorithm.

\[
ext\text{gcd}(x,y)\\
\text{if } y = 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
\quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \\
\quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Example: \(a - \lfloor x/y \rfloor \cdot b = \)

\[
ext\text{gcd}(35,12) \\
ext\text{gcd}(12, 11) \\
ext\text{gcd}(11, 1) \\
ext\text{gcd}(1,0) \\
\text{return } (1,1,0) ;; 1 = (1)1 + (0) 0
\]
Extended GCD Algorithm.

```
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
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        (d, a, b) := ext-gcd(y, mod(x, y))
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```

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1\)

```
ext-gcd(35, 12)
ext-gcd(12, 11)
ext-gcd(11, 1)
ext-gcd(1, 0)
return (1, 1, 0) ;; 1 = (1)1 + (0) 0
return (1, 0, 1) ;; 1 = (0)11 + (1)1
```
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

1. if \( y = 0 \) then return \((x, 1, 0)\)
2. else
   - \((d, a, b) := \text{ext-gcd}(y, \mod(x, y))\)
   - return \((d, b, a - \text{floor}(x/y) \times b)\)

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1\)

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
&= \text{ext-gcd}(12, 11) \\
&= \text{ext-gcd}(11, 1) \\
&= \text{ext-gcd}(1, 0) \\
&\quad \text{return } (1, 1, 0) ;; 1 = (1)1 + (0)0 \\
&\quad \text{return } (1, 0, 1) ;; 1 = (0)11 + (1)1 \\
&\quad \text{return } (1, 1, -1) ;; 1 = (1)12 + (-1)11
\end{align*}
\]
Extended GCD Algorithm.

\[ \text{ext-gcd}(x, y) \]
\[ \quad \text{if } y = 0 \text{ then return } (x, 1, 0) \]
\[ \quad \text{else} \]
\[ \quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \]
\[ \quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b) \]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 35/12 \rfloor \cdot (-1) = 3\)

\[ \text{ext-gcd}(35, 12) \]
\[ \quad \text{ext-gcd}(12, 11) \]
\[ \quad \quad \text{ext-gcd}(11, 1) \]
\[ \quad \quad \quad \text{ext-gcd}(1, 0) \]
\[ \quad \quad \quad \text{return } (1, 1, 0) \quad ;; \quad 1 = (1)1 + (0)0 \]
\[ \quad \quad \quad \text{return } (1, 0, 1) \quad ;; \quad 1 = (0)11 + (1)1 \]
\[ \quad \quad \text{return } (1, 1, -1) \quad ;; \quad 1 = (1)12 + (-1)11 \]
\[ \quad \text{return } (1, -1, 3) \quad ;; \quad 1 = (-1)35 + (3)12 \]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x,y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \mod(x,y)) \\
\quad \quad \text{return } (d, b, a - \floor{x/y} \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\text{ext-gcd}(35, 12) \\
\quad \text{ext-gcd}(12, 11) \\
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\quad \quad \text{ext-gcd}(1, 0) \\
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\quad \quad \text{return } (1, 1, -1) ;; 1 = (1)12 + (-1)11 \\
\quad \text{return } (1, -1, 3) ;; 1 = (-1)35 + (3)12
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

\[
\begin{align*}
\text{if } y &= 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
(d, a, b) &:= \text{ext-gcd}(y, \text{mod}(x, y)) \\
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\end{align*}
\]
Extended GCD Algorithm.

```python
def ext_gcd(x, y):
    if y == 0:
        return (x, 1, 0)
    else:
        (d, a, b) = ext_gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)
```

**Theorem:** Returns $(d, a, b)$, where $d = \gcd(a, b)$ and

$$d = ax + by.$$
Correctness.

**Proof:** Strong Induction.\(^1\)

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)
**Base:** \(\text{ext-gcd}(x,0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)
**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).
**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

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Ind hyp: \(\text{ext-gcd}(y, \mod (x, y))\) returns \((d, a, b)\) with

\[
d = ay + b\mod (x, y)
\]

---

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**Proof:** Strong Induction.\(^1\)

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**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \mod(x, y))\) returns \((d, a, b)\) with
\[ d = ay + b(\mod(x, y)) \]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod(x, y))\) so

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
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Ind hyp: \(\text{ext-gcd}(y, \mod(x, y))\) returns \((d, a, b)\) with
\[
d = ay + b(\mod(x, y))
\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod(x, y))\) so

\[
d = ay + b\cdot(\mod(x, y))
\]

---

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.¹

**Base:** `ext-gcd(x, 0)` returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: `ext-gcd(y, \mod(x, y))` returns \((d, a, b)\) with
\[
d = ay + b(\mod(x, y))
\]

`ext-gcd(x, y)` calls `ext-gcd(y, \mod(x, y))` so
\[
d = ay + b(\mod(x, y)) = ay + b(x - \lfloor \frac{x}{y} \rfloor y)
\]

---

¹Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.  
**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)  
Ind hyp: \(\text{ext-gcd}(y, \mod{(x, y)})\) returns \((d, a, b)\) with  
\[d = ay + b(\mod{(x, y)})\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod{(x, y)})\) so  
\[
d = ay + b(\mod{(x, y)})
\]
\[
= ay + b(\mod{(x - \left\lfloor \frac{x}{y} \right\rfloor y)}
\]
\[
= bx + (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b)y
\]

\(^1\)Assume \(d\) is \(\text{gcd}(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)

Base: \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

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\[d = ay + b(\mod(x, y))\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod(x, y))\) so

\[
    d = ay + b \cdot (\mod(x, y)) \\
    = ay + b \cdot (x - \left\lfloor \frac{x}{y} \right\rfloor y) \\
    = bx + (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b)y
\]

And \(\text{ext-gcd}\) returns \((d, b, (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b))\) so theorem holds!

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Ind hyp: \(\text{ext-gcd}(y, \text{mod}(x, y))\) returns \((d, a, b)\) with 
\[d = ay + b(\text{mod}(x, y))\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \text{mod}(x, y))\) so 
\[
\begin{align*}
d &= ay + b \cdot (\text{mod}(x, y)) \\
    &= ay + b \cdot (x - \left\lfloor \frac{x}{y} \right\rfloor y) \\
    &= bx + (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b)y
\end{align*}
\]

And \(\text{ext-gcd}\) returns \((d, b, (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b))\) so theorem holds! \(\square\)

---

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
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        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y) * b)
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
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        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y) * b)

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)$
ext-gcd(x, y)
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    else
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Recursively: \( d = ay + b(x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y) \implies d = b(\frac{x}{y} \cdot y - a - \left\lfloor \frac{x}{y} \right\rfloor b) \)
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
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        (d, a, b) := ext-gcd(y, mod(x,y))
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Recursively: \( d = ay + b(x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y) \implies d = bx - (a - \left\lfloor \frac{x}{y} \right\rfloor b)y \)

Returns \((d, b, (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b))\).
Example: gcd(7,60) = 1.
Example: \( \gcd(7, 60) = 1. \)
\( \text{egcd}(7, 60). \)
Hand Calculation Method for Inverses.

Example: $\text{gcd}(7, 60) = 1$.

$\text{egcd}(7, 60)$.

\[
7(0) + 60(1) = 60
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Example: \( \gcd(7, 60) = 1 \).
\[ \text{egcd}(7,60). \]

\[
\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7
\end{align*}
\]
Hand Calculation Method for Inverses.

Example: \( \text{gcd}(7, 60) = 1 \).
\[
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\[
\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7 \\
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\end{align*}
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7(1) + 60(0) &= 7 \\
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\end{align*}
\]

Confirm:

\[
-119 + 120 = 1
\]

Note: an "iterative" version of the egcd algorithm.
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7(0) + 60(1) & = 60 \\
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7(-17) + 60(2) & = 1
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Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!
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Very different from elementary school: try 1, try 2, try 3...
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$2^{n/2}$
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\[2^{n/2}\]
Inverse of 500,000,357 modulo 1,000,000,000,000?
Wrap-up

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Very different from elementary school: try 1, try 2, try 3...
$2^{n/2}$
Inverse of 500,000,357 modulo 1,000,000,000,000?
$\leq 80$ divisions.

Internet Security: 512 digits. 512 divisions vs. $(1000000000000000000000000000000000000000000)$^5 divisions.
Wrap-up

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Wrap-up

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  $\leq 80$ divisions.
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Internet Security.
Public Key Cryptography: 512 digits.
  512 divisions vs.
  $(100000000000000000000000000000000000000000000000000000000000000)^5$ divisions.

Internet Security: Soon.
Bijections

Bijection is **one to one** and **onto**. Bijection:

\[
\begin{align*}
\text{Domain:} & \quad A, \quad \text{Co-Domain:} \quad B \\
\text{Versus Range.} \\
\text{E.g.} & \quad \sin(x) \\
A = B = \text{reals}. \\
\text{Range is} & \quad [-1, 1]. \\
\text{Onto:} & \quad [-1, 1]. \\
\text{Not one-to-one.} & \quad \sin(\pi) = \sin(0) = 0. \\
\text{Range Definition always is onto.} \\
\text{Consider} & \quad f(x) = ax \mod m. \\
\end{align*}
\]
Bijections

Bijection is one to one and onto.

Bijection:

\[ f : A \rightarrow B. \]
**Bijections**

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**Bijection:**

\[ f : A \rightarrow B. \]

**Domain:** \(A\), **Co-Domain:** \(B\).
Bijections

**Bijection** is **one to one and onto**.

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Domain: \( A \), Co-Domain: \( B \).

Versus Range.

E.g. \( \sin(x) \).

\( A = B = \) reals.

Range is \( \left[ -1, 1 \right] \).

Onto: \( \left[ -1, 1 \right] \).

Not one-to-one.

\( \sin(\pi) = \sin(0) = 0. \)

Range Definition always is onto.

Consider \( f(x) = ax \mod m \).

\( f : \{0, ..., m-1\} \rightarrow \{0, ..., m-1\} \).

Domain/Co-Domain: \( \{0, ..., m-1\} \).

When is it a bijection?

\( \text{When } \gcd(a, m) = 1. \)

Not Example:

\( a = 2, m = 4, \) \( f(0) = f(2) = 0 \mod 4. \)
Bijections

**Bijection** is *one to one* and *onto*.

**Bijection:**

\[ f : A \rightarrow B. \]

**Domain:** \( A \), **Co-Domain:** \( B \).

**Versus Range.**

E.g. \( \sin (x) \).
Bijections

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E.g. \( \sin(x) \).

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Domain: \( A \), Co-Domain: \( B \).

Versus Range.

E.g. \( \text{sin} \ (x) \).

\( A = B = \text{reals}. \)

Range is \([−1, 1]\).
Bijection is one to one and onto.

Bijection:

\[ f : A \rightarrow B. \]

Domain: \( A \), Co-Domain: \( B \).

Versus Range.

E.g. \( \sin(x) \).

\( A = B = \) reals.

Range is \([-1, 1]\). Onto: \([-1, 1]\).
Bijections

**Bijection** is **one to one** and **onto**.

Bijection:  
\[ f : A \rightarrow B. \]

Domain: \( A \), Co-Domain: \( B \).

Versus Range.

E.g. \( \sin(x) \).

\( A = B = \text{reals}. \)

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Domain: \( A \), Co-Domain: \( B \).

Versus Range.

E.g. \( \sin (x) \).

\( A = B = \) reals.

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Not one-to-one. \( \sin(\pi) = \sin(0) = 0 \).
**Bijections**

**Bijection** is **one to one** and **onto**.

Bijection:

\[ f : A \rightarrow B. \]

Domain: \( A \), Co-Domain: \( B \).

Versus Range.

E.g. \( \sin(x) \).

\( A = B = \text{reals}. \)

Range is \([-1, 1]\). Onto: \([-1, 1]\).

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Range Definition always is onto.
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Consider \( f(x) = ax \mod m. \)

\[ f : \{0, \ldots, m-1\} \rightarrow \{0, \ldots, m-1\}. \]
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When is it a bijection?
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Domain/Co-Domain: \( \{0, \ldots, m - 1\} \).

When is it a bijection?

When \( \gcd(a, m) \) is ....
Bijectons

**Bijecton is one to one and onto.**

Bijecton:

\[ f : A \to B. \]

Domain: \( A \), Co-Domain: \( B \).

Versus Range.

E.g. \( \sin(x) \).

\( A = B = \text{reals}. \)

Range is \([-1, 1]\). Onto: \([-1, 1]\).

Not one-to-one. \( \sin(\pi) = \sin(0) = 0. \)

Range Definition always is onto.

Consider \( f(x) = ax \mod m \).

\( f : \{0, \ldots, m - 1\} \to \{0, \ldots, m - 1\}. \)

Domain/Co-Domain: \{0, \ldots, m – 1\}.

When is it a bijection?

When \( \gcd(a, m) \) is ....?
Bijections

**Bijection** is one to one and onto.

Bijection:
\[ f : A \rightarrow B. \]

Domain: \( A \), Co-Domain: \( B \).

Versus Range.

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When is it a bijection?

When \( \gcd(a, m) \) is ....? ... 1.

Not Example: \( a = 2, m = 4, \)
**Bijections**

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Bijection:

\[ f : A \to B. \]

Domain: \( A \), Co-Domain: \( B \).

Versus Range.

E.g. \( \sin(x) \).

\( A = B = \text{reals} \).

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Not one-to-one. \( \sin(\pi) = \sin(0) = 0 \).

Range Definition always is onto.

Consider \( f(x) = ax \mod m \).

\( f : \{0, \ldots, m-1\} \to \{0, \ldots, m-1\} \).

Domain/Co-Domain: \( \{0, \ldots, m-1\} \).

When is it a bijection?

When \( \gcd(a, m) \) is ....? ... 1.

Not Example: \( a = 2, m = 4, f(0) = f(2) = 0 \mod 4 \).
Lots of Mods

\[ x = 5 \pmod{7} \] and \[ x = 3 \pmod{5}. \]
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?

Let’s try 5.

Oh, only 33 is \( 3 \pmod{5} \).

Hmmm... only one solution.

A bit slow for large values.
$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let's try 5. Not 3 \pmod{5}!
$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let’s try 5. Not $3 \pmod{5}$!
Let’s try 3.
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?

Let’s try 5. Not 3 \( \pmod{5} \)!

Let’s try 3. Not 5 \( \pmod{7} \)!
$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let’s try 5. Not $3 \pmod{5}$!
Let’s try 3. Not $5 \pmod{7}$!
$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let’s try 5. Not $3 \pmod{5}$!
Let’s try 3. Not $5 \pmod{7}$!

If $x = 5 \pmod{7}$
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?

Let’s try 5. Not \( 3 \pmod{5}! \)
Let’s try 3. Not \( 5 \pmod{7}! \)

If \( x = 5 \pmod{7} \)
  then \( x \) is in \( \{5, 12, 19, 26, 33\} \).
$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let’s try 5. Not $3 \pmod{5}$!
Let’s try 3. Not $5 \pmod{7}$!

If $x = 5 \pmod{7}$
then $x$ is in $\{5, 12, 19, 26, 33\}$.
$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let’s try 5. Not 3 (mod 5)!

Let’s try 3. Not 5 (mod 7)!

If $x = 5 \pmod{7}$

then $x$ is in $\{5, 12, 19, 26, 33\}$.

Oh, only 33 is 3 (mod 5).
x = 5 (mod 7) and x = 3 (mod 5).

What is x (mod 35)?

Let’s try 5. Not 3 (mod 5)!
Let’s try 3. Not 5 (mod 7)!

If x = 5 (mod 7)
then x is in \{5, 12, 19, 26, 33\}.

Oh, only 33 is 3 (mod 5).
Hmmm...
Lots of Mods

$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let’s try 5. Not $3 \pmod{5}$!
Let’s try 3. Not $5 \pmod{7}$!

If $x = 5 \pmod{7}$

then $x$ is in $\{5, 12, 19, 26, 33\}$.

Oh, only 33 is $3 \pmod{5}$.

Hmmm... only one solution.
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?

Let’s try 5. Not 3 \( \pmod{5} \)!
Let’s try 3. Not 5 \( \pmod{7} \)!

If \( x = 5 \pmod{7} \)
   then \( x \) is in \( \{5, 12, 19, 26, 33\} \).

Oh, only 33 is 3 \( \pmod{5} \).
Hmmm... only one solution.

A bit slow for large values.
Simple Chinese Remainder Theorem.

My love is won.

Zero and One.

Nothing and nothing done.

Find \( x = a \mod m \) and \( x = b \mod n \) where \( \gcd(m, n) = 1 \).

**CRT Thm:** There is a unique solution \( x \mod mn \).

**Proof (solution exists):**

Consider \( u = n(n - 1) \mod m \).

\[ u = 0 \mod n \]
\[ u = 1 \mod m \]

Consider \( v = m(m - 1) \mod n \).

\[ v = 1 \mod n \]
\[ v = 0 \mod m \]

Let \( x = au + bv \).

\[ x = a \mod m \] since \( bv = 0 \mod m \) and \( au = a \mod m \)

\[ x = b \mod n \] since \( au = 0 \mod n \) and \( bv = b \mod n \)

This shows there is a solution.
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My love is won. Zero and One.
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My love is won. Zero and One. Nothing and nothing done.
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \)
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where \( \gcd(m, n) = 1 \).
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Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where \( \gcd(m, n) = 1 \).

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**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (solution exists):**
Consider $u = n(n^{-1} \pmod{m})$. 
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Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where \( \gcd(m, n) = 1 \).

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**Proof (solution exists):**
Consider \( u = n(n^{-1} \pmod{m}) \).
\[ u = 0 \pmod{n} \]
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- $u = 0 \pmod{n}$
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**CRT Thm:** There is a unique solution \( x \pmod{mn} \).

**Proof (solution exists):**
Consider \( u = n(n^{-1} \pmod{m}) \).
\[
\begin{align*}
u &= 0 \pmod{n} & u &= 1 \pmod{m} \end{align*}
\]
Consider \( v = m(m^{-1} \pmod{n}) \).
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where \( \gcd(m, n) = 1 \).

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**Proof (solution exists):**

Consider \( u = n(n^{-1} \pmod{m}) \).

\[
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u &= 0 \pmod{n} & u &= 1 \pmod{m} \\
\end{align*}
\]

Consider \( v = m(m^{-1} \pmod{n}) \).

\[
\begin{align*}
v &= 1 \pmod{n} \\
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- \( u = 0 \pmod{n} \)
- \( u = 1 \pmod{m} \)

Consider \( v = m(m^{-1} \pmod{n}) \).
- \( v = 1 \pmod{n} \)
- \( v = 0 \pmod{m} \)
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\end{align*}
\]
Consider \( v = m(m^{-1} \pmod{n}) \).
\[
\begin{align*}
  v &= 1 \pmod{n} & v &= 0 \pmod{m} \\
\end{align*}
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Let \( x = au + bv \).
Simple Chinese Remainder Theorem.

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Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where gcd\((m, n)\)=1.

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\[
\begin{align*}
    v &= 1 \pmod{n} & v &= 0 \pmod{m} \\
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Let \( x = au + bv \).
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\begin{align*}
    x &= a \pmod{m} \\
\end{align*}
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My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where $\gcd(m, n)=1$.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (solution exists):**

Consider $u = n(n^{-1} \pmod{m})$.

\[
u = 0 \pmod{n} \quad u = 1 \pmod{m}\]

Consider $v = m(m^{-1} \pmod{n})$.

\[
v = 1 \pmod{n} \quad v = 0 \pmod{m}\]

Let $x = au + bv$.

\[
x = a \pmod{m} \text{ since } bv = 0 \pmod{m} \text{ and } au = a \pmod{m}\]
My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where $\gcd(m,n)=1$.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (solution exists):**

Consider $u = n(n^{-1} \pmod{m})$.

- $u = 0 \pmod{n}$
- $u = 1 \pmod{m}$

Consider $v = m(m^{-1} \pmod{n})$.

- $v = 1 \pmod{n}$
- $v = 0 \pmod{m}$

Let $x = au + bv$.

- $x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where \( \gcd(m, n) = 1 \).

**CRT Thm:** There is a unique solution \( x \pmod{mn} \).

**Proof (solution exists):**

Consider \( u = n(n^{-1}) \pmod{m} \).
- \( u = 0 \pmod{n} \)
- \( u = 1 \pmod{m} \)

Consider \( v = m(m^{-1}) \pmod{n} \).
- \( v = 1 \pmod{n} \)
- \( v = 0 \pmod{m} \)

Let \( x = au + bv \).
- \( x = a \pmod{m} \) since \( bv = 0 \pmod{m} \) and \( au = a \pmod{m} \)
- \( x = b \pmod{n} \)
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where $\gcd(m,n)=1$.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (solution exists):**

Consider $u = n(n^{-1} \pmod{m})$.
\[
u = 0 \pmod{n} \quad u = 1 \pmod{m}\]

Consider $v = m(m^{-1} \pmod{n})$.
\[
v = 1 \pmod{n} \quad v = 0 \pmod{m}\]

Let $x = au + bv$.
\[
x = a \pmod{m} \text{ since } bv = 0 \pmod{m} \text{ and } au = a \pmod{m}
\]
\[
x = b \pmod{n} \text{ since } au = 0 \pmod{n} \text{ and } bv = b \pmod{n}\]
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where \( \gcd(m, n) = 1 \).

**CRT Thm:** There is a unique solution \( x \pmod{mn} \).

**Proof (solution exists):**
Consider \( u = n(n^{-1} \pmod{m}) \).
\[
u = 0 \pmod{n} \quad u = 1 \pmod{m}
\]
Consider \( v = m(m^{-1} \pmod{n}) \).
\[
v = 1 \pmod{n} \quad v = 0 \pmod{m}
\]
Let \( x = au + bv \).
\[
x = a \pmod{m} \quad \text{since } bv = 0 \pmod{m} \text{ and } au = a \pmod{m}
\]
\[
x = b \pmod{n} \quad \text{since } au = 0 \pmod{n} \text{ and } bv = b \pmod{n}
\]
This shows there is a solution.
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$. 

---

**Proof (uniqueness):** If not, two solutions, $x$ and $y$. 

$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.

$\implies (x - y)$ is multiple of $m$ and $n$.

$\gcd(m, n) = 1 \implies$ no common primes in factorization $m$ and $n$.

$mn | (x - y) \implies x - y \geq mn$.

$\implies x, y \not\in \{0, \ldots, mn - 1\}$.

Thus, only one solution modulo $mn$. 

Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \ (\text{mod } mn)$. 

Proof (uniqueness): If not, two solutions, $x$ and $y$.

$$(x - y) \equiv 0 \ (\text{mod } m)$$

and

$$(x - y) \equiv 0 \ (\text{mod } n).$$

$\Rightarrow (x - y)$ is a multiple of $m$ and $n$.

$\gcd(m, n) = 1 \Rightarrow$ no common primes in the factorization of $m$ and $n$.

$\Rightarrow mn \mid (x - y) \Rightarrow x - y \geq mn \Rightarrow x, y \notin \{0, \ldots, mn - 1\}$.

Thus, only one solution modulo $mn$. 


Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.

$\Rightarrow (x - y)$ is a multiple of $m$ and $n$.

$\gcd(m, n) = 1 \Rightarrow$ no common primes in the factorization of $m$ and $n$.

$\Rightarrow mn | (x - y) \Rightarrow x - y \geq mn$.

$\Rightarrow x, y \notin \{0, \ldots, mn - 1\}$.

Thus, only one solution modulo $mn$. 
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$$(x - y) \equiv 0 \pmod{m} \text{ and } (x - y) \equiv 0 \pmod{n}.$$
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.

$\implies (x - y)$ is multiple of $m$ and $n$.
**Simple Chinese Remainder Theorem.**

**CRT Thm:** There is a unique solution \( x \) (mod \( mn \)).

**Proof (uniqueness):**
If not, two solutions, \( x \) and \( y \).

\[
(x - y) \equiv 0 \pmod{m} \quad \text{and} \quad (x - y) \equiv 0 \pmod{n}.
\]

\[
\Rightarrow (x - y) \text{ is multiple of } m \text{ and } n
\]

\[
gcd(m, n) = 1 \quad \Rightarrow \text{ no common primes in factorization } m \text{ and } n
\]
**Simple Chinese Remainder Theorem.**

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$$(x - y) \equiv 0 \pmod{m} \text{ and } (x - y) \equiv 0 \pmod{n}.$$  
$\implies (x - y)$ is multiple of $m$ and $n$.

$\gcd(m, n) = 1 \implies$ no common primes in factorization $m$ and $n$  
$\implies mn|(x - y)$

Thus, only one solution modulo $mn$. 

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Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$$(x - y) \equiv 0 \pmod{m} \text{ and } (x - y) \equiv 0 \pmod{n}.$$  
\[\Rightarrow (x - y) \text{ is multiple of } m \text{ and } n\]

$\gcd(m, n) = 1 \Rightarrow \text{no common primes in factorization } m \text{ and } n$

\[\Rightarrow mn | (x - y)\]

\[\Rightarrow x - y \geq mn\]
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.

$\implies (x - y)$ is multiple of $m$ and $n$

$\gcd(m, n) = 1 \implies$ no common primes in factorization $m$ and $n$

$\implies mn|(x - y)$

$\implies x - y \geq mn \implies x, y \not\in \{0, \ldots, mn - 1\}$. 
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$.

$\implies (x - y)$ is multiple of $m$ and $n$

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Thus, only one solution modulo \( mn \).
Poll.

My love is won,
Zero and one.
Nothing and nothing done.
Poll.

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What is the rhyme saying?
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What is the rhyme saying?

(A) Multiplying by 1, gives back number. (Does nothing.)
(B) Adding 0 gives back number. (Does nothing.)
(C) Rao has gone mad.
(D) Multiplying by 0, gives 0.
(E) Adding one does, not too much.
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All are (maybe) correct.
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(C) Recall Polonius:
   “Though this be madness, yet there is method in ’t.”
CRT: isomorphism.

For \( m, n \), \( \gcd(m, n) = 1 \).
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\begin{align*}
x \mod mn & \leftrightarrow x = a \mod m \text{ and } x = b \mod n \\
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Mapping is “isomorphic”: corresponding addition (and multiplication) operations consistent with mapping.
Fermat’s Theorem: Reducing Exponents.

**Fermat’s Little Theorem**: For prime \( p \), and \( a \not\equiv 0 \pmod{p} \),
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Poll

Which was used in Fermat’s theorem proof?

(A) The mapping \( f(x) = ax \mod p \) is a bijection.
(B) Multiplying a number by 1, gives the number.
(C) All nonzero numbers mod \( p \), have an inverse.
(D) Multiplying a number by 0 gives 0.
(E) Multiplying elements of sets \( A \) and \( B \) together is the same if \( A = B \).
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(A), (C), and (E)
Fermat and Exponent reducing.

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Correct: $2^{101} = 2^{6 \times 16 + 5} = 2^5 = 32 = 4 \pmod{7}$. 
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For a prime modulus, we can reduce exponents modulo $p - 1$!
Lecture in a minute.

Euclid's Alg:
\[
\text{gcd}(x, y) = \text{gcd}(y, x \mod y)
\]

Fast cuz value drops by a factor of two every two recursive calls.

Extended Euclid: Find \(a, b\) where \(ax + by = \text{gcd}(x, y)\).

Idea: compute \(a, b\) recursively (euclid), or iteratively.

Inverse:
\[
ax + by = ax = \text{gcd}(x, y) \mod y.
\]

If \(\gcd(x, y) = 1\), we have \(ax = 1 \mod y\) → \(a = x - 1 \mod y\).

Chinese Remainder Theorem:
\[
\text{If gcd}(n, m) = 1, x = a \mod n, x = b \mod m \text{ unique sol.}
\]

Proof: Find \(u = 1 \mod n, u = 0 \mod m\), and \(v = 0 \mod n, v = 1 \mod m\).

Then:
\[
x = au + bv = a \mod n \ldots u = m(m - 1) \mod n \text{ works!}
\]

Fermat: Prime \(p\), \(a^p - 1 = 1 \mod p\).

Proof Idea:
\(f(x) = a(x) \mod p\): bijection on \(S = \{1, \ldots, p - 1\}\).

Product of elts == for range/domain:
\(a^{p - 1}\) factor in range.
Lecture in a minute.

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If \( \text{gcd}(x, y) = 1 \), we have \( ax = 1 \mod y \rightarrow a = x \mod y \).

Chinese Remainder Theorem: If \( \text{gcd}(n, m) = 1 \), \( x = a \mod n \), \( x = b \mod m \) unique sol.

Proof: Find \( u_1 = 1 \mod n \), \( u_2 = 0 \mod m \), and \( v_1 = 0 \mod n \), \( v_2 = 1 \mod m \).

Then:
\[
\begin{align*}
x &= au + bv \\
u &= m (m - 1) \mod n \\
v &= 0 \mod m \\
&
\end{align*}
\]

Works!

Fermat: Prime \( p \), \( a^p - 1 = 1 \mod p \).

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Lecture in a minute.

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Lecture in a minute.

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Lecture in a minute.

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Inverse: $ax + by = ax = gcd(x, y) \mod y$.

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Lecture in a minute.

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      Then: $x = au + bv = a \mod n$...
         $u = m(m^{-1} \mod n)) \mod n$ works!
Lecture in a minute.

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Fermat: Prime \( p \), \( a^{p-1} = 1 \mod p \).
Lecture in a minute.

Euclid’s Alg: \( \gcd(x, y) = \gcd(y, x \mod y) \)
Fast cuz value drops by a factor of two every two recursive calls.

Extended Euclid: Find \( a, b \) where \( ax + by = \gcd(x, y) \).
Idea: compute \( a, b \) recursively (euclid), or iteratively.
Inverse: \( ax + by = ax = \gcd(x, y) \mod y \).
If \( \gcd(x, y) = 1 \), we have \( ax = 1 \mod y \)
\( \rightarrow a = x^{-1} \mod y \).

Chinese Remainder Theorem:
If \( \gcd(n, m) = 1 \), \( x = a \mod n, x = b \mod m \) unique sol.
Proof: Find \( u = 1 \mod n, u = 0 \mod m \),
and \( v = 0 \mod n, v = 1 \mod m \).
Then: \( x = au + bv = a \mod n \)...
\( u = m(m^{-1} \mod n) \mod n \) works!

Fermat: Prime \( p \), \( a^{p-1} = 1 \mod p \).
Proof Idea: \( f(x) = a(x) \mod p \): bijection on \( S = \{1, \ldots, p - 1\} \).
Lecture in a minute.

Euclid’s Alg: $gcd(x, y) = gcd(y, x \mod y)$

Fast cuz value drops by a factor of two every two recursive calls.

Extended Euclid: Find $a, b$ where $ax + by = gcd(x, y)$.

Idea: compute $a, b$ recursively (euclid), or iteratively.

Inverse: $ax + by = ax = gcd(x, y) \mod y$.

If $gcd(x, y) = 1$, we have $ax = 1 \mod y$

$\rightarrow a = x^{-1} \mod y$.

Chinese Remainder Theorem:

If $gcd(n, m) = 1$, $x = a \pmod{n}, x = b \pmod{m}$ unique sol.

Proof: Find $u = 1 \pmod{n}, u = 0 \pmod{m}$,

and $v = 0 \pmod{n}, v = 1 \pmod{m}$.

Then: $x = au + bv = a \pmod{n}$

$u = m(m^{-1} \pmod{n})) \pmod{n}$ works!

Fermat: Prime $p$, $a^{p-1} = 1 \pmod{p}$.

Proof Idea: $f(x) = a(x) \pmod{p}$: bijection on $S = \{1, \ldots, p - 1\}$.

Product of elts $\equiv$ for range/domain: $a^{p-1}$ factor in range.