Introduction to Discrete Probability

CS70: Discrete Mathematics and Probability Theory

UC Berkeley – Summer 2026

Lecture 16

Ref: Note 13

CS70: On To Probability...

Dealing with events that happen by chance

Apologies in advance for the bad nerd-joke....

Today (and maybe a little of next time):

1. Key Ideas
2. Experiments, sample space, probability space, events, ...
3. Examples... some that are counter-intuitive

Key Ideas

Canonical example of an event that happens by chance: flip a fair coin

- Can you predict the outcome in advance?
- Can you say anything about the outcome of a flip?
- Can you say anything about the outcome of 20 flips?

Consider two trials with 20 flips each:

Trial 1 outcome: `HHHHHHHHHHHHHHHHHHHH`

Trial 2 outcome: `HHTTHTHHHTTHTHTHHTHT`

- Which is more likely?

- Which is "more random"?

What is Randomness?

Intuitive notion of randomness $\implies$ unpredictability

But is it equivalent? And unpredictable by who/what?

- Statistical tests? (i.e., specific computational processes)
- *Any efficient* computational process? (Cryptographic pseudorandomness)
- *Any* computational process? (Kolmogorov complexity)
- An arbitrary function (computable or not)?

Does real randomness exist or are events just too complex to predict?

- Flip a fair coin: If you knew precise initial position, force applied, and environmental conditions, is the outcome predictable?
- What does physics (quantum mechanics) tell us?

Perhaps "does randomness exist" is better left to physicists and philosophers...

Probability Theory and Statistics

Two related fields of study:

- Probability Theory
  - ⇒ Model driven
  - ⇒ Idealized
  - ⇒ Can prove things!
  - ⇒ An *approximation* of the real world
- Statistics
  - ⇒ Data driven
  - ⇒ Measurements from the real world
  - ⇒ Often *approximated* by ideal probability models
  - ⇒ With *assumptions* can make predictions

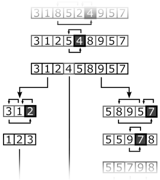
In this class: primarily probability theory

Understand idealized mathematics to give insight into statistical world

How is Probability Theory Used?
Analysis of algorithms

Consider *Quicksort*:

- Randomized divide-and-conquer
- Picks “pivot” elements randomly
  $\Rightarrow$ Most are good – some are not



Question 1: What is the *expected running time* of the algorithm?

Question 2: What is the *probability* that it takes much longer than expected?

Probability theory gives us tools to analyze and answer both questions

Note similarity to investment portfolio planning.

How is Probability Theory Used?

Calculating likelihood from samples

Poll 4,000 random Americans about preference in a two-person election

How well does this reflect current state or predict the election outcome?

Style of answer:

- "54% of respondents preferred candidate A"
- "With 95% probability, the fraction of all Americans supporting candidate A is in the range 51% to 57%"

Challenges:

- What is the distribution of the sample (ideally want uniform)
- Are people honest?
- Do people change their minds before the election?

Is this probability theory or statistics? Random sample from fixed population, fixed preferences, ...

How is Probability Theory Used?

Deriving probability of earlier, non-measured events

Shopping center 1:

3 Indian restaurants:

2 Moroccan restaurants:

Shopping center 2:

1 Indian restaurant:

1 Moroccan restaurants:

Joe picks a center randomly and then a restaurant at that center randomly

$\Rightarrow$ *In this case, "randomly" means "uniformly" (all outcomes equally likely)*

Joe comes back and says he ate at a Moroccan restaurant

What was the probability that he went to shopping center 1?

How is Probability Theory Used?

Analyzing likelihood from different measures

Company BigPharma developed a new test for COVID-19. Tests show that:

- Given to a person with COVID-19, says “positive” 90% of the time
- Given to a person without COVID-19, says “positive” 20% of the time
  - $\Rightarrow$ These are called *false positives*
- 5% of the population has COVID-19

Question: If you test “positive”, what is the chance you are actually infected?

Spoiler: Not nearly as high as you might think... (it’s 19%)

Note: Similar to machine-learning classification

Random Experiment: Flip One Fair Coin

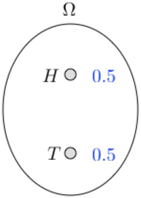
Flip a fair coin (*one flips or tosses a coin*):

- Possible outcomes: Heads (H) and Tails (T)
  (*One flip yields either ‘heads’ or ‘tails’.*)
- Likelihoods: $H : 50\%$ and $T : 50\%$

Random Experiment: Flip One Fair Coin

Flip a fair coin: Model

Physical Experiment



Probability Model

- The physical experiment is complex (shape, density, initial momentum and position, ...)
- The Probability model is simple:
  - A set Ω of outcomes of experiments: $\Omega = \{H, T\}$.
    $\Rightarrow$ The set Ω is called the sample space
  - A probability assigned to each outcome: $\mathbb{P}[H] = 0.5, \;\; \mathbb{P}[T] = 0.5$
    $\Rightarrow$ Ω with function $\mathbb{P} : \Omega \to [0, 1]$ is a probability space

Probability Space

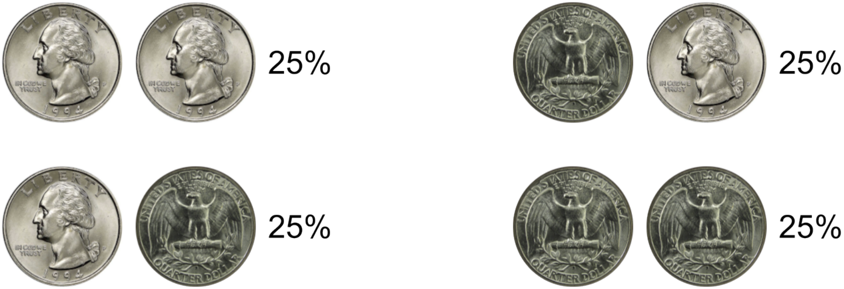
The probability function (or measure) $\mathbb{P}$ must meet certain conditions:

- (Non-negativity): $0 \leq \mathbb{P}[\omega] \leq 1$ for all $\omega \in \Omega$
- (Sum to 1): $\sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1$

The value $\mathbb{P}[\omega]$ is called the *probability* of outcome ω

Slightly more informal: the *likelihood* or *chance* of ω

Flip Two Fair Coins



- Possible outcomes: $\Omega = \{HH, HT, TH, TT\}$
- Probabilities all the same, so: $1/4$ each

"Probabilities all the same" is a uniform probability space

Is this uniform?

Random Experiment: Flip One Unfair Coin

Flip an unfair (biased, loaded) coin:

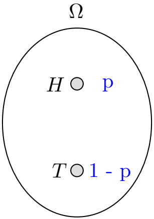
- Possible outcomes: $\Omega = \{H, T\}$ (just like before)
- Probabilities:
  - $\mathbb{P}[H] = p \in [0, 1]$

    What can you say about $\mathbb{P}[T]$?

Random Experiment: Flip one Unfair Coin

Flip an unfair (biased, loaded) coin: model

Physical Experiment



Probability Model

Poll

Flip two fair coins. Which describes outcomes?

(A) There are two outcomes.

(B) The outcomes are H and T, twice.

(C) $\Omega = \{HH, HT, TH, TT\}$.

(D) A sample point ω could be “H”

(E) A sample point ω could be “HT”

Flipping n times

Flip a fair coin $n \geq 1$ times.

Ω

2^{-n} TT ... TTT
2^{-n} TT ... TTH
2^{-n} TT ... THT
....
2^{-n} HH ... HHH

Probability Model

- Outcomes: $\Omega = \{\mathtt{H}, \mathtt{T}\}^n$

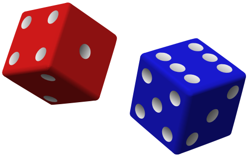
  n-way *Cartesian Product*, A^n has size $|A^n| = |A|^n$

  $\Rightarrow$ Here $|\Omega| = |\{\mathtt{H}, \mathtt{T}\}|^n = 2^n$
- Uniform distribution: for all $\omega \in \Omega$, $\mathbb{P}[\omega] = 1/|\Omega|$

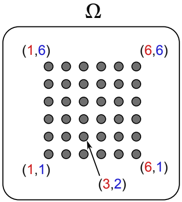
  $\Rightarrow$ Here: $\mathbb{P}[\omega] = 2^{-n}$

Roll Two Dice

Roll two fair 6-sided dice (one red, one blue):



Physical Experiment



Probability Model

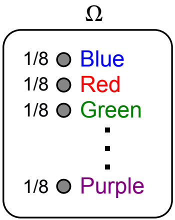
- Possible outcomes: $\Omega = \{1,2,3,4,5,6\}^2 = \{(a,b) \mid 1 \le a,b \le 6\}$.
- $|\Omega| = 36$, uniform probability space

  $\Rightarrow \mathbb{P}[\omega] = 1/36$ for all ω

Drawing Balls

Drawing a ball from a bag – all balls identical except color:

Physical Experiment



Probability Model

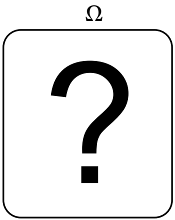
Uniform probability space:

- $|\Omega| = 8$
- For all $\omega \in \Omega$: $\mathbb{P}[\omega] = 1/|\Omega| = 1/8$

Drawing Balls

Drawing a ball from a bag – duplicate colors:

Physical Experiment



Probability Model

Is this non-uniform?

Consider:

$\mathbb{P}[\textit{red}] = 3/8$

$\mathbb{P}[\textit{blue}] = 1/8$

Is this the right experiment?

Are we drawing a *color* or a *ball*?

Drawing Balls

Drawing a ball from a bag – duplicate colors:

Physical Experiment

Ω

1/8 ● Ball 1: Blue
1/8 ● Ball 2: Red
1/8 ● Ball 3: Green
⋮
1/8 ● Ball 8: Red

Probability Model

Draw identified ball:

$\mathbb{P}[\text{Ball 1 (Blue)}] = 1/8$
$\mathbb{P}[\text{Ball 2 (Red)}] = 1/8$
$\ldots$
$\mathbb{P}[\text{Ball 8 (Red)}] = 1/8$

Experiment draws a ball uniformly
$\Rightarrow$ This is a uniform distribution!

If we are *interested* in colors, *group outcomes*

Enter “Events”!

Events

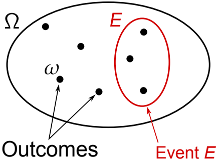
Sometimes we don't care about exact outcome – only a property of outcome

Idea: Define a set of outcomes with the desired property

Definition:

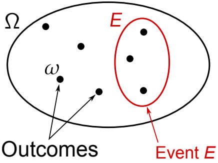
- An **event,** E**,** is a subset of outcomes: $E \subseteq \Omega$.
- The **probability of** E is defined as $\mathbb{P}[E] = \sum_{\omega \in E} \mathbb{P}[\omega]$.

Sample Space



$$\mathbb{P}[E] = \sum_{\omega \in E} \mathbb{P}[\omega]$$

Uniform Sample Space



$$\mathbb{P}[\omega] = \frac{1}{|\Omega|} \qquad \mathbb{P}[E] = \frac{|E|}{|\Omega|}$$

When sample space is uniform, finding $\mathbb{P}[E]$ is just a counting problem!

Drawing a ball from a bag – duplicate colors:

Physical Experiment

Ω

1/8 Ball 1: Blue
1/8 Ball 2: Red
1/8 Ball 3: Green
⋮
1/8 Ball 8: Red

Probability Model

Events:

$$E_{\text{blue}} = \{1\}$$
$$E_{\text{red}} = \{2,4,8\}$$
$$E_{\text{green}} = \{3,7\}$$
$$\ldots$$

Since uniform:

$$\mathbb{P}[E_{\text{blue}}] = |E_{\text{blue}}|/|\Omega| = 1/8$$
$$\mathbb{P}[E_{\text{red}}] = |E_{\text{red}}|/|\Omega| = 3/8$$
$$\mathbb{P}[E_{\text{green}}] = |E_{\text{green}}|/|\Omega| = 2/8$$
$$\ldots$$

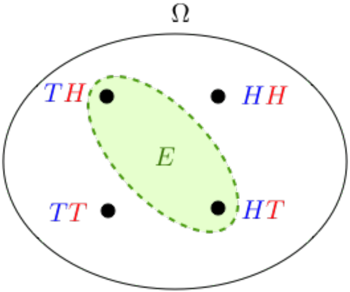
Probability space is still uniform – *events* have different probabilities

Events: Exactly one heads in two coin flips?

Sample Space, $\Omega = \{HH, HT, TH, TT\}$.

Uniform probability space: $\mathbb{P}[HH] = \mathbb{P}[HT] = \mathbb{P}[TH] = \mathbb{P}[TT] = \frac{1}{4}$.

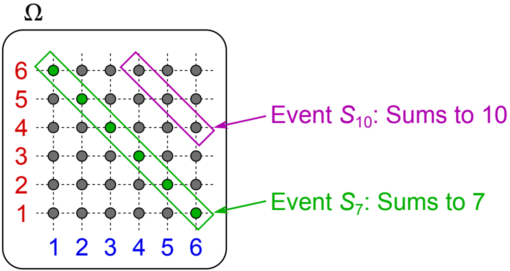
Event, E, “exactly one heads”: $\{TH, HT\}$.



$$\mathbb{P}[E] = \sum_{\omega \in E} \mathbb{P}[\omega] = \frac{|E|}{|\Omega|} = \frac{2}{4} = \frac{1}{2}.$$

Events: Based on a rolled pair of dice

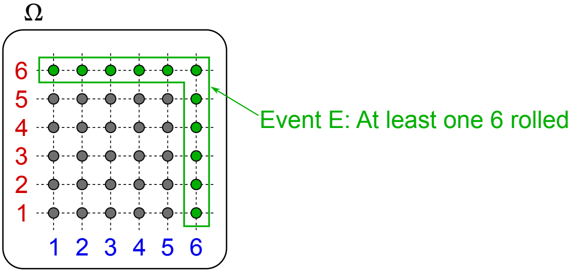
Define events based on the *sum* of two dice



$\mathbb{P}[S_7] =$ $\mathbb{P}[S_{10}] =$

Events: Based on a rolled pair of dice

Define events based on whether values appear or not



$\mathbb{P}[E] =$

Review from Counting Lecture...

How many poker hands? (*52 cards in a deck, all different, 5 in a poker hand*)

$52 \times 51 \times 50 \times 49 \times 48 = \frac{52!}{47!}$???

Are $A, K, Q, 10, J$ of spades
and $10, J, Q, K, A$ of spades the same?

Second Rule of Counting: If order doesn't matter, count ordered objects and then divide by the number of equivalent orderings of an ordered object.

"the" number of equivalent... assumes same for all ordered objects!

Number of orderings for a poker hand: 5!

By second rule of counting, number of poker hands is

$$\frac{52!}{47!} \cdot \frac{1}{5!} = \frac{52!}{47! \cdot 5!} = \binom{52}{5}$$

Events: Poker Flush

Probability space: uniformly chosen poker hand $\Rightarrow |\Omega| = \binom{52}{5}$

Example hands:

$\{A\heartsuit, 2\clubsuit, 7\spadesuit, 5\heartsuit, J\clubsuit\}$ $\{2\clubsuit, 5\clubsuit, 9\clubsuit, J\clubsuit, K\clubsuit\}$ $\{8\heartsuit, 8\clubsuit, 8\diamondsuit, 8\spadesuit, Q\clubsuit\}$

A flush is a hand in which all cards are the same suit:

$\{2\clubsuit, 5\clubsuit, 9\clubsuit, J\clubsuit, K\clubsuit\}$ $\{3\heartsuit, 4\heartsuit, 9\heartsuit, J\heartsuit, Q\heartsuit\}$ $\{10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit, A\spadesuit\}$

Define event $F \subseteq \Omega$ as hands that are flushes – what is $|F|$?

Heart flushes: $\binom{13}{5}$
Club flushes: $\binom{13}{5}$
Diamond flushes: $\binom{13}{5}$
Spade flushes: $\binom{13}{5}$

$\Longrightarrow \quad |F| = 4 \cdot \binom{13}{5}$

$$\mathbb{P}[F] = |F|/|\Omega| = 4 \cdot \binom{13}{5} / \binom{52}{5} = 4 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \approx 0.002 \quad \text{(or 0.2\%)}$$

Interesting Examples...

We close the intro to probability with two counter-intuitive results:

- The Birthday Paradox
- The Monty Hall Problem

Birthday Paradox

Problem: There are 30 people in a room. What is the probability that two (or more) have the same birthday?

Intuition: 365 days in a year[1] and only 30 people – surely small probability!

Sample space: 365 choices for person 1, 365 choices for person 2, 365 choices for person 3, ...
$\Rightarrow$ $|\Omega| = 365^{30}$

Event D is "duplicate value" – persons 1 and 5, persons 7 and 8, ...
Maybe 2 with the same, 3 with the same, ... How do we count this?????

Better: $\overline{D}$ is *no* duplicate
365 choices for person 1, 364 for person 2, 363 for person 3, ...
$\Rightarrow |\overline{D}| = 365 \cdot 364 \cdot 363 \cdots 336$

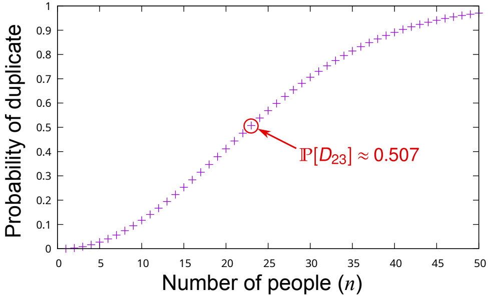
$$\mathbb{P}[D] = 1 - \mathbb{P}[\overline{D}] = 1 - \frac{|\overline{D}|}{|\Omega|} = 1 - \frac{365 \cdot 364 \cdot 363 \cdots 336}{365^{30}} \approx 0.7063 \quad \text{(or 70.53\%)}$$

[1] We're ignoring leap years

Birthday Paradox

<u>Problem:</u> There are n people in a room. What is the probability that two have the same birthday?

$$\mathbb{P}[D_n] = 1 - \frac{365 \cdot 364 \cdots (366 - n)}{365^n}$$



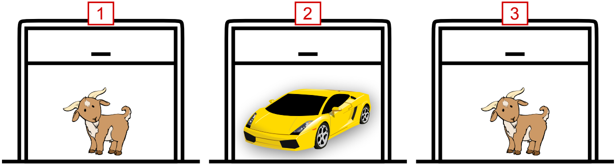
Generalizes with lots of C.S. applications!

Example: Probability of a collision in a hash function

We will return to this in the "Applications" lecture

The Monty Hall Problem

Game show! There are three doors, with a car and two goats:

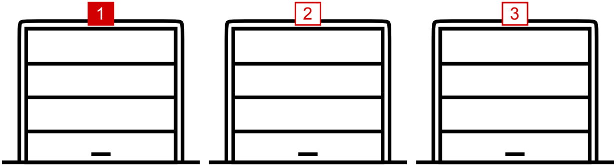


You want the car (no matter how cute the goat is)

Game:

1. The doors are closed, and three items randomly placed
2. You choose a door
3. The host (Monty Hall) opens one of the other doors with a goat
   Important: Monty knows where the prizes are and always shows a goat
4. You have a choice: switch your choice, or stick with it

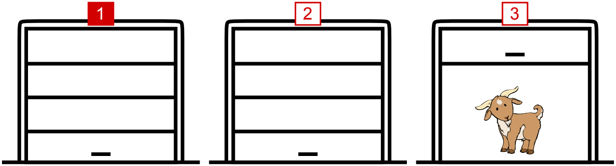
The Monty Hall Problem



Example:

1. You pick Door 1

The Monty Hall Problem

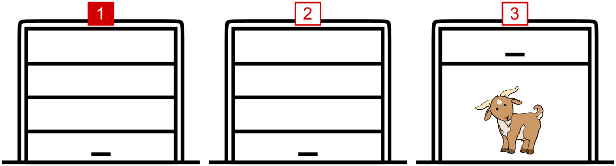


Example:

1. You pick Door 1
2. Monty opens Door 3

Stick with Door 1 or switch to Door 2?

The Monty Hall Problem



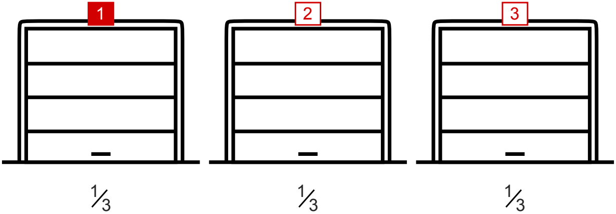
First, some history:

- "World's Highest IQ" Marilyn vos Savant had write-in column in *Parade*
- In 1990 someone sent this puzzle, and vos Savant answered (correctly!)
- People went berserk – including many with Ph.D.'s

See the responses:

https://web.archive.org/web/20130121183432/http:/marilynvossavant.com/game-show-problem/

The Monty Hall Problem



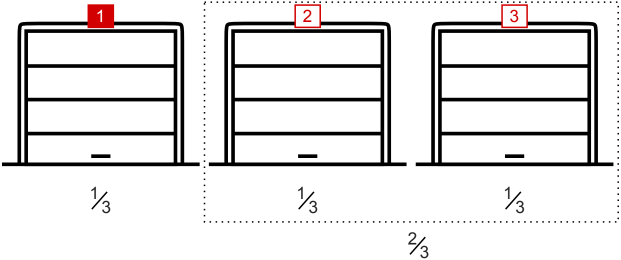
Back to the beginning...

When you pick Door 1, probability of the car behind that door is $1/3$

$\mathbb{P}[\text{car is behind Door 1}] = 1/3$

If you “stay” nothing will ever change this probability!

The Monty Hall Problem

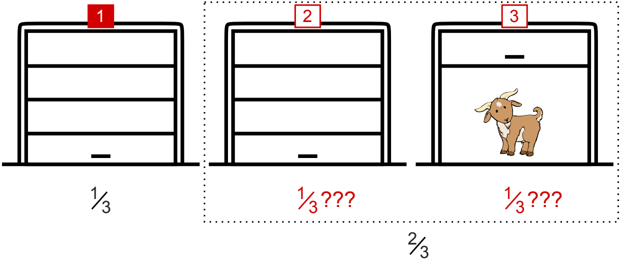


$\mathbb{P}[\text{car is behind Door 1}] = 1/3$

$\mathbb{P}[\text{car is behind Door 2 or Door 3}] = 2/3$

$\Rightarrow$ *Split evenly between the two possibilities*

The Monty Hall Problem



$\mathbb{P}[\text{car is behind Door 1}] = 1/3$

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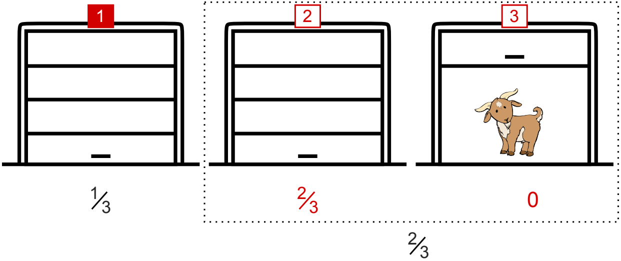
$\Rightarrow$ *Split evenly between the two possibilities*

After reveal: Only one possibility remains...

The Monty Hall Problem



$\mathbb{P}[\text{car is behind Door 1}] = 1/3$

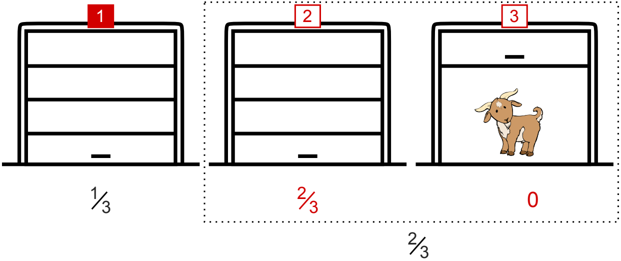
$\mathbb{P}[\text{car is behind Door 2 } \cancel{\text{or Door 3}}] = 2/3$

$\Rightarrow$ *Split evenly between the two possibilities*

After reveal: Only one possibility remains...

All $2/3$ chance concentrated in Door 2

The Monty Hall Problem



$\mathbb{P}[\text{car is behind Door 1}] = 1/3$

$\mathbb{P}[\text{car is behind Door 2}] = 2/3$ $\quad\leftarrow$ Switch to get this!

So always better to switch!

Summary

1. Random Experiment
2. Probability Space: Ω; $\quad \mathbb{P}[\omega] \in [0,1]$; $\quad \sum_{\omega} \mathbb{P}[\omega] = 1$
3. Uniform Probability Space: $\mathbb{P}[\omega] = 1/|\Omega|$, for all $\omega \in \Omega$
4. Event, "subset of outcomes": $A \subseteq \Omega$, and $\mathbb{P}[A] = \sum_{\omega \in A} \mathbb{P}[\omega]$

   $\Rightarrow$ On a uniform space (common!), just counting $|A|$ since $\mathbb{P}[A] = \frac{|A|}{|\Omega|}$
5. Examples!