Today

Homework/No-Homework option.
Today

Homework/No-Homework option. Deadline is set for after Midterm.
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Finish Euclid.
Today

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Finish Euclid.

Bijection/CRT/Isomorphism.
Today

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Finish Euclid.

Bijection/CRT/Isomorphism.

Fermat’s Little Theorem.
Today

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Finish Euclid.

Bijection/CRT/Isomorphism.

Fermat’s Little Theorem.
Quick review

Review runtime proof.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$. 

---

Runtime Proof.

---

**Fact:** First arg decreases by at least factor of two in two recursive calls. After $2 \log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number. One more recursive call to finish. $O(n)$ divisions.
Runtime Proof.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.
(define (euclid x y)
  (if (= y 0)
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      (euclid y (mod x y))))

**Theorem:** \( (\text{euclid } x \ y) \) uses \( O(n) \) ”divisions” where \( n = b(x) \).

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After \( 2\log_2 x = O(n) \) recursive calls, argument \( x \) is 1 bit number.
Runtime Proof.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Theorem: (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

Proof:

Fact:
First arg decreases by at least factor of two in two recursive calls.

After $2 \log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
One more recursive call to finish.
(define (euclid x y)
 (if (= y 0)
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     (euclid y (mod x y))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.
After $2\log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

**Theorem:** (euclid x y) uses $O(n)$ ”divisions” where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After $2 \log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
$O(n)$ divisions.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
  \[ \implies \text{true in one recursive call;} \]
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
\[ \implies \text{true in one recursive call; } \]
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
\[ \implies \text{true in one recursive call;} \]

Case 2: Will show \( y \geq x/2 \) \( \implies \) \( \text{mod}(x, y) \leq x/2. \)
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
  \( \implies \) true in one recursive call;

Case 2: Will show “\( y \geq \frac{x}{2} \)” \( \implies \) “\( \text{mod}(x, y) \leq x/2 \).”
  \( \text{mod} (x, y) \) is second argument in next recursive call,
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
\[ \Rightarrow \text{true in one recursive call}; \]

Case 2: Will show “\( y \geq x/2 \) \[ \Rightarrow \text{“mod}(x, y) \leq x/2.” \]”

\( \text{mod} (x, y) \) is second argument in next recursive call, and becomes the first argument in the next one.
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
  \( \implies \) true in one recursive call;

Case 2: Will show “\( y \geq \frac{x}{2} \) \( \implies \) “\( \text{mod}(x, y) \leq \frac{x}{2} \)”

  \( \text{mod} (x, y) \) is second argument in next recursive call,
  and becomes the first argument in the next one.

When \( y \geq \frac{x}{2} \), then
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

**Proof of Fact:** Recall that first argument decreases every call.

**Case 1:** \( y < \frac{x}{2} \), first argument is \( y \)

\[ \Rightarrow \text{true in one recursive call;} \]

**Case 2:** Will show \( y \geq \frac{x}{2} \) \( \Rightarrow \) \( \text{mod}(x, y) \leq \frac{x}{2}. \)

\( \text{mod} (x, y) \) is second argument in next recursive call, and becomes the first argument in the next one.

**When \( y \geq \frac{x}{2} \), then**

\[ \left\lfloor \frac{x}{y} \right\rfloor = 1, \]
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
  \( \implies \) true in one recursive call;

Case 2: Will show “\( y \geq \frac{x}{2} \)” \( \implies \) “\( \text{mod}(x, y) \leq \frac{x}{2} \).”

\( \text{mod} (x, y) \) is second argument in next recursive call, and becomes the first argument in the next one.

When \( y \geq \frac{x}{2} \), then
\[ \left\lfloor \frac{x}{y} \right\rfloor = 1, \]
\[ \text{mod} (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = \]
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < x/2 \), first argument is \( y \)
  \( \implies \) true in one recursive call;

Case 2: Will show \( y \geq x/2 \) \( \implies \) \( \text{mod}(x, y) \leq x/2. \)

\( \text{mod} (x, y) \) is second argument in next recursive call,
  and becomes the first argument in the next one.

When \( y \geq x/2 \), then
\[
\left\lfloor \frac{x}{y} \right\rfloor = 1,
\]
\[
\text{mod} (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - x/2
\]
Runtime Proof (continued.)

```scheme
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))
```

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
\[ \implies \text{true in one recursive call; } \]

Case 2: Will show \( y \geq \frac{x}{2} \) \( \implies \) \( \mod(x, y) \leq x/2. \)

\( \mod(x, y) \) is second argument in next recursive call, and becomes the first argument in the next one.

When \( y \geq \frac{x}{2} \), then

\[ \lfloor \frac{x}{y} \rfloor = 1, \]

\[ \mod(x, y) = x - y \lfloor \frac{x}{y} \rfloor = x - y \leq x - \frac{x}{2} = \frac{x}{2}, \]
Runtime Proof (continued.)

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Fact:
First arg decreases by at least factor of two in two recursive calls.

Proof of Fact: Recall that first argument decreases every call.

Case 1: \( y < \frac{x}{2} \), first argument is \( y \)
\[ \implies \text{true in one recursive call} \]

Case 2: Will show \( \frac{y}{\frac{x}{2}} \) \( \implies \) \( \text{mod}(x, y) \leq \frac{x}{2} \).
\( \text{mod} (x, y) \) is second argument in next recursive call, and becomes the first argument in the next one.

When \( y \geq \frac{x}{2} \), then
\[ \left\lfloor \frac{x}{y} \right\rfloor = 1, \]
\[ \text{mod} (x, y) = x - y \left\lfloor \frac{x}{y} \right\rfloor = x - y \leq x - \frac{x}{2} = \frac{x}{2} \]
Mark correct answers.
Note: Mod(x,y) is the remainder of x divided by y and y < x.
Mark correct answers.
Note: Mod(x,y) is the remainder of x divided by y and y < x.

(A)  $\text{mod} (x, y) < y$
(B) If euclid(x,y) calls euclid(u,v) calls euclid (a,b) then $a \leq x/2$.
(C) euclid(x,y) calls euclid (u,v) means $u = y$.
(D) if $y > x/2$,  $\text{mod} (x, y) = (x - y)$
(E) if $y > x/2$,  $\text{mod} (x, y) < x/2$
Mark correct answers.
Note: Mod(x,y) is the remainder of x divided by y and y < x.

(A) \( \text{mod} (x, y) < y \)
(B) If euclid(x,y) calls euclid(u,v) calls euclid (a,b) then \( a \leq x/2 \).
(C) euclid(x,y) calls euclid (u,v) means u = y.
(D) if \( y > x/2 \), \( \text{mod} (x, y) = (x - y) \)
(E) if \( y > x/2 \), \( \text{mod} (x, y) < x/2 \)
Finding an inverse?

We showed how to efficiently tell if there is an inverse.
Finding an inverse?

We showed how to efficiently tell if there is an inverse.

Extend euclid to find inverse.
Euclid’s GCD algorithm.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))
Euclid’s GCD algorithm.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))

Computes the gcd(x, y) in \(O(n)\) divisions. (Remember \(n = \log_2 x\).)
Euclid’s GCD algorithm.

(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y)))))

Computes the gcd(x, y) in $O(n)$ divisions. (Remember $n = \log_2 x$.)
For $x$ and $m$, if gcd($x, m$) = 1 then $x$ has an inverse modulo $m$. 
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse.
Multiplicative Inverse.

GCD algorithm used to tell if there is a multiplicative inverse. How do we find a multiplicative inverse?
Extended GCD

**Euclid’s Extended GCD Theorem:** For any $x, y$ there are integers $a, b$ such that

$$ax + by$$
Extended GCD

Euclid’s Extended GCD Theorem: For any \( x, y \) there are integers \( a, b \) such that
\[
ax + by = d \quad \text{where } d = \gcd(x, y).
\]
Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that
\[ ax + by = d \]
where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

\[ (3) \cdot 12 + (−1) \cdot 35 = 1 \]

$a = 3$ and $b = −1$.

The multiplicative inverse of $12$ (mod $35$) is $3$.

Check: $3 \cdot (12) \equiv 36 \equiv 1 \pmod{35}$. 

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Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

$$ax + by = d$$

where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$."

What is multiplicative inverse of $x$ modulo $m$?
Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that
\[ ax + by = d \quad \text{where } d = \gcd(x, y). \]

“Make $d$ out of sum of multiples of $x$ and $y.$”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$. 
Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that
$$ax + by = d$$ where $d = \gcd(x, y)$.

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Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that
\[ ax + by = d \]
where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.
\[
ax + bm = 1 \\
ax \equiv 1 - bm \equiv 1 \pmod{m}.
\]
Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that
$$ax + by = d$$
where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$  

So $a$ multiplicative inverse of $x \pmod{m}$!!
Extended GCD

Euclid’s Extended GCD Theorem: For any \( x, y \) there are integers \( a, b \) such that
\[
ax + by = d
\]
where \( d = \gcd(x, y) \).

“Make \( d \) out of sum of multiples of \( x \) and \( y \).”

What is multiplicative inverse of \( x \) modulo \( m \)?

By extended GCD theorem, when \( \gcd(x, m) = 1 \).
\[
ax + bm = 1
\]
\[
ax \equiv 1 - bm \equiv 1 \pmod{m}.
\]

So \( a \) multiplicative inverse of \( x \) (mod \( m \))!

Example: For \( x = 12 \) and \( y = 35 \), \( \gcd(12, 35) = 1 \).
Extended GCD

**Euclid’s Extended GCD Theorem:** For any $x, y$ there are integers $a, b$ such that
\[ ax + by = d \quad \text{where } d = \text{gcd}(x, y). \]

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\text{gcd}(x, m) = 1$.
\[
ax + bm = 1
\]
\[
ax \equiv 1 - bm \equiv 1 \pmod{m}.
\]

So $a$ multiplicative inverse of $x \pmod{m}$!!

Example: For $x = 12$ and $y = 35$, $\text{gcd}(12, 35) = 1$.

\[
(3)12 + (-1)35 = 1.
\]
Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

$$ax + by = d \quad \text{where } d = \gcd(x, y).$$

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$

$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$ 

So $a$ multiplicative inverse of $x \pmod{m}$!!

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$ 

$a = 3$ and $b = -1$. 

Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that
\[ ax + by = d \]
where $d = \gcd(x, y)$.

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.
\[ ax + bm = 1 \]
\[ ax \equiv 1 - bm \equiv 1 \pmod{m}. \]

So $a$ multiplicative inverse of $x \pmod{m}$!!

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.
\[(3)12 + (-1)35 = 1.\]

$a = 3$ and $b = -1$.
The multiplicative inverse of 12 (mod 35) is 3.
Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

$$ax + by = d \quad \text{where} \quad d = \gcd(x, y).$$

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$

$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$  

So $a$ multiplicative inverse of $x \pmod{m}!!$

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$  

$a = 3$ and $b = -1$.

The multiplicative inverse of 12 (mod 35) is 3.

Check: $3(12)$
Extended GCD

Euclid’s Extended GCD Theorem: For any $x, y$ there are integers $a, b$ such that

$$ax + by = d \quad \text{where } d = \gcd(x, y).$$

“Make $d$ out of sum of multiples of $x$ and $y$.”

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$

$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$ 

So $a$ multiplicative inverse of $x$ $(\mod m)$!!

Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$ 

$a = 3$ and $b = -1$.

The multiplicative inverse of 12 $(\mod 35)$ is 3.

Check: $3(12) = 36$
Extended GCD

Euclid’s Extended GCD Theorem: For any \( x, y \) there are integers \( a, b \) such that
\[
ax + by = d \quad \text{where} \quad d = \gcd(x, y).
\]

“Make \( d \) out of sum of multiples of \( x \) and \( y \).”

What is multiplicative inverse of \( x \) modulo \( m \)?

By extended GCD theorem, when \( \gcd(x, m) = 1 \).

\[
ax + bm = 1
\]
\[
ax \equiv 1 - bm \equiv 1 \pmod{m}.
\]

So \( a \) multiplicative inverse of \( x \pmod{m} \)!!

Example: For \( x = 12 \) and \( y = 35 \), \( \gcd(12, 35) = 1 \).

\[
(3)12 + (-1)35 = 1.
\]

\( a = 3 \) and \( b = -1 \).

The multiplicative inverse of 12 \pmod{35} is 3.

Check: \( 3(12) = 36 = 1 \pmod{35} \).
Make $d$ out of multiples of $x$ and $y$..?

\[
gcd(35,12)
\]
Make $d$ out of multiples of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) \quad ;; \quad gcd(12, 35 \% 12)
\]
Make $d$ out of multiples of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35\%12) \\
gcd(11, 1) ;; gcd(11, 12\%11)
\]
Make \( d \) out of multiples of \( x \) and \( y \)?

\[
gcd(35, 12)
\]
\[
gcd(12, 11) ;; gcd(12, 35 \mod 12)
\]
\[
gcd(11, 1) ;; gcd(11, 12 \mod 11)
\]
\[
gcd(1, 0)
\]
\[
1
\]

How did gcd get 11 from 35 and 12?

\[
35 - \lfloor \frac{35}{12} \rfloor \times 12 = 35 - (2 \times 12) = 11
\]

How does gcd get 1 from 12 and 11?

\[
12 - \lfloor \frac{12}{11} \rfloor \times 11 = 12 - (1 \times 11) = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

\[
1 = 12 - (1 \times 11)
\]

Get 11 from 35 and 12 and plugin....

Simplify.

\[
a = 3 \quad \text{and} \quad b = -1.
\]
Make $d$ out of multiples of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; gcd(12, 35\%12) \\
gcd(11, 1) ;; gcd(11, 12\%11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
Make $d$ out of multiples of $x$ and $y$..?

\[
\text{gcd}(35, 12)
\]
\[
\text{gcd}(12, 11) \;; \; \text{gcd}(12, 35 \% 12)
\]
\[
\text{gcd}(11, 1) \;; \; \text{gcd}(11, 12 \% 11)
\]
\[
\text{gcd}(1, 0)
\]
\[
1
\]

How did gcd get 11 from 35 and 12?

\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]
Make $d$ out of multiples of $x$ and $y$..?

```
gcd(35, 12)
gcd(12, 11) ;; gcd(12, 35\%12)
gcd(11, 1) ;; gcd(11, 12\%11)
gcd(1, 0)
    1
```

How did gcd get 11 from 35 and 12?

$35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

$1 = 12 - (1)(11)
= (3)12 + (−1)35$
Make \( d \) out of multiples of \( x \) and \( y \)...

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;; \gcd(12, 35 \% 12) \\
gcd(11, 1) ;; \gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]
Make \( d \) out of multiples of \( x \) and \( y \)..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; \gcd(12, 35 \% 12) \\
gcd(11, 1) ;; \gcd(11, 12 \% 11) \\
gcd(1, 0) \\
1
\]

How did \( \gcd \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( \gcd \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.
Make $d$ out of multiples of $x$ and $y$..?

\[
gcd(35, 12)
\]
\[
gcd(12, 11) ;; gcd(12, 35 \% 12)
\]
\[
gcd(11, 1) ;; gcd(11, 12 \% 11)
\]
\[
gcd(1, 0)
\]
1

How did gcd get 11 from 35 and 12?
35 – $\left\lfloor \frac{35}{12} \right\rfloor 12 = 35 – (2)12 = 11$

How does gcd get 1 from 12 and 11?
12 – $\left\lfloor \frac{12}{11} \right\rfloor 11 = 12 – (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Make $d$ out of multiples of $x$ and $y$..?

\[
gcd(35, 12) \\
gcd(12, 11) ;; \gcd(12, \ 35 \mod 12) \\
gcd(11, 1) ;; \gcd(11, \ 12 \mod 11) \\
gcd(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \lfloor \frac{35}{12} \rfloor \times 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \lfloor \frac{12}{11} \rfloor \times 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?
Get 1 from 12 and 11.
Make $d$ out of multiples of $x$ and $y$..?

```
gcd(35,12)
gcd(12, 11) ;; gcd(12, 35%12)
gcd(11, 1) ;; gcd(11, 12%11)
gcd(1,0)
1
```

How did gcd get 11 from 35 and 12?
$$35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11$$

How does gcd get 1 from 12 and 11?
$$12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1$$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$$1 = 12 - (1)11$$
Make $d$ out of multiples of $x$ and $y$..?

$\gcd(35, 12)$
$\gcd(12, 11) \;;; \gcd(12, 35 \% 12)$
$\gcd(11, 1) \;;; \gcd(11, 12 \% 11)$
$\gcd(1, 0)$
1

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor \times 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor \times 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1) (35 - (2)12)$
Get 11 from 35 and 12 and plugin....
Make $d$ out of multiples of $x$ and $y$..?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) ;;&\gcd(12, 35\%12) \\
gcd(11, 1) ;;&\gcd(11, 12\%11) \\
gcd(1, 0) \\
1
\end{align*}
\]

How did $gcd$ get $11$ from $35$ and $12$?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does $gcd$ get $1$ from $12$ and $11$?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns $1$.

But we want $1$ from sum of multiples of $35$ and $12$?

Get $1$ from $12$ and $11$.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (\text{-}1)35
\]

Get $11$ from $35$ and $12$ and plugin.... Simplify.
Make $d$ out of multiples of $x$ and $y$..?

\[
\text{gcd}(35, 12) \\
\text{gcd}(12, 11) \quad ;; \quad \text{gcd}(12, 35 \mod 12) \\
\text{gcd}(11, 1) \quad ;; \quad \text{gcd}(11, 12 \mod 11) \\
\text{gcd}(1, 0) \\
1
\]

How did gcd get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does gcd get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin....  Simplify.
Make \( d \) out of multiples of \( x \) and \( y \)··?

\[
\begin{align*}
gcd(35, 12) \\
gcd(12, 11) & ;; \quad gcd(12, 35 \mod 12) \\
gcd(11, 1) & ;; \quad gcd(11, 12 \mod 11) \\
gcd(1, 0) & \\
1
\end{align*}
\]

How did \( gcd \) get 11 from 35 and 12?
\[
35 - \left\lfloor \frac{35}{12} \right\rfloor 12 = 35 - (2)12 = 11
\]

How does \( gcd \) get 1 from 12 and 11?
\[
12 - \left\lfloor \frac{12}{11} \right\rfloor 11 = 12 - (1)11 = 1
\]

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
\[
1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35
\]

Get 11 from 35 and 12 and plugin… Simplify. \( a = 3 \) and \( b = -1 \).
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]
Extended GCD Algorithm.

ext-gcd(x,y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y) * b)

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) =
\begin{cases} 
  (x, 1, 0) & \text{if } y = 0 \\
  \text{return } (d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \text{ and } \text{return } (d, b, a - \text{floor}(x/y) \times b) & \text{else}
\end{cases}
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\text{ext-gcd}(35, 12)
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]
\[
\text{if } y = 0 \text{ then return } (x, 1, 0)
\]
\[
\text{else}
\]
\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y))
\]
\[
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\text{ext-gcd}(35, 12)
\]
\[
\text{ext-gcd}(12, 11)
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

\[
\text{if } y = 0 \text{ then return } (x, 1, 0)
\]

\[
\text{else}
\]

\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]

\[
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\text{ext-gcd}(35, 12)
\]

\[
\text{ext-gcd}(12, 11)
\]

\[
\text{ext-gcd}(11, 1)
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]
\[
\text{if } y = 0 \text{ then return } (x, 1, 0)
\]
\[
\text{else}
\]
\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]
\[
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\text{ext-gcd}(35, 12)
\]
\[
\text{ext-gcd}(12, 11)
\]
\[
\text{ext-gcd}(11, 1)
\]
\[
\text{ext-gcd}(1, 0)
\]
Extended GCD Algorithm.

\[
ext\text{gcd}(x, y) \\
\text{if } y = 0 \text{ then return } (x, 1, 0) \\
\text{else} \\
\quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example: \(a - \lfloor x/y \rfloor \cdot b =\)

\[
ext\text{gcd}(35, 12) \\
ext\text{gcd}(12, 11) \\
ext\text{gcd}(11, 1) \\
ext\text{gcd}(1, 0) \\
\text{return } (1,1,0) ;; 1 = (1)1 + (0) 0
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return }(x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
\quad \quad \text{return } (d, b, a - \lfloor x/y \rfloor \cdot b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).
Example: \(a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1\)

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
\quad \text{ext-gcd}(12, 11) \\
\quad \quad \text{ext-gcd}(11, 1) \\
\quad \quad \quad \text{ext-gcd}(1, 0) \\
\quad \quad \quad \quad \text{return } (1, 1, 0) \;;\; 1 = (1)1 + (0)0 \\
\quad \quad \quad \quad \text{return } (1, 0, 1) \;;\; 1 = (0)11 + (1)1
\end{align*}
\]
Extended GCD Algorithm.

```
ext-gcd(x,y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y) * b)
```

Claim: Returns $(d, a, b)$: $d = \gcd(a, b)$ and $d = ax + by$.

Example: $a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

```
ext-gcd(35,12)
ext-gcd(12, 11)
ext-gcd(11, 1)
ext-gcd(1,0)
return (1,1,0) ;; 1 = (1)1 + (0) 0
return (1,0,1) ;; 1 = (0)11 + (1)1
return (1,1,-1) ;; 1 = (1)12 + (-1)11
```
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y) \\
\quad \text{if } y = 0 \text{ then return } (x, 1, 0) \\
\quad \text{else} \\
\quad \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x,y)) \\
\quad \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \( (d, a, b) \): \( d = \gcd(a, b) \) and \( d = ax + by \).

Example: \( a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 35/12 \rfloor \cdot (-1) = 3 \)

\[
\begin{align*}
\text{ext-gcd}(35, 12) \\
\quad \text{ext-gcd}(12, 11) \\
\quad \quad \text{ext-gcd}(11, 1) \\
\quad \quad \quad \text{ext-gcd}(1, 0) \\
\quad \quad \quad \quad \text{return } (1, 1, 0) \; ; \; 1 = (1)1 + (0) 0 \\
\quad \quad \quad \text{return } (1, 0, 1) \; ; \; 1 = (0)11 + (1)1 \\
\quad \quad \text{return } (1, 1, -1) \; ; \; 1 = (1)12 + (-1)11 \\
\quad \text{return } (1, -1, 3) \; ; \; 1 = (-1)35 + (3)12
\end{align*}
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]

\[
\text{if } y = 0 \text{ then return } (x, 1, 0)
\]

\[
\text{else}
\]

\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]

\[
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]

Claim: Returns \((d, a, b)\): \(d = \gcd(a, b)\) and \(d = ax + by\).

Example:

\[
\text{ext-gcd}(35, 12)
\]

\[
\text{ext-gcd}(12, 11)
\]

\[
\text{ext-gcd}(11, 1)
\]

\[
\text{ext-gcd}(1, 0)
\]

\[
\text{return } (1, 1, 0) \quad ;; \quad 1 = (1)1 + (0)0
\]

\[
\text{return } (1, 0, 1) \quad ;; \quad 1 = (0)11 + (1)1
\]

\[
\text{return } (1, 1, -1) \quad ;; \quad 1 = (1)12 + (-1)11
\]

\[
\text{return } (1, -1, 3) \quad ;; \quad 1 = (-1)35 + (3)12
\]
Extended GCD Algorithm.

\[
\text{ext-gcd}(x, y)
\]
\[
\text{if } y = 0 \text{ then return } (x, 1, 0)
\]
\[
\text{else}
\]
\[
(d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y))
\]
\[
\text{return } (d, b, a - \text{floor}(x/y) \times b)
\]
Extended GCD Algorithm.

\[\text{ext-gcd}(x, y)\]
  \[
  \text{if } y = 0 \text{ then return } (x, 1, 0) \\
  \text{else} \\
  \quad (d, a, b) := \text{ext-gcd}(y, \text{mod}(x, y)) \\
  \quad \text{return } (d, b, a - \text{floor}(x/y) \times b)
  \]

**Theorem:** Returns \((d, a, b)\), where \(d = \gcd(a, b)\) and

\[d = ax + by.\]
Correctness.

**Proof:** Strong Induction.\(^1\)

---

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

---

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)

Base: \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

Induction Step: Returns \((d, A, B)\) with \(d = Ax + By\)
Ind hyp: \(\text{ext-gcd}(y, \text{mod}(x, y))\) returns \((d, a, b)\) with
\[d = ay + b(\text{mod}(x, y))\]

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)

Base: \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

Induction Step: Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \mod(x, y))\) returns \((d, a, b)\) with
\[
d = ay + b(\mod(x, y))
\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod(x, y))\) so

---

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.\(^1\)

**Base:** \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \mod (x, y))\) returns \((d, a, b)\) with

\[
d = ay + b \cdot (\mod (x, y))
\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \mod (x, y))\) so

\[
d = ay + b \cdot (\mod (x, y))
\]

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof**: Strong Induction.\(^1\)

**Base**: ext-gcd\((x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step**: Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: ext-gcd\((y, \text{ mod }(x, y))\) returns \((d, a, b)\) with 
\[
d = ay + b(\text{ mod }(x, y))
\]

ext-gcd\((x, y)\) calls ext-gcd\((y, \text{ mod }(x, y))\) so
\[
d = ay + b\cdot(\text{ mod }(x, y))
\]
\[
= ay + b\cdot(x - \left\lfloor \frac{x}{y} \right\rfloor y)
\]

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

**Proof:** Strong Induction.¹

**Base:** ext-gcd(x, 0) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

**Induction Step:** Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: ext-gcd(y, mod(x, y)) returns \((d, a, b)\) with

\[d = ay + b \cdot \text{mod}(x, y)\]

ext-gcd(x, y) calls ext-gcd(y, mod(x, y)) so

\[d = ay + b \cdot \text{mod}(x, y)\]

\[= ay + b \cdot (x - \left\lfloor \frac{x}{y} \right\rfloor y)\]

\[= bx + (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b)y\]

¹Assume \(d\) is gcd\((x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)

Base: \(\text{ext-gcd}(x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

Induction Step: Returns \((d, A, B)\) with \(d = Ax + By\)

Ind hyp: \(\text{ext-gcd}(y, \text{mod}(x, y))\) returns \((d, a, b)\) with 
\[d = ay + b(\text{mod}(x, y))\]

\(\text{ext-gcd}(x, y)\) calls \(\text{ext-gcd}(y, \text{mod}(x, y))\) so
\[
d = ay + b(\text{mod}(x, y))
\]
\[
= ay + b(x - \lfloor \frac{x}{y} \rfloor y)
\]
\[
= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y
\]

And \(\text{ext-gcd}\) returns \((d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))\) so theorem holds!

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
Correctness.

Proof: Strong Induction.\(^1\)
Base: ext-gcd\((x, 0)\) returns \((d = x, 1, 0)\) with \(x = (1)x + (0)y\).

Induction Step: Returns \((d, A, B)\) with \(d = Ax + By\)
Ind hyp: ext-gcd\((y, \text{ mod } (x, y))\) returns \((d, a, b)\) with \(d = ay + b( \text{ mod } (x, y))\)

ext-gcd\((x, y)\) calls ext-gcd\((y, \text{ mod } (x, y))\) so

\[
\begin{align*}
d &= ay + b \cdot ( \text{ mod } (x, y)) \\
&= ay + b \cdot (x - \left\lfloor \frac{x}{y} \right\rfloor y) \\
&= bx + (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b)y
\end{align*}
\]

And ext-gcd returns \((d, b, (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b))\) so theorem holds! \(\square\)

\(^1\)Assume \(d\) is \(gcd(x, y)\) by previous proof.
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)

\[
\text{ext-gcd}(x, y)
\]

if \( y = 0 \) then return \((x, 1, 0)\)
else

\[
(d, a, b) := \text{ext-gcd}(y, \mod(x, y))
\]
return \((d, b, a - \text{floor}(x/y) \times b)\)

Recursively: \( d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \)
ext-gcd(x, y)
  if y = 0 then return(x, 1, 0)
  else
      (d, a, b) := ext-gcd(y, mod(x,y))
      return (d, b, a - floor(x/y) * b)

Recursively: \( d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y \)
ext-gcd(x, y)
    if y = 0 then return(x, 1, 0)
    else
        (d, a, b) := ext-gcd(y, mod(x, y))
        return (d, b, a - floor(x/y) * b)

Recursively: \( d = ay + b(x - \left\lfloor \frac{x}{y} \right\rfloor \cdot y) \) \( \implies d = bx - (a - \left\lfloor \frac{x}{y} \right\rfloor b)y \)

Returns \((d, b, (a - \left\lfloor \frac{x}{y} \right\rfloor \cdot b))\).
Hand Calculation Method for Inverses.

Example: \( \text{gcd}(7, 60) = 1 \).
Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.
$\text{egcd}(7, 60)$. 

Confirm: 
$-119 + 120 = 1$ 

Note: an "iterative" version of the e-gcd algorithm.
Example: \( \gcd(7, 60) = 1 \).
\[ \text{egcd}(7, 60). \]

\[
7(0) + 60(1) = 60
\]
Example: \( \gcd(7, 60) = 1 \).
\[\text{egcd}(7, 60).\]

\[
\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7
\end{align*}
\]
Hand Calculation Method for Inverses.

Example: gcd(7, 60) = 1.

egcd(7, 60).

\[
\begin{align*}
7(0) + 60(1) & = 60 \\
7(1) + 60(0) & = 7 \\
7(-8) + 60(1) & = 4
\end{align*}
\]
Hand Calculation Method for Inverses.

Example: $\text{gcd}(7, 60) = 1$.

$\text{egcd}(7, 60)$.

\[
egin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7 \\
7(-8) + 60(1) &= 4 \\
7(9) + 60(-1) &= 3
\end{align*}
\]
Hand Calculation Method for Inverses.

Example: \( \text{gcd}(7, 60) = 1 \).
\[
\text{egcd}(7, 60).
\]

\[
\begin{align*}
7(0) + 60(1) &= 60 \\
7(1) + 60(0) &= 7 \\
7(-8) + 60(1) &= 4 \\
7(9) + 60(-1) &= 3 \\
7(-17) + 60(2) &= 1
\end{align*}
\]
Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.

$\text{egcd}(7, 60)$.

\[
7(0) + 60(1) = 60
\]
\[
7(1) + 60(0) = 7
\]
\[
7(-8) + 60(1) = 4
\]
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7(9) + 60(-1) = 3
\]
\[
7(-17) + 60(2) = 1
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Confirm:

$-119 + 120 = 1$

Note: an “iterative” version of the e-gcd algorithm.
Hand Calculation Method for Inverses.

Example: \( \text{gcd}(7, 60) = 1 \).
\( \text{egcd}(7, 60) \).

\[
\begin{align*}
7(0) + 60(1) &= 60 \\
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Fundamental Theorem of Arithmetic.

Thm: Every natural number can be written as the product of primes.
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Proof: $n$ is either prime (base cases)
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Generalization: things with a “division algorithm”.
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Fundamental Theorem of Arithmetic: Every natural number can be written as the a unique (up to reordering) product of primes.

Generalization: things with a “division algorithm”.

One example: polynomial division.
Claim: For $x, y, z \in \mathbb{Z}^+$ with $gcd(x, y) = 1$ and $x | yz$ then $x | z$. 

Idea: $x$ doesn't share common factors with $y$ so it must divide $z$.

Euclid: $1 = ax + by$.

Observe: $x | axz$ and $x | byz$ (since $x | yz$), and $x$ divides the sum.

$= \Rightarrow x | axz + byz$

And $axz + byz = z$, thus $x | z$.

Used to prove:

That if $gcd(x, m) = 1$, then $x$ has multiplicative inverse modulo $m$.

So used Extended Euclid through this lemma.

Used to prove that the prime factorization of a number is unique.

Induction: Divide by largest power of a prime. And induct, on remaining.
Claim: For $x, y, z \in \mathbb{Z}^+$ with $gcd(x, y) = 1$ and $x|yz$ then $x|z$.

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Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!
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Very different from elementary school: try 1, try 2, try 3...
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$2^{n/2}$
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Inverse of 500,000,357 modulo 1,000,000,000,000,000?
Wrap-up

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$\leq 80$ divisions.
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$(10000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000}^5$ divisions.
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$512$ divisions vs.

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Inverse of 500,000,357 modulo 1,000,000,000,000?

$\leq 80$ divisions.

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Public Key Cryptography: 512 digits.

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Internet Security: Soon.
Fundamental Theorem of Arithmetic: uniqueness

Thm: The prime factorization of \( n \) is unique up to reordering.

Proof by induction.

Base case: If \( l = 1 \), \( p_1 \cdots p_k = q_1 \).

But if \( q_1 \) is prime, only prime factor is \( q_1 \) and \( p_1 = q_1 \) and \( l = k = 1 \).

Induction step:

From Fact: \( p_1 = q_j \) for some \( j \).

\( n / p_1 = p_2 \cdots p_k \) and \( n / q_j = \prod_{i \neq j} q_i \).

These two expressions are the same up to reordering by induction.

And \( p_1 \) is matched to \( q_j \).
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Fact: If \( p \mid q_1 \cdots q_l \), then \( p = q_j \) for some \( j \).
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If \( \gcd(p, q_l) = 1 \), \( p \mid q_1 \cdots q_{l-1} \) by Claim.
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Assume not.

$n = p_1 \cdot p_2 \cdots p_k$ and $n = q_1 \cdot q_2 \cdots q_l$.

Fact: If $p | q_1 \ldots q_l$, then $p = q_j$ for some $j$.

- If $gcd(p, q_l) = 1$, $p_1 | q_1 \cdots q_{l-1}$ by Claim.
- If $gcd(p, q_l) = d$, then $d$ is a common factor.
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- If both prime, both only have 1 and themselves as factors.
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Thus, \( p = q_l = d \).
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$n/p_1 = p_2 \cdots p_k$ and $n/q_j = q_1 \cdots q_i$.

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These two expressions are the same up to reordering by induction.

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End Proof.
x = 5 (mod 7) and x = 3 (mod 5).
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?
Lots of Mods

\[ x = 5 \pmod{7} \] and \[ x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?

Let’s try 5.
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?

Let’s try 5. Not 3 \( \pmod{5} \)!
Lots of Mods

\[ x = 5 \pmod{7} \] and \[ x = 3 \pmod{5} \].

What is \( x \pmod{35} \)?

Let’s try 5. Not 3 \( \pmod{5} \)!
Let’s try 3.
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?

Let’s try 5. Not \( 3 \pmod{5} \)!
Let’s try 3. Not \( 5 \pmod{7} \)!
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?

Let's try 5. Not 3 \( \pmod{5} \)!
Let's try 3. Not 5 \( \pmod{7} \)!
Lots of Mods

$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod{5}$!

Let's try 3. Not $5 \pmod{7}$!

If $x = 5 \pmod{7}$
$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod{5}$!

Let's try 3. Not $5 \pmod{7}$!

If $x = 5 \pmod{7}$
then $x$ is in $\{5, 12, 19, 26, 33\}$.
Lots of Mods

\[ x = 5 \pmod{7} \] and \[ x = 3 \pmod{5} \].

What is \( x \pmod{35} \)?

Let’s try 5. Not 3 \( \pmod{5} \)!
Let’s try 3. Not 5 \( \pmod{7} \)!

If \( x = 5 \pmod{7} \),
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$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let’s try 5. Not $3 \pmod{5}$!
Let’s try 3. Not $5 \pmod{7}$!

If $x = 5 \pmod{7}$
  then $x$ is in $\{5, 12, 19, 26, 33\}$.

Oh, only 33 is $3 \pmod{5}$.
$x = 5 \pmod{7}$ and $x = 3 \pmod{5}$.

What is $x \pmod{35}$?

Let’s try 5. Not $3 \pmod{5}$!
Let’s try 3. Not $5 \pmod{7}$!

If $x = 5 \pmod{7}$
  then $x$ is in $\{5, 12, 19, 26, 33\}$.

Oh, only 33 is $3 \pmod{5}$.
Hmmm...
x = 5 (mod 7) and x = 3 (mod 5).

What is $x$ (mod 35)?

Let’s try 5. Not 3 (mod 5)!

Let’s try 3. Not 5 (mod 7)!

If $x = 5$ (mod 7)
   then $x$ is in \{5, 12, 19, 26, 33\}.

Oh, only 33 is 3 (mod 5).

Hmmm... only one solution.
Lots of Mods

\[ x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}. \]

What is \( x \pmod{35} \)?

Let’s try 5. Not 3 \( \pmod{5} \)!
Let’s try 3. Not 5 \( \pmod{7} \)!

If \( x = 5 \pmod{7} \)
    then \( x \) is in \( \{5, 12, 19, 26, 33\} \).

Oh, only 33 is 3 \( \pmod{5} \).
Hmmm... only one solution.

A bit slow for large values.
Simple Chinese Remainder Theorem.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where $\gcd(m, n) = 1$.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (solution exists):**
Consider $u = n(n - 1 \pmod{m})$.

- $u = 0 \pmod{n}$
- $u = 1 \pmod{m}$

Consider $v = m(m - 1 \pmod{n})$.

- $v = 1 \pmod{n}$
- $v = 0 \pmod{m}$

Let $x = au + bv$.

- $x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$

- $x = b \pmod{n}$ since $au = 0 \pmod{n}$ and $bv = b \pmod{n}$

This shows there is a solution.
Simple Chinese Remainder Theorem.

My love is won.
Simple Chinese Remainder Theorem.

My love is won. Zero and One.
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.
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My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where \( \gcd(m, n) = 1 \).
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where $\gcd(m, n) = 1$.

**CRT Thm:** There is a unique solution $x \pmod{mn}$. 
Simple Chinese Remainder Theorem.

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Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where \( \gcd(m, n)=1 \).

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**Proof (solution exists):**
Consider $u = n(n^{-1} \pmod{m})$. 

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My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where $\gcd(m, n) = 1$.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (solution exists):**
Consider $u = n(n^{-1} \pmod{m})$.
\[ u = 0 \pmod{n} \]
Simple Chinese Remainder Theorem.

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**CRT Thm:** There is a unique solution \( x \pmod{mn} \).

**Proof (solution exists):**
Consider \( u = n(n^{-1} \pmod{m}) \).

\[
\begin{align*}
u &= 0 \pmod{n} & u &= 1 \pmod{m} \\
\end{align*}
\]
Simple Chinese Remainder Theorem.

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Consider $u = n(n^{-1} \pmod{m})$.

\[
\begin{align*}
    u &= 0 \pmod{n} &
    u &= 1 \pmod{m}
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\]
Consider $v = m(m^{-1} \pmod{n})$. 

Simple Chinese Remainder Theorem.

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Consider \( v = m(m^{-1} \pmod{n}) \).

\[ v = 1 \pmod{n} \quad v = 0 \pmod{m} \]
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\end{align*}
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\[
\begin{align*}
  v &= 1 \pmod{n} \quad v = 0 \pmod{m} \\
\end{align*}
\]
Let \( x = au + bv \).
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

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Consider \( u = n(n^{-1}) \pmod{m} \).
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\begin{align*}
  v & = 1 \pmod{n} & v & = 0 \pmod{m} \\
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\end{align*}
\]

Consider \( v = m(m^{-1} \pmod{n}) \).

\[
\begin{align*}
  v & = 1 \pmod{n} & v & = 0 \pmod{m} \\
\end{align*}
\]

Let \( x = au + bv \).

\[
\begin{align*}
  x & = a \pmod{m} \quad \text{since} \quad bv = 0 \pmod{m} \quad \text{and} \quad au = a \pmod{m} \\
\end{align*}
\]
Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where $\gcd(m, n)=1$.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (solution exists):**

Consider $u = n(n^{-1} \pmod{m})$.

$u = 0 \pmod{n}$  \hspace{1cm} $u = 1 \pmod{m}$

Consider $v = m(m^{-1} \pmod{n})$.

$v = 1 \pmod{n}$  \hspace{1cm} $v = 0 \pmod{m}$

Let $x = au + bv$.

$x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
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- $v = 1 \pmod{n}$
- $v = 0 \pmod{m}$

Let $x = au + bv$.

- $x = a \pmod{m}$ since $bv = 0 \pmod{m}$ and $au = a \pmod{m}$
- $x = b \pmod{n}$
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- $v = 1 \pmod{n}$
- $v = 0 \pmod{m}$

Let $x = au + bv$.
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Find \( x = a \pmod{m} \) and \( x = b \pmod{n} \) where \( \gcd(m,n)=1 \).

**CRT Thm:** There is a unique solution \( x \pmod{mn} \).

**Proof (solution exists):**
Consider \( u = n(n^{-1} \pmod{m}) \).
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 u = 0 \pmod{n} \quad u = 1 \pmod{m}
\]
Consider \( v = m(m^{-1} \pmod{n}) \).
\[
 v = 1 \pmod{n} \quad v = 0 \pmod{m}
\]
Let \( x = au + bv \).
\[
 x = a \pmod{m} \quad \text{since } bv = 0 \pmod{m} \text{ and } au = a \pmod{m}
\]
\[
 x = b \pmod{n} \quad \text{since } au = 0 \pmod{n} \text{ and } bv = b \pmod{n}
\]
This shows there is a solution. \( \square \)
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$. 

Proof (uniqueness): If not, two solutions, $x$ and $y$. 

$(x - y) \equiv 0 \pmod{m}$ and $(x - y) \equiv 0 \pmod{n}$. 

$\Rightarrow (x - y)$ is a multiple of $m$ and $n$. 

$\gcd(m, n) = 1 \Rightarrow$ no common primes in factorization of $m$ and $n$. 

$\Rightarrow mn \mid (x - y) \Rightarrow x - y \geq mn \Rightarrow x, y \not\in \{0, \ldots, mn - 1\}$. 

Thus, only one solution modulo $mn$. 
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$. 
Simple Chinese Remainder Theorem.

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**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$$(x - y) \equiv 0 \pmod{m} \text{ and } (x - y) \equiv 0 \pmod{n}.$$
CRT Thm: There is a unique solution $x \pmod{mn}$.

Proof (uniqueness):
If not, two solutions, $x$ and $y$.

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Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$$(x - y) \equiv 0 \pmod{m} \text{ and } (x - y) \equiv 0 \pmod{n}.$$  

$$\implies (x - y) \text{ is multiple of } m \text{ and } n$$

$$\text{gcd}(m, n) = 1 \implies \text{no common primes in factorization } m \text{ and } n$$
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

\[(x - y) \equiv 0 \pmod{m} \text{ and } (x - y) \equiv 0 \pmod{n}.\]

\[\implies (x - y) \text{ is multiple of } m \text{ and } n.\]

\[\gcd(m, n) = 1 \implies \text{no common primes in factorization } m \text{ and } n\]

\[\implies mn|(x - y)\]
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$$(x - y) \equiv 0 \pmod{m} \text{ and } (x - y) \equiv 0 \pmod{n}.$$  
$\Rightarrow (x - y)$ is multiple of $m$ and $n$

$\gcd(m, n) = 1 \Rightarrow$ no common primes in factorization $m$ and $n$

$\Rightarrow mn | (x - y)$

$\Rightarrow x - y \geq mn$
**Simple Chinese Remainder Theorem.**

**CRT Thm:** There is a unique solution \(x \pmod{mn}\).

**Proof (uniqueness):**
If not, two solutions, \(x\) and \(y\).

\[(x - y) \equiv 0 \pmod{m} \quad \text{and} \quad (x - y) \equiv 0 \pmod{n}.
\]

\[\implies (x - y) \text{ is multiple of } m \text{ and } n\]

\[\gcd(m, n) = 1 \implies \text{no common primes in factorization } m \text{ and } n\]

\[\implies mn | (x - y)\]

\[\implies x - y \geq mn \implies x, y \notin \{0, \ldots, mn - 1\}.
\]
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution \( x \pmod{mn} \).

**Proof (uniqueness):**
If not, two solutions, \( x \) and \( y \).

\[(x - y) \equiv 0 \pmod{m} \quad \text{and} \quad (x - y) \equiv 0 \pmod{n}.
\]

\[\implies (x - y) \text{ is multiple of } m \text{ and } n\]

\[\gcd(m, n) = 1 \implies \text{no common primes in factorization of } m \text{ and } n\]

\[\implies mn | (x - y)\]

\[\implies x - y \geq mn \implies x, y \not\in \{0, \ldots, mn - 1\}.
\]

Thus, only one solution modulo \( mn \).
Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution \( x \) (mod \( mn \)).

**Proof (uniqueness):**

If not, two solutions, \( x \) and \( y \).

\[
(x - y) \equiv 0 \pmod{m} \quad \text{and} \quad (x - y) \equiv 0 \pmod{n}.
\]

\[
\implies (x - y) \text{ is multiple of } m \text{ and } n
\]

\[
\gcd(m, n) = 1 \implies \text{no common primes in factorization } m \text{ and } n
\]

\[
\implies mn | (x - y)
\]

\[
\implies x - y \geq mn \implies x, y \not\in \{0, \ldots, mn - 1\}.
\]

Thus, only one solution modulo \( mn \). \(\blacksquare\)
My love is won,
Zero and one.
Nothing and nothing done.
My love is won,
Zero and one.
Nothing and nothing done.

What is the rhyme saying?
My love is won,
Zero and one.
Nothing and nothing done.

What is the rhyme saying?

(A) Multiplying by 1, gives back number. (Does nothing.)
(B) Adding 0 gives back number. (Does nothing.)
(C) Rao has gone mad.
(D) Multiplying by 0, gives 0.
(E) Adding one does, not too much.
My love is won,
Zero and one.
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What is the rhyme saying?

(A) Multiplying by 1, gives back number. (Does nothing.)
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All are (maybe) correct.
My love is won,
Zero and one.
Nothing and nothing done.

What is the rhyme saying?

(A) Multiplying by 1, gives back number. (Does nothing.)
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(E) Adding one does, not too much.

All are (maybe) correct.
(E) doesn’t have to do with the rhyme.
My love is won,
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Nothing and nothing done.

What is the rhyme saying?

(A) Multiplying by 1, gives back number. (Does nothing.)
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All are (maybe) correct.
(E) doesn’t have to do with the rhyme.
(C) Recall Polonius:
My love is won,
Zero and one.
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What is the rhyme saying?

(A) Multiplying by 1, gives back number. (Does nothing.)
(B) Adding 0 gives back number. (Does nothing.)
(C) Rao has gone mad.
(D) Multiplying by 0, gives 0.
(E) Adding one does, not too much.

All are (maybe) correct.
(E) doesn’t have to do with the rhyme.
(C) Recall Polonius:
   “Though this be madness, yet there is method in ’t.”
CRT: isomorphism.

For $m, n$, $\gcd(m, n) = 1$. 
CRT: isomorphism.

For $m, n$, $\gcd(m, n) = 1$.

$x \mod mn \leftrightarrow x = a \mod m$ and $x = b \mod n$
For $m, n$, $\gcd(m, n) = 1$.

\[x \mod mn \leftrightarrow x = a \mod m \text{ and } x = b \mod n\]
\[y \mod mn \leftrightarrow y = c \mod m \text{ and } y = d \mod n\]
CRT: isomorphism.

For $m, n$, $\gcd(m, n) = 1$.

\[
\begin{align*}
x \mod mn & \leftrightarrow x = a \mod m \text{ and } x = b \mod n \\
y \mod mn & \leftrightarrow y = c \mod m \text{ and } y = d \mod n
\end{align*}
\]

Also, true that $x + y \mod mn \leftrightarrow a + c \mod m \text{ and } b + d \mod n$. 
CRT: isomorphism.

For $m, n$, $\gcd(m, n) = 1$.

$x \mod mn \leftrightarrow x = a \mod m$ and $x = b \mod n$

$y \mod mn \leftrightarrow y = c \mod m$ and $y = d \mod n$

Also, true that $x + y \mod mn \leftrightarrow a + c \mod m$ and $b + d \mod n$.

Mapping is “isomorphic”: corresponding addition (and multiplication) operations consistent with mapping.
Fermat’s Theorem: Reducing Exponents.

Fermat’s Little Theorem: For prime $p$, and $a \not\equiv 0 \pmod{p}$,
Fermat’s Theorem: Reducing Exponents.

**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,

$$a^{p-1} \equiv 1 \pmod{p}.$$
Fermat’s Theorem: Reducing Exponents.

Fermat’s Little Theorem: For prime $p$, and $a \not\equiv 0 \pmod{p}$,

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Proof:
Fermat’s Theorem: Reducing Exponents.

**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,

$$a^{p-1} \equiv 1 \pmod{p}.$$  

**Proof:** Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$. 
Fermat’s Theorem: Reducing Exponents.

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**Proof:** Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$.

All different modulo $p$ since $a$ has an inverse modulo $p$. 
Fermat’s Theorem: Reducing Exponents.

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Proof: Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$.

All different modulo $p$ since $a$ has an inverse modulo $p$.

$S$ contains representative of $\{1, \ldots, p-1\}$ modulo $p$. 

solve to get...
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**Proof**: Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$.

All different modulo $p$ since $a$ has an inverse modulo $p$. $S$ contains representative of $\{1, \ldots, p-1\}$ modulo $p$.

$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \pmod{p},$$
Fermat’s Theorem: Reducing Exponents.

**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$, 
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**Proof:** Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$.

All different modulo $p$ since $a$ has an inverse modulo $p$.

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\[ (a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \pmod{p}, \]

Since multiplication is commutative.
Fermat’s Theorem: Reducing Exponents.

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$$a^{(p-1)}(1 \cdots (p-1)) \equiv (1 \cdots (p-1)) \pmod{p}.$$
Fermat’s Theorem: Reducing Exponents.

**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,
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**Proof:** Consider $S = \{a \cdot 1, \ldots, a \cdot (p - 1)\}$.

All different modulo $p$ since $a$ has an inverse modulo $p$.

$S$ contains representative of $\{1, \ldots, p - 1\}$ modulo $p$.

\[ (a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p - 1)) \equiv 1 \cdot 2 \cdots (p - 1) \pmod{p}, \]

Since multiplication is commutative.

\[ a^{(p-1)}(1 \cdots (p - 1)) \equiv (1 \cdots (p - 1)) \pmod{p}. \]

Each of $2, \ldots (p - 1)$ has an inverse modulo $p$. 

Fermat’s Theorem: Reducing Exponents.

Fermat’s Little Theorem: For prime $p$, and $a \not\equiv 0 \pmod{p}$, $a^{p-1} \equiv 1 \pmod{p}$.

Proof: Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$.

All different modulo $p$ since $a$ has an inverse modulo $p$. $S$ contains representative of $\{1, \ldots, p-1\}$ modulo $p$.

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Each of $2, \ldots (p-1)$ has an inverse modulo $p$, solve to get...
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**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$, 
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**Proof:** Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$.

All different modulo $p$ since $a$ has an inverse modulo $p$. 
$S$ contains representative of $\{1, \ldots, p-1\}$ modulo $p$.

\[(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \pmod{p}, \]

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\[ a^{(p-1)}(1 \cdots (p-1)) \equiv (1 \cdots (p-1)) \pmod{p}. \]

Each of $2, \ldots (p-1)$ has an inverse modulo $p$, solve to get...

\[ a^{(p-1)} \equiv 1 \pmod{p}. \]
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All different modulo $p$ since $a$ has an inverse modulo $p$.

$S$ contains representative of $\{1, \ldots, p-1\}$ modulo $p$.

$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \pmod{p},$$

Since multiplication is commutative.

$$a^{(p-1)}(1 \cdots (p-1)) \equiv (1 \cdots (p-1)) \pmod{p}.$$  

Each of $2, \ldots (p-1)$ has an inverse modulo $p$, solve to get...

$$a^{(p-1)} \equiv 1 \pmod{p}.$$  

□
Poll

Which was used in Fermat’s theorem proof?

(A) The mapping \( f(x) = ax \mod p \) is a bijection.

(B) Multiplying a number by 1, gives the number.

(C) All nonzero numbers \( \mod p \), have an inverse.

(D) Multiplying a number by 0 gives 0.

(E) Multiplying elements of sets \( A \) and \( B \) together is the same if \( A = B \).

(A), (C), and (E)
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Fermat and Exponent reducing.

**Fermat’s Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,
Fermat’s Little Theorem: For prime $p$, and $a \not\equiv 0 \pmod{p}$,
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What is $2^{101} \pmod{7}$?

Wrong: $2^{101} = 2^7 \cdot 14 + 3 = 2^3 \pmod{7}$.

Fermat: 2 is relatively prime to 7.

⇒ $2^6 \equiv 1 \pmod{7}$.

Correct: $2^{101} = 2^6 \cdot 16 + 5 = 2^5 = 32 = 4 \pmod{7}$. 

For a prime modulus, we can reduce exponents modulo $p - 1$. 

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For a prime modulus, we can reduce exponents modulo $p - 1$!
Lecture in a minute.

Extended Euclid: Find $a, b$ where $ax + by = \gcd(x, y)$.
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Fundamental Theorem of Algebra: Unique prime factorization of any natural number.

Claim: any prime that divides a number $n$, divides a number in any factorization of $n$.

From Extended Euclid.

Induction.

Chinese Remainder Theorem: If $gcd(n, m) = 1$, $x = a \pmod{n}$, $x = b \pmod{m}$ unique sol.
Proof: Find $u = 1 \pmod{n}$, $u = 0 \pmod{m}$, and $v = 0 \pmod{n}$, $v = 1 \pmod{m}$.

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Proof Idea: $f(x) = a^x \pmod{p}$: bijection on $S = \{1, \ldots, p-1\}$.

Product of elts == for range/domain: $a^{p-1}$ factor in range.
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