Lecture 5: Cardinality and Countability

Today’s Plan

- Functions.
  - Bijection / Surjection / Injection
  - Composition
- Cardinality
  - Countable
  - Uncountable
  - Diagonalization

Hilbert's Infinite Hotel

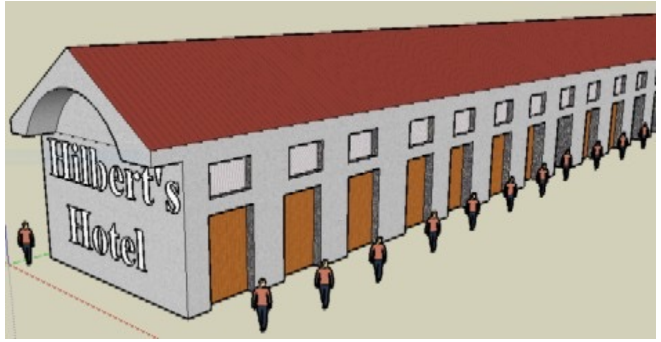
- Suppose there is a hotel, with the same number of rooms as natural numbers.
- Rooms are marked with 1, 2, 3, ..., n, ... All rooms are occupied.

- Now number of new guest = 1.
  - We tell the guest in room n to move in room $n + 1$.
  - The new guest can then take room 1.

Hilbert’s Infinite Hotel

- Suppose there is a hotel, with the same number of rooms as natural numbers.
- Rooms are marked with 1, 2, 3, ..., n, ... All rooms are occupied.

- Now number of new guest = k.
  - We tell the guest in room n to move in room $n + k$.
  - The new guest can then take room 1, 2, $\cdots, k$.



Hilbert’s Infinite Hotel

- Suppose there is a hotel, with the same number of rooms as natural numbers.
- Rooms are marked with 1, 2, 3, ..., n, ... All rooms are occupied.

- Now number of new guest = number of natural numbers.
  - We tell the guest in room n to move in room ???.
  - There is no Room n + ∞. Because it is not a natural number.

Hilbert’s Infinite Hotel

- Suppose there is a hotel, with the same number of rooms as natural numbers.
- Rooms are marked with 1, 2, 3, ..., n, ... All rooms are occupied.

- Now number of new guest = number of natural numbers.
  - We tell the guest in room n to move in room $2n$.
  - The new guest can then take room 1, 3, 5, $\cdots$.

To Infinity!

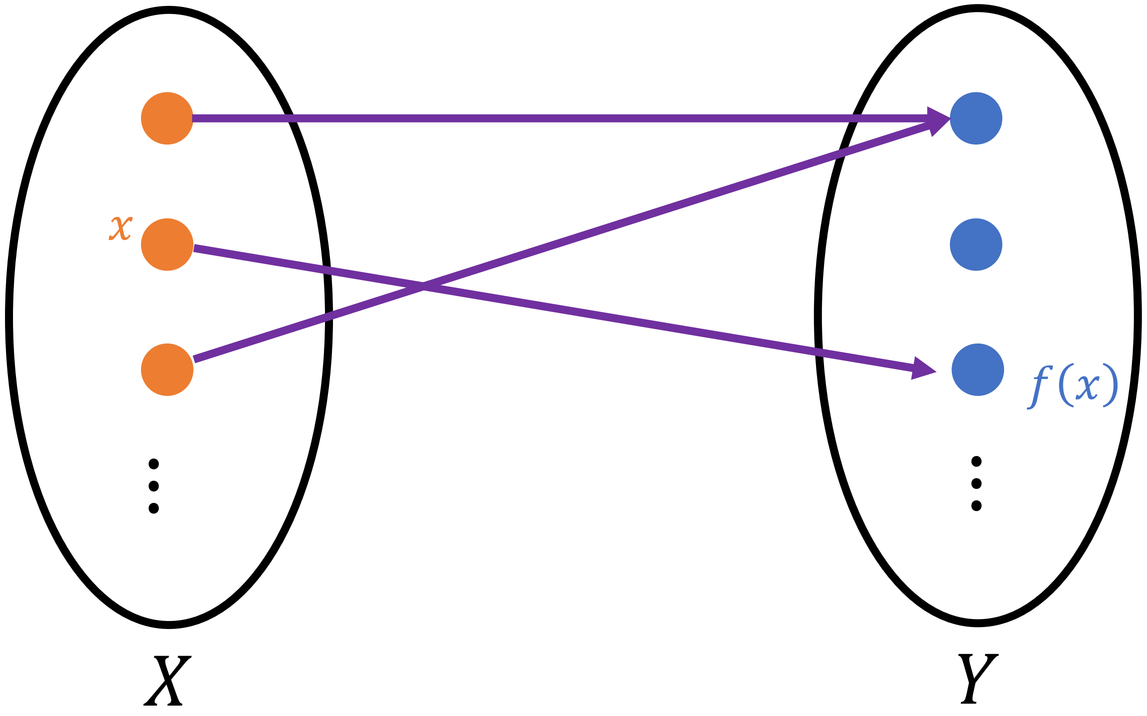
- How do we compare sizes of infinite sets?
- How do we add one to infinite sets?
- How do we ``multiply'' the size of infinite sets?

Functions

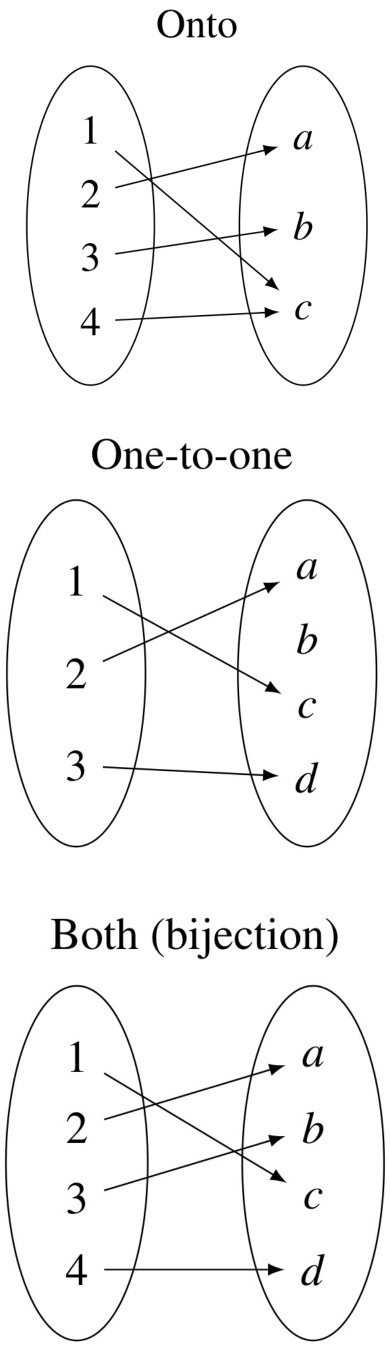
Definition

A function $f: X \to Y$ has a unique value $f(x) \in Y$ for every $x \in X$.
We say f maps x to $f(x)$.

x is called a preimage.
$f(x)$ is called an image.



Surjection / Injection.



Definition

A function $f: X \to Y$ is surjective (onto) if and only if

$$\forall\, y \in Y \quad |\{x \mid f(x) = y\}| \geq 1\,.$$

Definition

A function $f: X \to Y$ is injective (one-to-one) if and only if

$$\forall\, y \in Y \quad |\{x \mid f(x) = y\}| \leq 1.$$

Definition

A function $f: X \to Y$ is bijective if and only if

$$\forall\, y \in Y \quad |\{x \mid f(x) = y\}| = 1.$$

Equivalently, if and only if f is both surjective and injective.

Bijection between Finite sets.

Definition

A function $f: X \to Y$ is bijective if and only if

$$\forall\, y \in Y \;\; |\{x \mid f(x) = y\}| = 1.$$

Claim

If X and Y are finite and has bijection f, we must have $|X| = |Y|$.

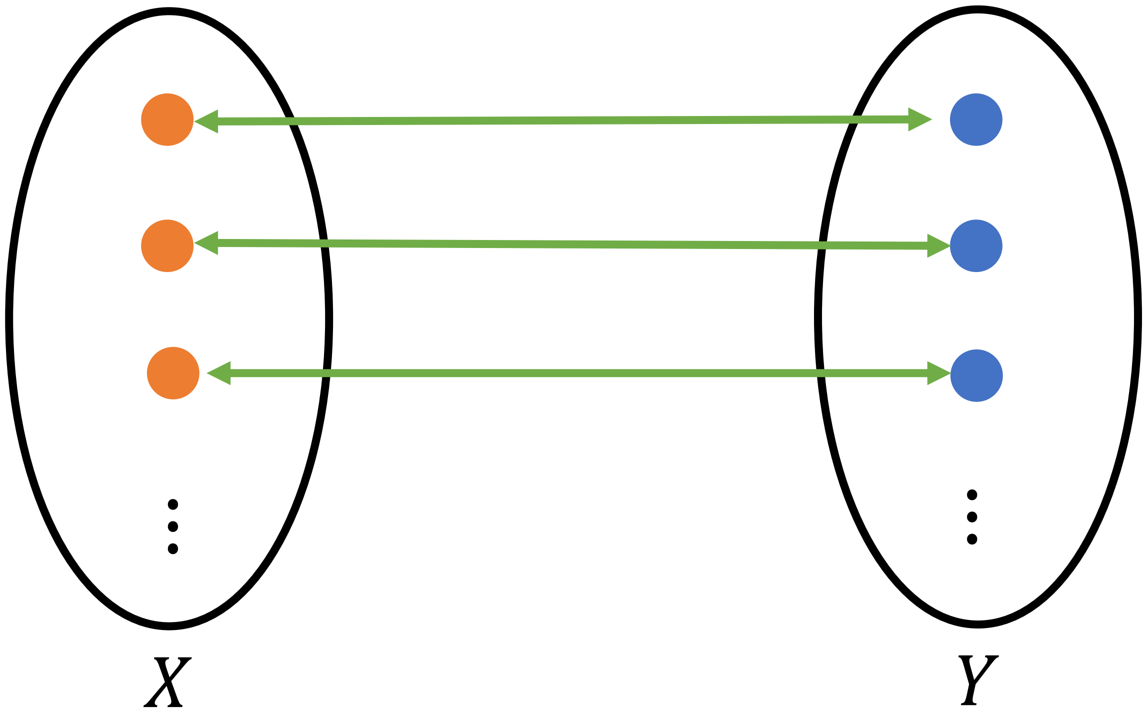
Proof.

$$|Y| = \sum_{y \in Y} 1 = \sum_{y \in Y} |\{x \mid f(x) = y\}| = \sum_{x \in X} 1 = |X|$$

Cardinality.

Definition

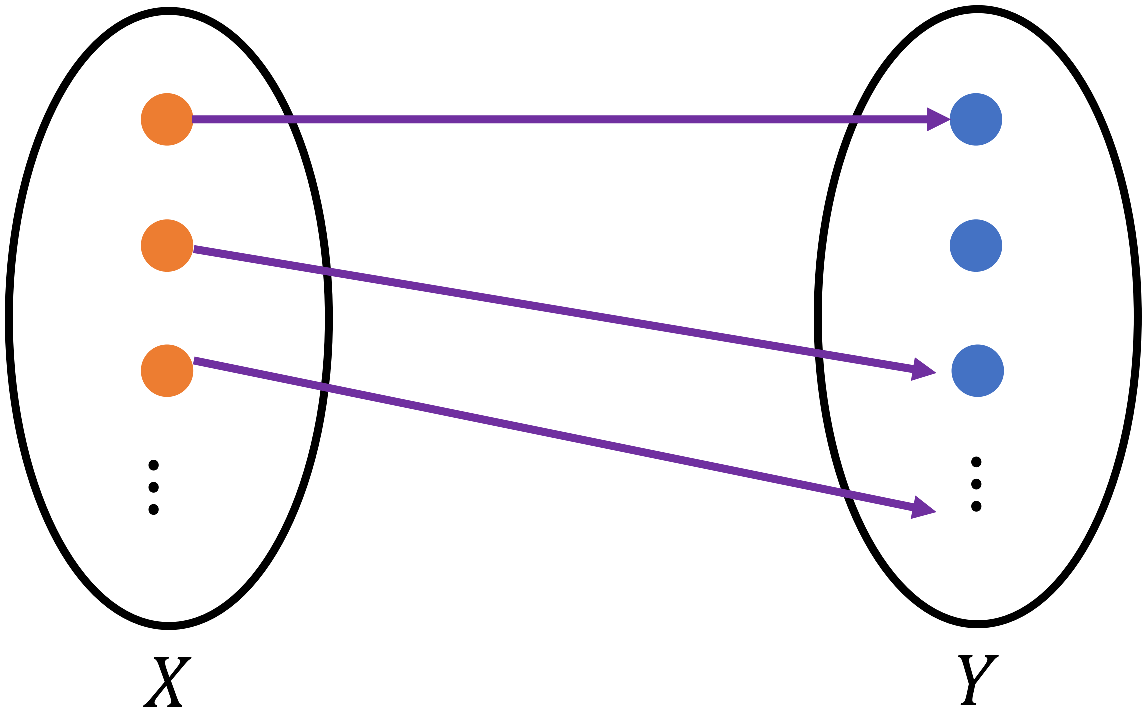
If X and Y are infinite sets and has bijection f, we say X and Y have the same cardinality.



Cardinality.

Definition

If there is an injection f from X to Y, then X has smaller or equal cardinality than Y. $|X| \leq |Y|$

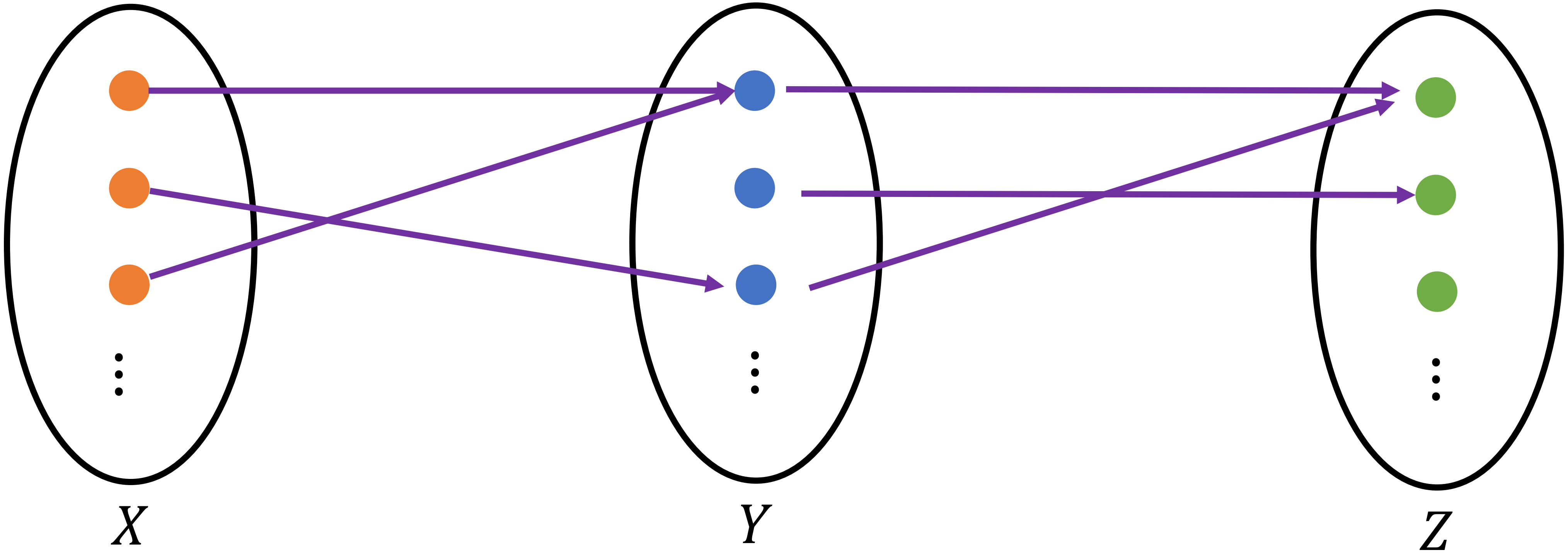


Function composition

Definition

The composition of a function $f: X \to Y$ and g: $Y \to Z$ is defined as:

$$g \circ f(x) = g(f(x)).$$



Function composition

Theorem

The composition of injection / surjection / bijection is still a injection / surjection / bijection.

Proof.

Implication: If $|X| \leq |Y|$ and $|Y| \leq |Z|$, then $|X| \leq |Z|$!

For example for injection,

$$g \circ f(x) = g\big(f(x)\big).$$

If for any z, there is a unique y such that g(y) = z.
For every y there is a unique x such that f(x) = y.

Then for any z, there is a unique x such that g(f(x)) = z.

Cardinality.

Theorem (Schröder–Bernstein Theorem)

If there is an injection f from X to Y, and a injection f' from Y to X.
Then there is a bijection between X and Y.

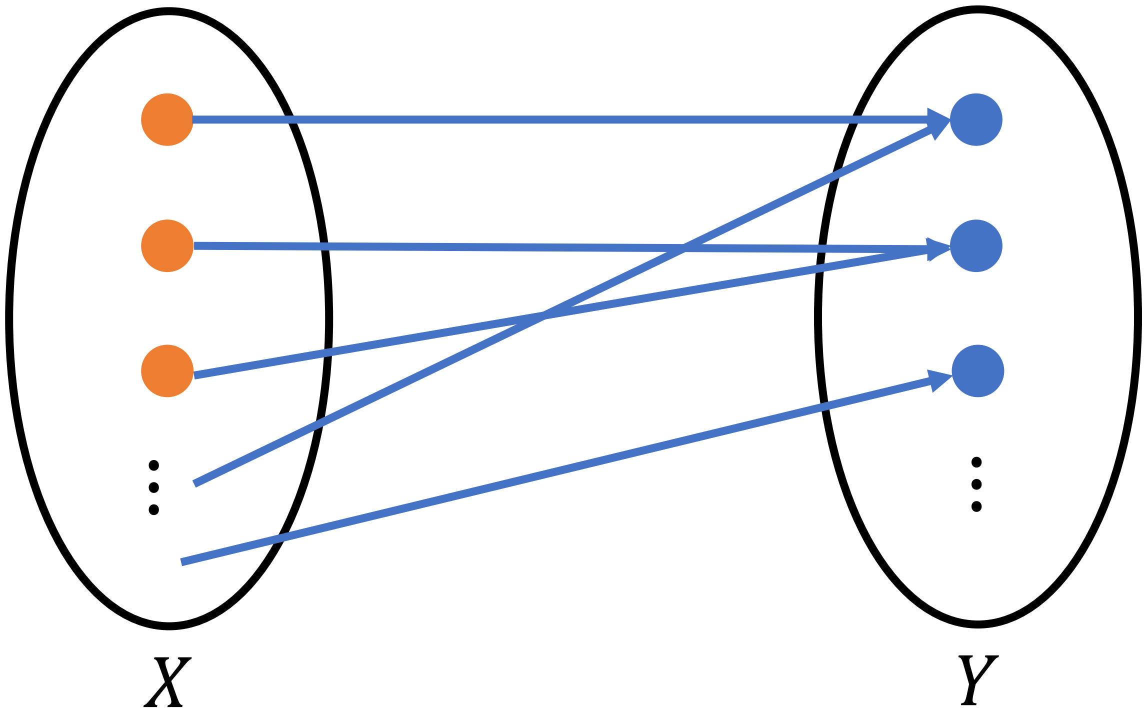
We will not cover its proof.

Implication: If $|X| \leq |Y|$ and $|Y| \leq |X|$, Then $|X| = |Y|$.

Cardinality.

Definition

If there is a surjection f from X to Y, then X has greater or equal cardinality than Y.



Natural Numbers

- Back to the infinite hotel:
  - Having one extra customer:

    $\mathbb{N}_+ \cup \{e\}$ has the same cardinality as $\mathbb{N}_+$.

    $$f : \mathbb{N}_+ \cup \{e\} \to \mathbb{N}_+ \text{ defined as } f(x) = \begin{cases} x + 1 & \text{if } x \in \mathbb{N}_+ \\ 1 & \text{if } x = e \end{cases}$$

    - Why is it a bijection?
  - Having k extra customers:

    $\mathbb{N}_+ \cup \{e_1, e_2, \ldots, e_k\}$ has the same cardinality as $\mathbb{N}_+$.

    $$f : \mathbb{N}_+ \cup \{e_1, e_2, \ldots, e_k\} \to \mathbb{N}_+ \text{ defined as } f(x) = \begin{cases} x + k & \text{if } x \in \mathbb{N}_+ \\ i & \text{if } x = e_i \end{cases}$$

Natural Numbers

- Back to the infinite hotel:
  - Having $\mathbb{N}_+$ extra customer:

    $\mathbb{N}_+ \sqcup \mathbb{N}_+$ has the same cardinality as $\mathbb{N}_+$.

    Here $\mathbb{N}_+ \sqcup \mathbb{N}_+$ is the ``disjoint union'' of two copies of $\mathbb{N}_+$.

    $\mathbb{N}_+ \sqcup \mathbb{N}_+ = \{1, 2, 3, 4, \cdots\} \cup \{1', 2', 3', 4', \cdots\}$

    Can we still use

    $f : \mathbb{N}_+ \cup \{e_1, e_2, \dots, e_k\} \to \mathbb{N}_+$ defined as $f(x) = \begin{cases} x + k & \text{if } x \in \mathbb{N}_+ \\ i & \text{if } x = e_i \end{cases}$ and take $k = \infty$?

    No! What is the image for 1?
    Remember $\infty \notin \mathbb{N}$!

Natural Numbers

- Back to the infinite hotel:
  - Having $\mathbb{N}_+$ extra customer:

    $\mathbb{N}_+ \sqcup \mathbb{N}_+$ has the same cardinality as $\mathbb{N}_+$.

    Here $\mathbb{N}_+ \sqcup \mathbb{N}_+$ is the ``disjoint union'' of two copies of $\mathbb{N}_+$.

    $\mathbb{N}_+ \sqcup \mathbb{N}_+ = \{1, 2, 3, 4, \cdots\} \cup \{1', 2', 3', 4', \cdots\}$

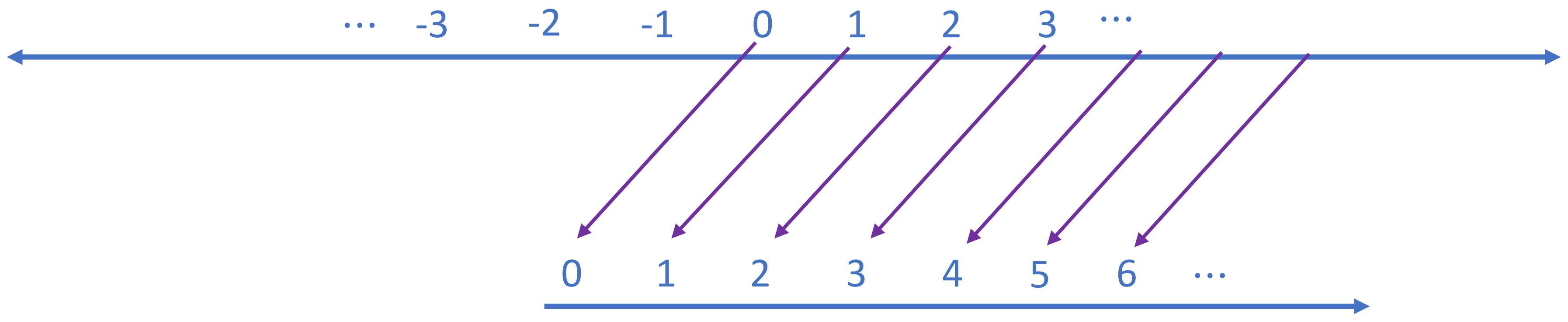
$$f : \mathbb{N}_+ \sqcup \mathbb{N}_+ \to \mathbb{N}_+ \text{ defined as } f(x) = \begin{cases} 2x & \text{if } x \in \mathbb{N}_+ \\ 2x + 1 & \text{if } x \in \mathbb{N}'_+ \end{cases}$$

Natural Numbers

- Back to the infinite hotel:
  - Having $\mathbb{N}$ extra customer:

    $\mathbb{Z}$ has the same cardinality as $\mathbb{N}_+$.

    The following will not work.

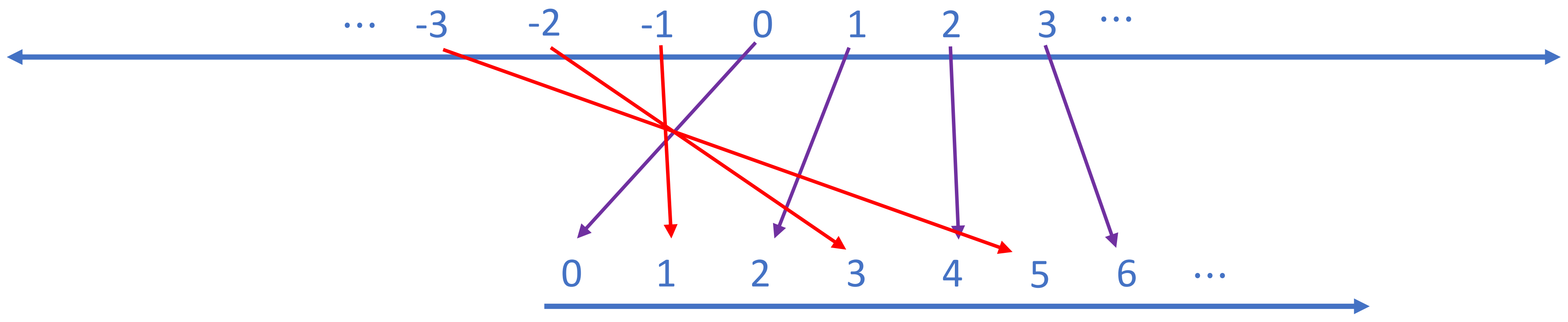


Natural Numbers

- Back to the infinite hotel:
  - Having $\mathbb{N}_+$ extra customer:

    $\mathbb{Z}$ has the same cardinality as $\mathbb{N}_+$.

$$f : \mathbb{Z} \rightarrow \mathbb{N}_+ \text{ defined as } f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 2(-x) - 1 & \text{if } x < 0 \end{cases}$$



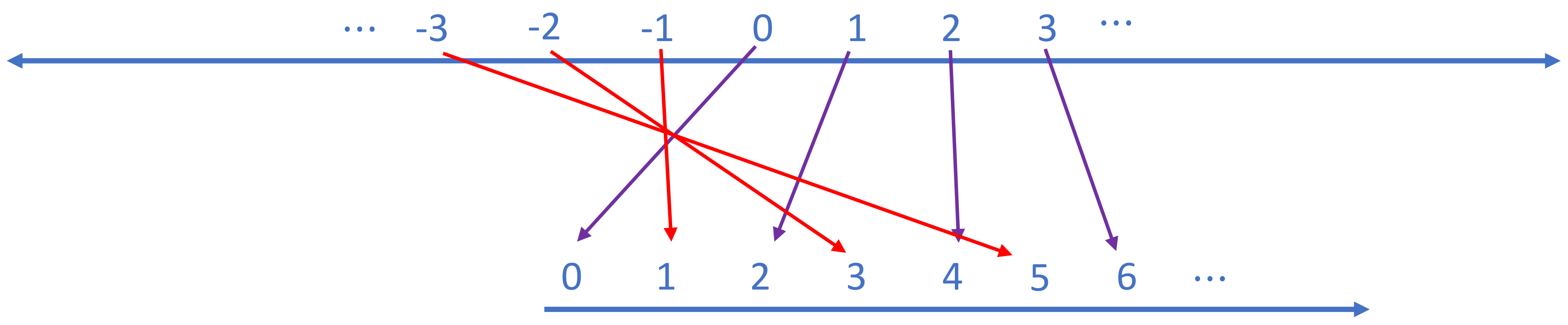
Another view: Enumerate

- Back to the infinite hotel:
  - Having $\mathbb{N}$ extra customer:

    We can enumerate all integers in $\mathbb{Z}$ as follows:

    0, -1, 1, -2, 2, -3, 3, $\cdots\cdots$ where all integers are reached in finite steps.

    $x \in \mathbb{Z}$ is reached in $\leq 2|x| + 1$ steps.

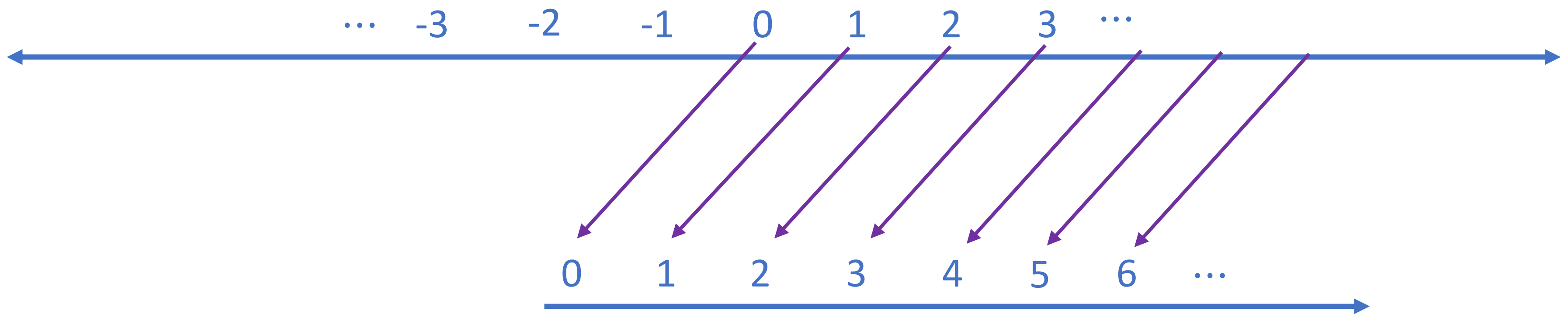


Another view: Enumerate

- Back to the infinite hotel:
  - Having $\mathbb{N}$ extra customer:

    We can NOT enumerate all integers in $\mathbb{Z}$ as follows:

    $0,\ 1,\ 2,\ 3,\ \cdots\cdots, -1, -2, \cdots\cdots$ because -1, -2, ... are NOT reached in finite steps.



Countability = Enumerability

Definition

A set S is said to be countable if $|S| \leq |\mathbb{N}|$.

- If we can enumerate a set S,

  we can map $s \in S$ to the number of steps (which is finite) it takes to reach s.

  This is injective. Thus $|S| \leq |\mathbb{N}|$.

- If S is countable, there must be surjection $f: \mathbb{N} \rightarrow S$.

  we enumerate $f(i)$ in the i-th step.

Because this is surjection, for every $s \in \mathrm{S}$, there exists $n \in \mathbb{N}$ such that $f(n) = s$.

Note $\infty \notin \mathbb{N}$, s is reachable in n, which is finite, steps.

Subsets

Theorem

A subsets of a countable set is still countable.

Proof (using injection)

Let S’ be a subset of S.

$f(x) = x$ is an injection mapping $S' \to S$.

Proof (using enumeration)

Suppose we have an enumeration for S.

If we only output what is in S’….

Strings

Can we enumerate all strings?
A string is a finite length sequence of letters
(either 0/1 or a/b/c/d/… depending on your finite alphabet)

YES!

What won’t work: lexicographical order

What would work: We first enumerate all strings of length 1. (a/b/c/d/…)
Then all strings of length 2. (aa/ab/ac/…)
…….

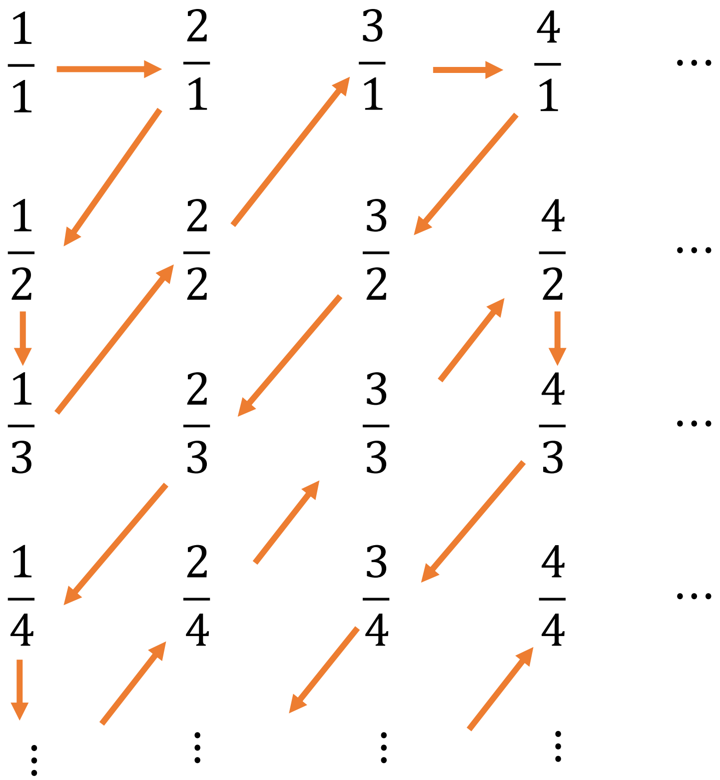
Rational Numbers

- How about $Q = \frac{p}{q}$?   This will NOT work:

$$
\begin{array}{ccccccccc}
\frac{1}{1} & \rightarrow & \frac{2}{1} & \rightarrow & \frac{3}{1} & \rightarrow & \frac{4}{1} & & \cdots \\
\frac{1}{2} & \rightarrow & \frac{2}{2} & \rightarrow & \frac{3}{2} & \rightarrow & \frac{4}{2} & & \cdots \\
\frac{1}{3} & \rightarrow & \frac{2}{3} & \rightarrow & \frac{3}{3} & \rightarrow & \frac{4}{3} & & \cdots \\
\frac{1}{4} & \rightarrow & \frac{2}{4} & \rightarrow & \frac{3}{4} & \rightarrow & \frac{4}{4} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

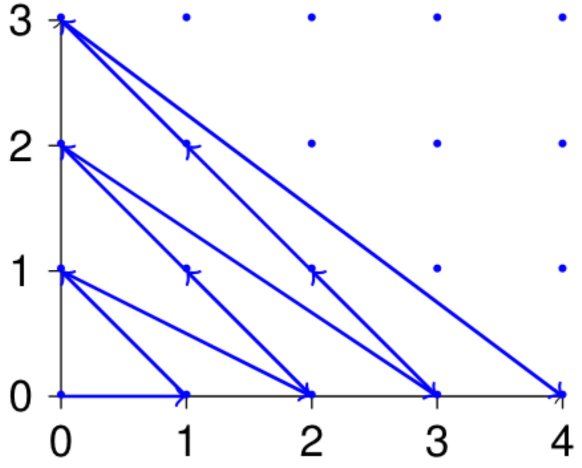
Rational Numbers

- How about $Q = \frac{p}{q}$?



Pairs of natural numbers

- $\mathbb{N} \times \mathbb{N} = (p, q)$?



Real numbers

- Real numbers $\mathbb{R}$ can be defined as countably long decimals.
  - E.g. 0.0023242321……, 131.42345324….., 3.1415926……..
  - Caveat: 1 = 0.999999……..
    3.3 = 3.29999……….
- $[1, \infty)$ vs (0,1] ?

  Bijection $f(x) = \frac{1}{x}$

  Same cardinality

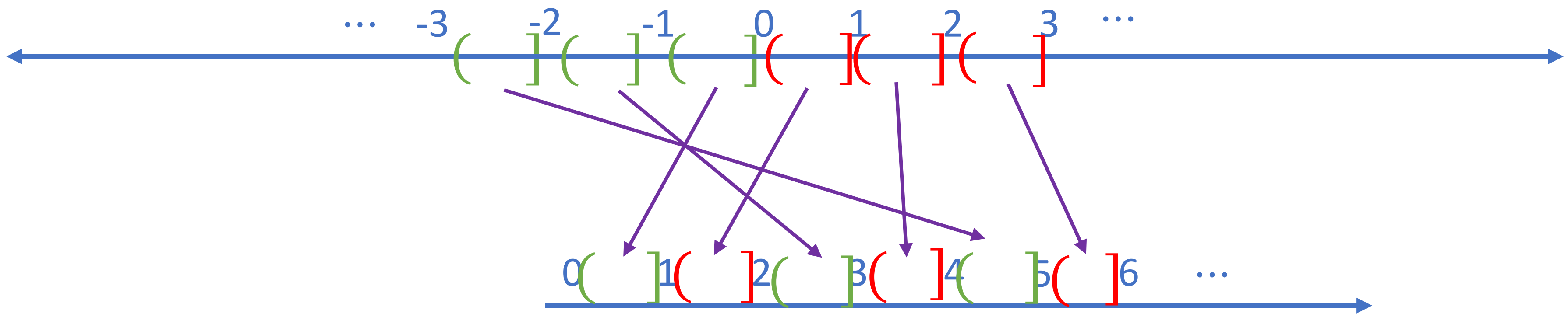
Real numbers

- $\mathbb{R}$ vs (0,1] ?

  $\mathbb{R}_+ = [1, \infty) \cup (0,1]$ has same cardinality as $(0,1]$

$$f(x) = \begin{cases} \dfrac{x}{2} \text{ for } x \in (0,1] \\ \dfrac{1}{2} + \dfrac{1}{2x} \text{ for } x \in [1, \infty) \end{cases}$$

  $\mathbb{R}_+$ has the same cardinality as $\mathbb{R}$.



Diagonalization

- Real numbers $\mathbb{R}$ can be defined as countably long decimals.
  - E.g. 0.0023242321……, 131.42345324….., 3.1415926……..
  - Caveat: 1 = 0.999999……..

    3.3 = 3.29999……….

- Is $\mathbb{R}$ countable?

Diagonalization

- Is $\mathbb{R}$ countable?
  - NO!

- Proof.

  Assume $\mathbb{R}$ is countable, then $\mathbb{R}$ is enumerable.

  Take any enumeration,

  0.32123435……

  0.34255235……

  0.12342551……

  0.59285225……

  …………….

Diagonalization

- Is $\mathbb{R}$ countable?
  - NO! (Equivalent to proving $[0,1)$ is uncountable.)

- Proof.

Assume $[0,1)$ is countable, then $[0,1)$ is enumerable.

Take any enumeration,

0.32123435……

0.36255235……

0.12642551……

0.59285225……

……………

0.6776……

We construct a real number not in the list:

If the i-th row's i-th digit is 6, we put 7 in the i-th digit of our number.

Otherwise we put 6.

Diagonalization

- Is $\mathbb{R}$ countable?
  - NO! (Equivalent to proving $[0,1)$ is uncountable.)

- Proof.

  Assume $[0,1)$ is countable, then $[0,1)$ is enumerable.

  Take any enumeration,

  0.32123435……

  0.36255235……

  0.12642551……

  0.59285225……

  ……………

  0.6776……

Why is not in the list? Proof by contradiction.

Why 6 and 7?

Wait a minute…

- We have seen
  - The set of all strings are countable.
    - This includes every English sentence.

  - The set of all real numbers are uncountable.
    - => Most of the real numbers cannot be described / named / said!

  - A philosophical question: Do they really exist?
    - If a tree falls in a forest….

Power set

- In the same way, we can prove:

  Let $2^S = \{T \mid T \subseteq S\}$ be the powerset of S.

- 2^S must be of a larger cardinality than S for any infinite set S.

- Proof: For any mapping $f: S \to 2^S$,

  $$\{x \in S \mid x \notin f(x)\} \in 2^S$$

  is not an image of f.

Power set

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- Actual Proof: For any mapping $f: S \to 2^S$,

	s_1	s_2	s_3	
$f(s_1)$	1	0	1	……
$f(s_2)$	0	1	1	……
$f(s_3)$	0	1	0	……

i-th row and j-th column = 1
if $s_j \in f(s_i)$

Power set

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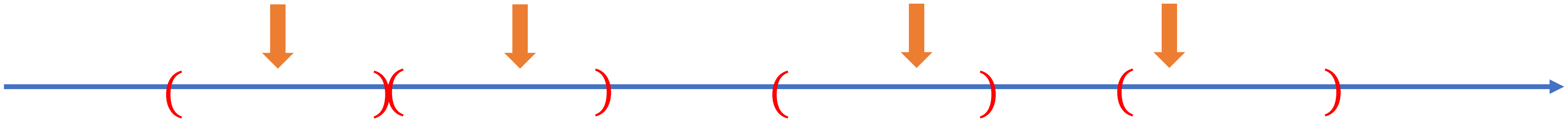
$\{x \in S \mid x \notin f(x)\}$	0	0	1	

i-th row and j-th column = 1 if $s_j \in f(s_i)$

How about… disjoint Intervals?

- Suppose S is a set of disjoint intervals.

  (e.g. S = $\{(1,2), (e, \pi), (4, \sqrt{29})$ ….})



S is countable!

Each interval contains at least one rational number.
We can construct injection $f: S \to \mathbb{Q}$.
Mapping intervals to that rational number. So $|S| \leq |\mathbb{Q}|$.