Motivating Puzzle

Suppose we have two shopping centers.

Shopping center 1:

3 Indian restaurants
2 Moroccan restaurants

Shopping center 2:

1 Indian restaurant
1 Moroccan restaurant

Suppose we pick a center randomly with probability 50%, and then pick a restaurant at that center with equal probability.

- If we eat at a Moroccan restaurant, what is the chance we picked center 1?

Motivating Puzzle

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Shopping center 1:

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1 Indian restaurant
1 Moroccan restaurant

Suppose we pick a center randomly with probability 50%, and then pick a restaurant from the list with equal probability.

- If we eat at a Moroccan restaurant ,what's the chance we picked center 1?

Might be tempting to say that the chance is $2/3$ since 2 out of 3 Moroccan restaurants are in center 1, but this is wrong. We'll come back to this problem later...

Conditional Probability of a Sample

Lecture 17, CS70 Summer 2025

Fruit and Bins, Review #1: Chance of a Specific Outcome

Suppose we have 4 fruits that we randomly throw into 3 large bins, durian first, then peach, then orange, then strawberry.

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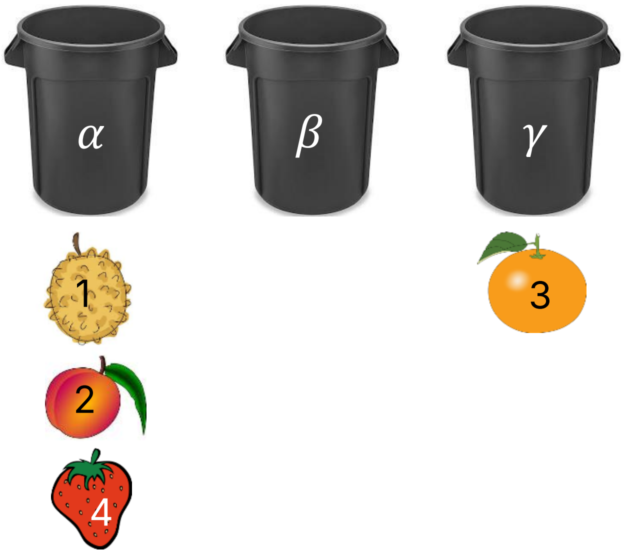
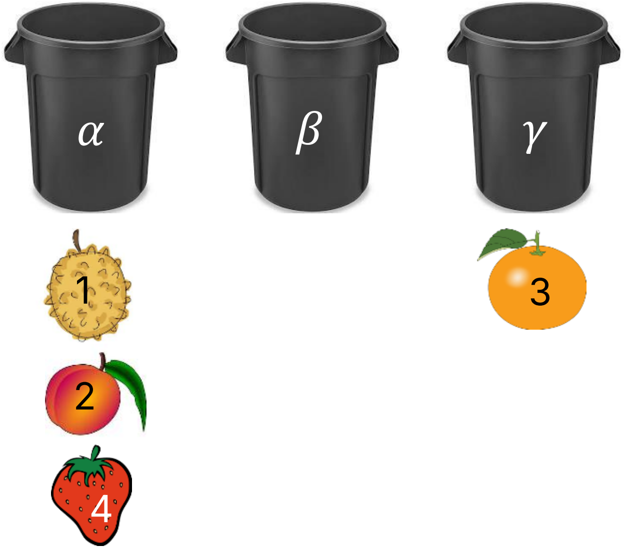
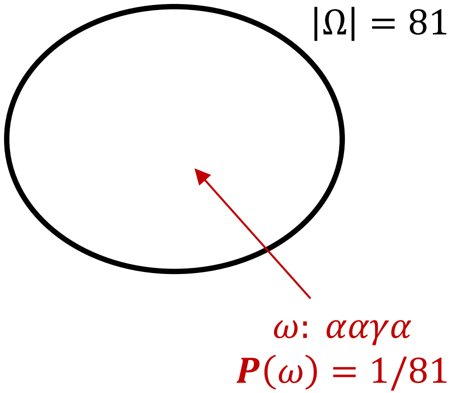


What is the chance that the durian goes in α, the peach goes in α, the orange goes in γ, and finally the strawberry goes in α, i.e., what is the probability that we get $\omega = \alpha\alpha\gamma\alpha$?

Fruit and Bins, Review #1: Chance of a Specific Outcome

Suppose we have 4 fruits that we randomly throw into 3 large bins, durian first, then peach, then orange, then strawberry.



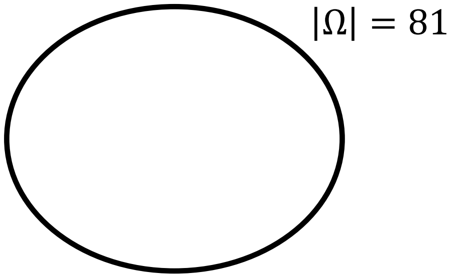


What is the chance that we get $\omega = \alpha\alpha\gamma\alpha$?

- This is sampling with replacement where order matters, i.e., total number of outcomes is $3^4 = 81$. Thus, chance is $1/81$.

Fruit and Bins, Review #2: Chance of an Event

Suppose we have 4 fruits that we randomly throw into 3 large bins, durian first, then peach, then orange, then strawberry.

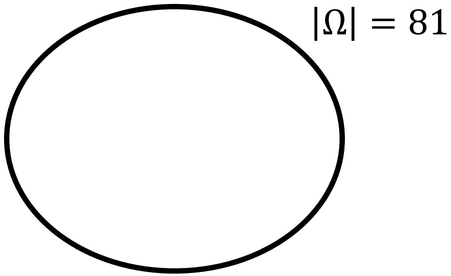


Let B be the event where second bin is empty, i.e., there are no βs.

What is $P(B)$?

Fruit and Bins, Review #2: Chance of an Event

Suppose we have 4 fruits that we randomly throw into 3 large bins, durian first, then peach, then orange, then strawberry.



Let B be the event where second bin is empty, i.e., there are no βs.

What is $P(B)$?

- Count $|B|$, the number of outcomes with no β and divide that by $|\Omega| = 3^4$.

Fruit and Bins, Review #2: Chance of an Event

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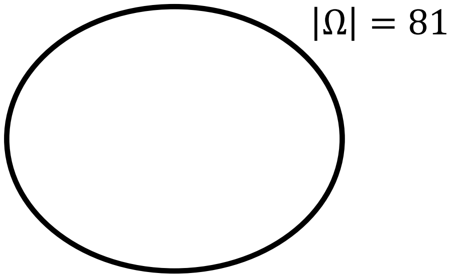


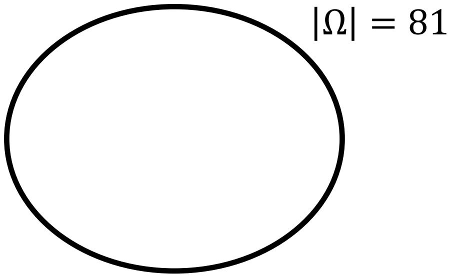
Let B be the event where second bin is empty, i.e., there are no βs.

What is $P(B)$?

- Count $|B|$, the number of outcomes with no β and divide that by $|\Omega| = 3^4$.
- $|B| = 2^4$ (this is the balls and bins problem), so $\mathrm{P(B)} = \frac{|\mathrm{B}|}{|\Omega|} = \frac{2^4}{3^4} = \frac{16}{81}$.

Fruit and Bins: Conditional Probability of an Outcome

Suppose we have 4 fruits that we randomly throw into 3 large bins, durian first, then peach, then orange, then strawberry.



What is the chance that we get ω=$\alpha\alpha\gamma\alpha$ if we somehow already know that there are no βs?

- As before, B be the event where there are no βs, and we know $|B| = 16$.

Fruit and Bins: Conditional Probability of an Outcome

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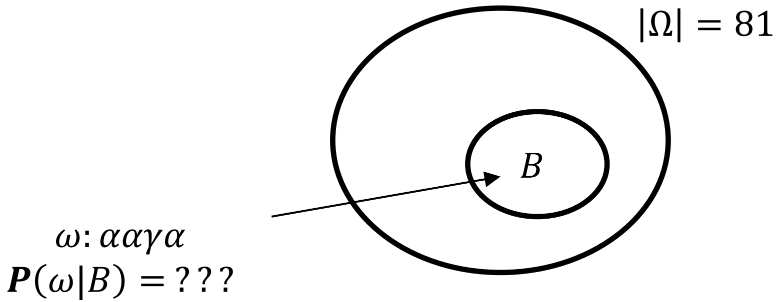

What is the chance that we get ω=$\alpha\alpha\gamma\alpha$ if we somehow already know that there are no βs?

- As before, B be the event where there are no βs, and we know $|B| = 16$.
- What is $\boldsymbol{P}(\omega = \alpha\alpha\gamma\alpha | B)$? Read this as the conditional probability of ω=$\alpha\alpha\gamma\alpha$, given that the event B is true. Here a vertical bar means "given that".

Fruit and Bins: Conditional Probability of an Outcome

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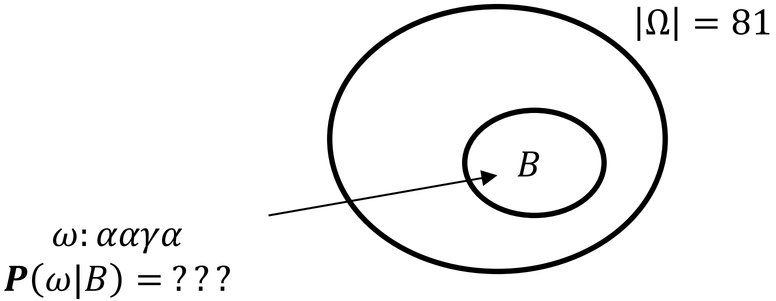


What is the chance that we get ω=$\alpha\alpha\gamma\alpha$ if we somehow already know that there are no βs?

- As before, B be the event where there are no βs, and we know $|B| = 16$.
- What is $\boldsymbol{P}(\omega=\alpha\alpha\gamma\alpha|B)$? Read this as the conditional probability of ω=$\alpha\alpha\gamma\alpha$, given that the event B is true. Here a vertical bar means "given that".
  - This is one out of 16 outcomes, so $\boldsymbol{P}(\omega=\alpha\alpha\gamma\alpha|B)$ is 1/16.

The Conditional Probability Formula

If ω is a single outcome in B, the following is always true:

$$P(\omega|B) = \frac{P(\omega)}{P(B)}$$

For our three bins example:

- $P(\omega) = 1/81$
- $P(B) = 16/81$
- $P(\omega|B) = \frac{1}{81} \div \frac{16}{81} = \frac{1}{16}$

The Conditional Probability Formula

If ω is a single outcome in B, the following is always true:

$$P(\omega|B) = \frac{P(\omega)}{P(B)}$$

What if $\omega \notin B$ – what is $P(\omega|B)$?

The Conditional Probability Formula

If ω is a single outcome in B, the following is always true:

$$P(\omega|B) = \frac{P(\omega)}{P(B)}$$

What if $\omega \notin B$ – what is $P(\omega|B)$? 0

Conditional Probability of an Event

Lecture 17, CS70 Summer 2025

Another Fruits and Bins Problem: Warmup #1

Again, suppose we have 4 fruits that we randomly throw into 3 large bins, durian first, then peach, then orange, then strawberry.

Let A be the event where the first bin contains at least 3 fruits. What is $P(A)$?

Another Fruits and Bins Problem: Warmup #1

Again, suppose we have 4 fruits that we randomly throw into 3 large bins, durian first, then peach, then orange, then strawberry.

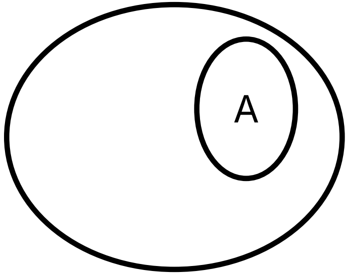
Let A be the event where the first bin contains at least 3 fruits. What is $P(A)$?

- This is just $\underbrace{8 \times 1/81}_{\text{Chance of 3 } \alpha\text{s and 1 something else.}} + \underbrace{1/81}_{\text{Chance of 4 } \alpha\text{s.}} = 9/81.$

Chance of 3 αs and 1 something else.

Chance of 4 αs.

$\beta\alpha\alpha\alpha$	$\gamma\alpha\alpha\alpha$	$\alpha\alpha\alpha\alpha$
$\alpha\beta\alpha\alpha$	$\alpha\gamma\alpha\alpha$	
$\alpha\alpha\beta\alpha$	$\alpha\alpha\alpha\gamma$	
$\alpha\alpha\alpha\beta$	$\alpha\alpha\gamma\alpha$	

$\Omega =$ 

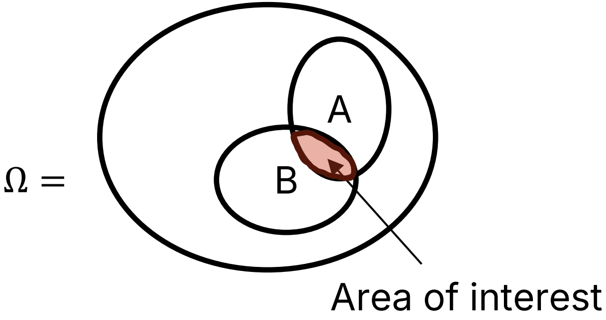
Another Fruits and Bins Problem: Warmup #2

Again, suppose we have 4 fruits that we randomly throw into 3 large bins, durian first, then peach, then orange, then strawberry.

Let A be the event where the first bin contains at least 3 fruits, and let B be the event where the second bin is empty. What is $P(A \cap B)$?

Hint: A is given below.

$\beta\alpha\alpha\alpha$	$\gamma\alpha\alpha\alpha$	$\alpha\alpha\alpha\alpha$
$\alpha\beta\alpha\alpha$	$\alpha\gamma\alpha\alpha$	
$\alpha\alpha\beta\alpha$	$\alpha\alpha\alpha\gamma$	
$\alpha\alpha\alpha\beta$	$\alpha\alpha\gamma\alpha$	



Another Fruits and Bins Problem: Warmup #2

Again, suppose we have 4 fruits that we randomly throw into 3 large bins, durian first, then peach, then orange, then strawberry.

Let A be the event where the first bin contains at least 3 fruits, and let B be the event where the second bin is empty. What is $P(A \cap B)$?

- 5/81

Answer: $A \cap B$ is A without the red items.

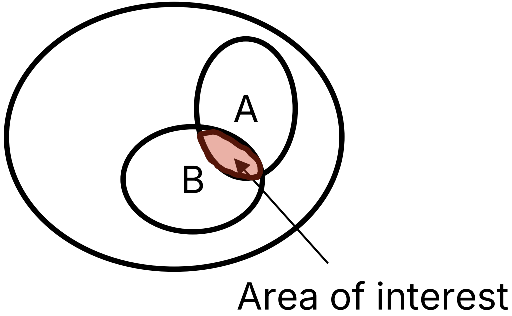
$\gamma\alpha\alpha\alpha$ $\quad$ $\alpha\alpha\alpha\alpha$

$\alpha\gamma\alpha\alpha$

$\alpha\alpha\alpha\gamma$

$\alpha\alpha\gamma\alpha$

$\Omega =$

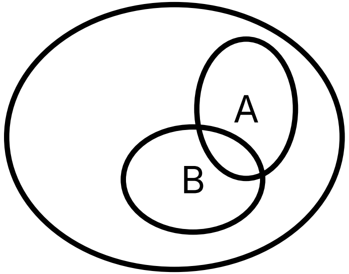


Another Fruits and Bins Problem: Summary So Far

Let A be the event where the first bin contains at least 3 fruits, and let B be the event where the second bin is empty. Things we know:

- $|\Omega| = 81$ $\quad P(\omega) = 1/81$
- $|A| = 9$ $\quad P(A) = 9/81$
- $|B| = 16$ $\quad P(B) = 16/81$ $\quad P(\omega|B) = 1/16$
- $|A \cap B| = 5$ $\quad P(A \cap B) = 5/81$

$\Omega =$



Another Fruits and Bins Problem: Conditional Probability of an Event

Let A be the event where the first bin contains at least 3 fruits, and let B be the event where the second bin is empty. Things we know:

- $|\Omega| = 81$ $\quad P(\omega) = 1/81$
- $|A| = 9$ $\quad P(A) = 9/81$
- $|B| = 16$ $\quad P(B) = 16/81$ $\quad P(\omega|B) = 1/16$
- $|A \cap B| = 5$ $\quad P(A \cap B) = 5/81$

Question: What is $P(A|B)$? Which is correct?

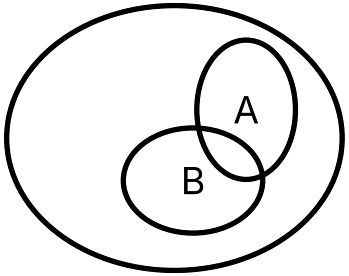
A. $P(A|B) = \sum_{\omega \in A} P(\omega|B)$

B. $P(A|B) = \sum_{\omega \in B} P(\omega|B)$

C. $P(A|B) = \sum_{\omega \in A \cup B} P(\omega|B)$

D. $P(A|B) = \sum_{\omega \in A \cap B} P(\omega|B)$

$\Omega =$



Another Fruits and Bins Problem: Conditional Probability of an Event

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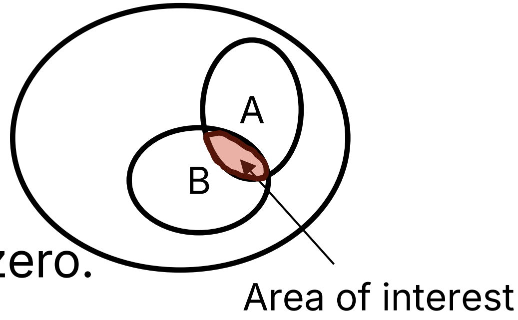
C. $P(A|B) = \sum_{\omega \in A \cup B} P(\omega|B)$

D. $P(A|B) = \sum_{\omega \in A \cap B} P(\omega|B)$

Technically, A and D are both correct.

- D is the best answer.
- Outcomes outside of B have probability $P(\omega|B)$ equal to zero.

$\Omega =$ 

Another Fruits and Bins Problem: Conditional Probability of an Event

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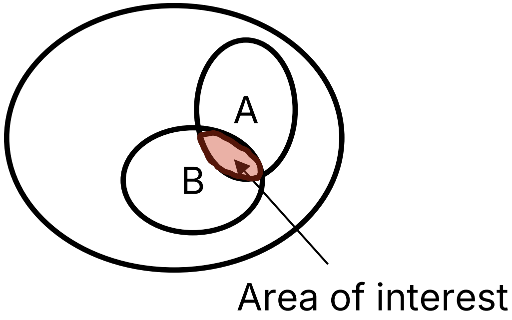
- $|A| = 9$ $\quad P(A) = 9/81$
- $|B| = 16$ $\quad P(B) = 16/81$ $\quad P(\omega|B) = 1/16$
- $|A \cap B| = 5$ $\quad P(A \cap B) = 5/81$

Since $P(A|B) = \sum_{\omega \in A \cap B} P(\omega|B)$

We have 5 outcomes in $A \cap B$, so $P(A|B) = 5 \times \frac{1}{16} = \frac{5}{16}$

$\Omega =$



Formula for Conditional Probability of Events

We usually write this equation $P(A|B) = \sum_{\omega \in A \cap B} P(\omega|B)$ differently.

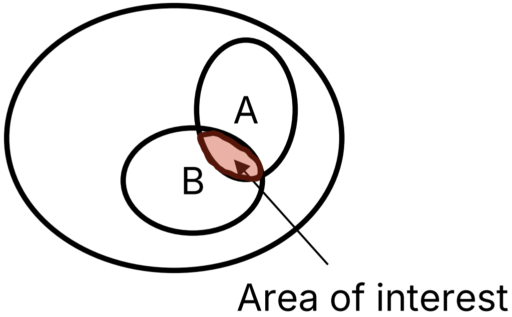
$$P(A|B) = \sum_{\omega \in A \cap B} P(\omega|B) = \sum_{\omega \in A \cap B} \frac{P(\omega)}{P(B)}$$

$$= \frac{\sum_{\omega \in A \cap B} P(\omega)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

Or simply:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$\Omega =$



Definition 14.1 from the Notes

For events $A, B \subseteq \Omega$ in the same probability space such that $P(B) > 0$, the conditional probability of A given B is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This expression is very common in the real world. Know it well.

Example Using Conditional Probability Formula: Dealing Cards

Suppose we are dealt an ace in cards and want to know the chance our next card will also be an ace. Here:

- B: Event that first card was an ace.
- A: Event that second card was an ace.

Can also reason directly: One ace is gone, so 3 out of 51 cards are aces, so $P(A|B) = 3/51$

Let's show that $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

- $P(B) = 4/52$ since 4 out of 52 cards is an ace.
- $P(A \cap B) = \frac{12}{52 \times 51}$ since there are 52×51 ways of drawing two cards, and $4 \times 3 = 12$ ways of getting two aces.
- Thus, chance that next card will be an ace is $\frac{12}{52 \times 51} / \frac{4}{52} = \frac{3}{51}$

Back to the Shopping Center Puzzle

Suppose we have two shopping centers.

Shopping center 1:

3 Indian restaurants
2 Moroccan restaurants

Shopping center 2:

1 Indian restaurant
1 Moroccan restaurant

Suppose we pick a center randomly with probability 50%, and then pick a restaurant from the list with equal probability.

- If we eat at a Moroccan restaurant ,what’s the chance we picked center 1?

Hint, let C_1 be the event we picked center 1, and C_2 be the event we picked center 2. Then:

$$P(C_1 | \bullet) = \frac{P(C_1 \cap \bullet)}{P(\bullet)}$$

Back to the Shopping Center Puzzle

Suppose we have two shopping centers.

Shopping center 1:

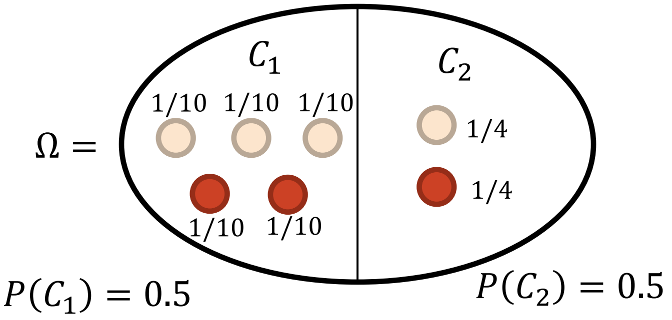
3 Indian restaurants ○○○
2 Moroccan restaurants ●●

Shopping center 2:

1 Indian restaurant ○
1 Moroccan restaurant ●

Suppose we pick a center randomly with probability 50%, and then pick a restaurant from the list with equal probability.

- If we eat at a Moroccan restaurant ●,what's the chance we picked center 1?



$$P(C_1 | ●) = \frac{P(C_1 \cap ●)}{P(●)}$$

$$= \frac{1/10 + 1/10}{1/10 + 1/10 + 1/4}$$

$$= \frac{2/10}{9/20} = \frac{4}{9}$$

Bayes Rule and the Total Probability Rule (TPR)

Lecture 17, CS70 Summer 2025

Bayesian Inference

Bayesian Inference: A technique for updating knowledge based on observation.

- Conditional probability is an important concept for making this possible.

In this context:

- $P(A)$: Assessment of the likelihood of A without making any observations, also known as a prior probability.
- $P(A|B)$: Assessment of the likelihood of A given the observation B, also known as a posterior probability. Includes the knowledge we gained from the observation.

Motivating Example: Coronavirus Testing

Let A be the event where a person is infected with the Coronavirus, and B be the event where a person tests positive on some test.

Suppose we know the following about the test:

- When taken by a person who is infected, comes up positive 90% of the time, and negative 10% of the time (false negatives).

If you test positive, what is the chance you are actually infected?

- Not enough information!

A: Infected
B: Tests Positive

In Terms of Conditional Probability

Let A be the event where a person is infected with the Coronavirus, and B be the event where a person tests positive on some test.

Suppose we know the following about the test:

- $P(B|A)$: When taken by a person who is infected, comes up positive 90% of the time, and negative 10% of the time (false negatives).

If you test positive, what is the chance you are actually infected? $P(A|B)$

- Not enough information! No reason that $P(A|B) = P(B|A)$

$$P(B|A) = 0.9$$

A: Infected
B: Tests Positive

What If We Know the False Positive Rate?

Suppose we have a test for Coronavirus where we know:

- $P(B|A)$: When taken by a person who is infected, comes up positive 90% of the time, and negative 10% of the time (false negatives).
- $P(B|\bar{A})$: When taken by a person who is not infected, comes up positive 20% of the time (false positives), and negative 80% of the time.

If you test positive, what is the chance you are actually infected?

- Not enough information!

$P(B|A) = 0.9$ $\quad$ $P(B|\bar{A}) = 0.2$

A: Infected
B: Tests Positive

Example: Coronavirus Testing

Suppose we have a test for Coronavirus where we know:

- $P(B|A)$: When taken by a person who is infected, comes up positive 90% of the time, and negative 10% of the time (false negatives).
- $P(B|\bar{A})$: When taken by a person who is not infected, comes up positive 20% of the time (false positives), and negative 80% of the time.

$P(A)$: Suppose that 5% of the people in a region currently have Coronavirus.

If you (in that region) test positive, what is the chance you are actually infected?

- Now we finally know enough! The answer, it may surprise you is 19%. Let's talk about how I computed that answer.

$P(A) = 0.05$ $P(B|A) = 0.9$ $P(B|\bar{A}) = 0.2$

A: Infected
B: Tests Positive

Example: Coronavirus Testing

Let A be the event where a person is infected and B be the event where a person tests positive. Then we know:

$$P(A) = 0.05 \qquad P(B|A) = 0.9 \qquad P(B|\bar{A}) = 0.2$$

Using this information, we want to compute $P(A|B)$, i.e., the chance that if a person tests positive that they have Coronavirus.

We know that $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

- Need to somehow compute these two quantities: $P(A \cap B)$ and $P(B)$!

A: Infected
B: Tests Positive

Example: Coronavirus Testing

Let A be the event where a person is infected and B be the event where a person tests positive. Then we know:

$$P(A) = 0.05 \qquad P(B|A) = 0.9 \qquad P(B|\bar{A}) = 0.2$$

Using this information, we want to compute $P(A|B)$, i.e., the chance that if a person tests positive that they have Coronavirus.

We know that $P(A|B) = \frac{P(A \cap B)}{P(B)}$. Computing $P(A \cap B)$ is easy:

- We know that $P(B|A) = \frac{P(A \cap B)}{P(A)}$
- Therefore, $P(A \cap B) = P(B|A)P(A) = 0.9 \times 0.05 = 0.045$

A: Infected
B: Tests Positive

Example: Coronavirus Testing

Let A be the event where a person is infected and B be the event where a person tests positive. Then we know:

$$P(A) = 0.05 \quad P(B|A) = 0.9 \quad P(B|\bar{A}) = 0.2 \quad P(A \cap B) = 0.045$$

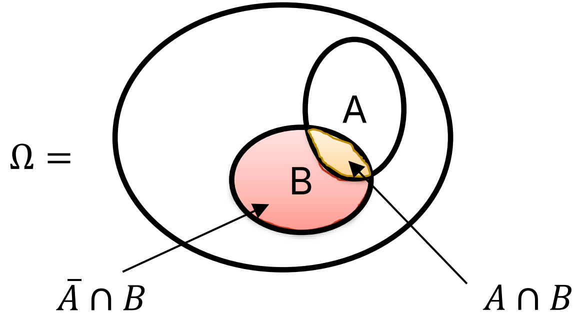
Using this information, we want to compute $P(A|B)$, i.e., the chance that if a person tests positive that they have Coronavirus.

We know that $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.045}{P(B)}$. Computing $P(B)$ is a little trickier:

- $P(B) = P(A \cap B) + P(\bar{A} \cap B)$
- $P(\bar{A} \cap B) = P(B|\bar{A})P(\bar{A})$

$$= P(B|\bar{A})\big(1 - P(A)\big)$$

$$= 0.2(1 - 0.05) = 0.19$$



Example: Coronavirus Testing

Let A be the event where a person is infected and B be the event where a person tests positive. Then we know:

$$P(A) = 0.05 \quad P(B|A) = 0.9 \quad P(B|\bar{A}) = 0.2 \quad P(A \cap B) = 0.045$$

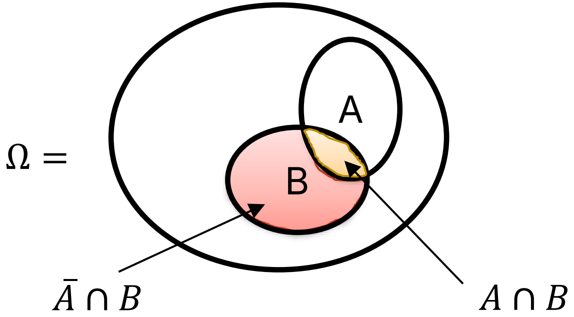
Using this information, we want to compute $P(A|B)$, i.e., the chance that if a person tests positive that they have Coronavirus.

We know that $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0.045}{P(B)}$. Computing $P(B)$ is a little trickier:

- $P(B) = P(A \cap B) + P(\bar{A} \cap B) = 0.045 + 0.19 = 0.235$
- $P(\bar{A} \cap B) = P(B|\bar{A})P(\bar{A})$

$$= P(B|\bar{A})\big(1 - P(A)\big)$$

$$= 0.2(1 - 0.05) = 0.19$$



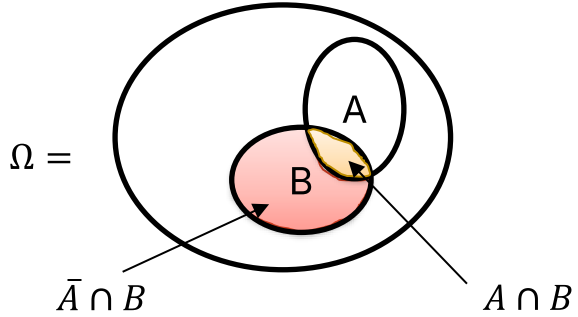
Example: Coronavirus Testing

Let A be the event where a person is infected and B be the event where a person tests positive. Then we know:

$$P(A) = 0.05 \quad P(B|A) = 0.9 \quad P(B|\bar{A}) = 0.2 \quad P(A \cap B) = 0.045$$

Using this information, we want to compute $P(A|B)$, i.e., the chance that if a person tests positive that they have Coronavirus.

We know that $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.045}{0.235} = 0.191$, i.e., 19%.



Example: Coronavirus Testing

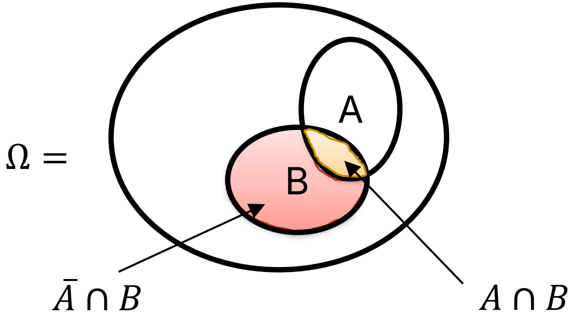
Let A be the event where a person is infected and B be the event where a person tests positive. Then we know:

$$P(A) = 0.05 \quad P(B|A) = 0.9 \quad P(B|\bar{A}) = 0.2 \quad P(A \cap B) = 0.045$$

Using this information, we want to compute $P(A|B)$, i.e., the chance that if a person tests positive that they have Coronavirus.

We know that $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.045}{0.235} = 0.191$, i.e., 19%.

In other words, even though the test has a 90% true positive rate and only a 20% false negative rate, if we get a positive result, we only have a 19% chance of having Coronavirus.



...Or Entirely Symbolically

We can also write what we've just done symbolically. Given $P(A)$, $P(B|A)$, $P(B|\bar{A})$, we can compute $P(A|B)$ as follows:

We know that $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(B|A)P(A)}{P(B)}$

$$= \frac{P(B|A)P(A)}{P(A \cap B) + P(\bar{A} \cap B)}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(\bar{A} \cap B)}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})\big(1 - P(A)\big)}$$

Note: Formulas are great. Memorizing formulas not so much. Remember where this comes from.

...Or Entirely Symbolically

We can also write what we've just done symbolically. Given $P(A)$, $P(B|A)$, $P(B|\bar{A})$, we can compute $P(A|B)$ as follows:

We know that $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)} = \dfrac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\bar{A})\big(1 - P(A)\big)}$

Example:

$P(A) = 0.05$ $\quad P(B|A) = 0.9$ $\quad P(B|\bar{A}) = 0.2$

$$P(A|B) = \frac{0.9 \times 0.05}{0.9 \times 0.05 + 0.2 \times (1 - 0.05)}$$

$$= 0.191$$

Bayes Rule and the Total Probability Rule

We've just derived two commonly used formulas:

Bayes Rule: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B)}$

Total Probability Rule: $P(B) = P(B|A)P(A) + P(B|\bar{A})\left(1 - P(A)\right)$

Note: this is just $P(\bar{A})$

Applications of Bayes and TPR

Lecture 17, CS70 Summer 2025

Example: Applying the Total Probability Rule

Suppose you are trying to enter a country and there are two immigration officers A and Nota. Let B be the event where you are allowed entry. Let A be the event where you are interviewed by officer A, and $\bar{A}$ is the event you are interview by Nota.

Suppose officer A will let you in with probability 20% and Nota with probability 70%. Suppose that you will be assigned officer A with probability 60% and officer Nota with probability 40%. What is $P(B)$?

$$P(B) =$$

$$P(B) = P(B|A)P(A) + P(B|\bar{A})\big(1 - P(A)\big)$$

$P(B|A) = 0.2$

$P(B|\bar{A}) = 0.7$

Example: Applying the Total Probability Rule

Suppose you are trying to enter a country and there are two immigration officers A and Nota. Let B be the event where you are allowed entry. Let A be the event where you are interviewed by officer A, and $\bar{A}$ is the event you are interview by Nota.

Suppose officer A will let you in with probability 20% and Nota with probability 70%. Suppose that you will be assigned officer A with probability 60% and officer Nota with probability 40%. What is $P(B)$?

$$P(B) = 0.2 \times 0.6 + 0.7 \times 0.4 = 0.4$$

$$P(B) = P(B|A)P(A) + P(B|\bar{A})\big(1 - P(A)\big)$$

$P(B|A) = 0.2$

$P(B|\bar{A}) = 0.7$

Shopping Center Puzzle Using Bayes Rule

Suppose we have two shopping centers.

Shopping center 1:

3 Indian restaurants
2 Moroccan restaurants

Shopping center 2:

1 Indian restaurant
1 Moroccan restaurant

Suppose we pick a center randomly with probability 50%, and then pick a restaurant from the list with equal probability.

- If we eat at a Moroccan restaurant ,what's the chance we picked center 1?

$$P(C_1|\bullet) = \frac{P(\bullet|\, C_1\,)P(C_1)}{P(\bullet)}$$

Shopping Center Puzzle Using Bayes Rule

Suppose we have two shopping centers.

Shopping center 1:

3 Indian restaurants ○ ○ ○
2 Moroccan restaurants ● ●

Shopping center 2:

1 Indian restaurant ○
1 Moroccan restaurant ●

Suppose we pick a center randomly with probability 50%, and then pick a restaurant from the list with equal probability.

- If we eat at a Moroccan restaurant ● ,what's the chance we picked center 1?

$$P(C_1 | \bullet) = \frac{P(\bullet | C_1)P(C_1)}{P(\bullet)}$$

$$= \frac{0.4 \times 0.5}{P(\bullet)}$$

Shopping Center Puzzle Using Bayes Rule

Suppose we have two shopping centers.

Shopping center 1:

3 Indian restaurants ○○○
2 Moroccan restaurants ●●

Shopping center 2:

1 Indian restaurant ○
1 Moroccan restaurant ●

Suppose we pick a center randomly with probability 50%, and then pick a restaurant from the list with equal probability.

- If we eat at a Moroccan restaurant ● ,what's the chance we picked center 1?

$$P(C_1|\bullet) = \frac{P(\bullet|\ C_1\ )P(C_1)}{P(\bullet)}$$

$$= \frac{0.4 \times 0.5}{P(\bullet)}$$

$$= \frac{0.4 \times 0.5}{0.4 \times 0.5 + 0.5 \times 0.5} = \frac{0.2}{0.20 + 0.25} = \frac{0.2}{0.45} = 4/9$$

Generalized Bayes and TPR

Lecture 17, CS70 Summer 2025

Multiple Events

Suppose we have more than 2 shopping centers:

Shopping center 1:

3 Indian restaurants
2 Moroccan restaurants

Shopping center 2:

1 Indian restaurant
1 Moroccan restaurant

Shopping center 3:

1 Indian restaurant
1 Moroccan restaurant

To handle such situations, let's define a "partition of an event".

We say that an event A is partitioned into n events $A_1, \dots, A_n$ if:

- $A = A_1 \cup A_2 \cup \cdots \cup A_n$
- $A_i \cup A_j = \emptyset$ for all $i \neq j$. That is, the events are mutually exclusive.

For the example, above we have three events, one for each shopping center.

Total Probability Rule

Our TPR (total probability rule) for events A and B was:

$$P(B) = P(B|A)P(A) + P(B|\bar{A})\big(1 - P(A)\big)$$

When considering a partition $A_1, \dots, A_n$ of the sample space Ω. The TPR for an event B is:

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

Question: Why are these two formulas basically saying the same thing?

Total Probability Rule

Partitions Ω into two events, $A_1 = A$ and $A_2 = \bar{A}$

Our TPR (total probability rule) for events A and B was:

$$P(B) = P(B|A)P(A) + P(B|\bar{A})\big(1 - P(A)\big)$$

When considering a partition $A_1, \dots, A_n$ of the sample space Ω. The TPR for an event B is:

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

Can also write this as:

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

Bayes Rule

For two events A and B, Bayes rule was:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

When considering a partition $A_1, \dots, A_n$ of the sample space Ω, Bayes Rule is:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$$

In both cases, you can naturally replace $P(B)$ by the TPR.

Bayes Rule

For two events A and B, Bayes rule was:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

When considering a partition $A_1, \dots, A_n$ of the sample space Ω, Bayes Rule is:

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B)}$$

In both cases, you can naturally replace $P(B)$ by the TPR.

- For partitions of space into A and $\bar{A}$, $\mathrm{P(B)} = P(B|A)P(A) + P(B|\bar{A})\big(1 - P(A)\big)$
- For partitions into $A_1, \dots, A_n$, $P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$

Independence

Lecture 17, CS70 Summer 2025

Independence

Earlier, we talked about how the probability of two events is sometimes the product of their individual probabilities.

For example, consider rolling a weighted four-sided die twice, where the probability of a 1 is $5/8$.

- Here, A_1 is the event that the first die is a 1, and we know that $P(A_1) = 5/8$.
- A_2 is the event that the second die is a 1, and we know that $P(A_2) = 5/8$.
- We said that the probability of both events happening was $P(A_1 \cap A_2) = \frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$.

What is notable about these two events is that they are independent!

- The outcome of each roll does not affect the other.

Independence More Formally

Two events A and B are said to be independent if:

$$P(A \cap B) = P(A) \times P(B)$$

Let's consider what this implies about $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \times P(B)}{P(B)} = P(A)$$

That is, knowing B tells us literally nothing new about A.

Mutual Independence

We can also define the mutual independence of more than 2 events. Let's start with 3 events: A_1, A_2, and A_3.

These 3 events are mutually independent if all of the following are true:

$$P(A_1 \cap A_2) = P(A_1) \times P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1) \times P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2) \times P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$$

Coin Flipping

Example: Imagine we flip a fair coin twice. Define the events below:

- A_1: First flip is heads.
- A_2: Second flip is heads.
- A_3: Both flips are the same.

Questions:

- Are these events pairwise independent?
- Are they mutually independent (satisfy all four constraints below)?

$$P(A_1 \cap A_2) = P(A_1) \times P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1) \times P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2) \times P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$$

Coin Flipping

Example: Imagine we flip a fair coin twice. Define the events below:

- A_1: First flip is heads.
- A_2: Second flip is heads.
- A_3: Both flips are the same.

Questions:

- Are these events pairwise independent? Yes!
  - Knowing first flip is heads says nothing about second being heads (and vice versa).
  - Knowing first flip heads says nothing about both flips being same (and v-v).
  - Knowing second flip heads says nothing about both flips being same (and v-v).
- Are they mutually independent (satisfy all four constraints)?

Coin Flipping

Example: Imagine we flip a fair coin twice. Define the events below:

- A_1: First flip is heads.
- A_2: Second flip is heads.
- A_3: Both flips are the same.

Questions:

- Are these events pairwise independent? Yes!
- Are they mutually independent (satisfy all four constraints)?
  - No. For example, if you know first and second flips are heads, you know that both flips are the same.

Generalizing to 4 Events

We can expand our definition of mutually independent. For four events, we'd have 11 constraints: Six pairwise, four three-way, one four-way.

$$P(A_1 \cap A_2) = P(A_1) \times P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1) \times P(A_3)$$
$$P(A_1 \cap A_4) = P(A_1) \times P(A_4)$$
$$P(A_2 \cap A_3) = P(A_2) \times P(A_3)$$
$$P(A_2 \cap A_4) = P(A_2) \times P(A_4)$$
$$P(A_3 \cap A_4) = P(A_3) \times P(A_4)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \times P(A_2) \times P(A_3)$$
$$P(A_1 \cap A_2 \cap A_4) = P(A_1) \times P(A_2) \times P(A_4)$$
$$P(A_1 \cap A_3 \cap A_4) = P(A_1) \times P(A_3) \times P(A_4)$$
$$P(A_2 \cap A_3 \cap A_4) = P(A_2) \times P(A_3) \times P(A_4)$$
$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1) \times P(A_2) \times P(A_3) \times P(A_4)$$

Generalizing to N Events

We can expand our definition of mutually independent. For N events, we have $2^n - n - 1$: $\binom{n}{2}$ pairwise, $\binom{n}{3}$ three-way, $\binom{n}{4}$ four-way,...

Can represent with compact notation that is a bit awkward. Define I to be the set of all subsets of $\{1, \ldots, n\}$ with size $|I| \geq 2$. Then:

$$P\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} P(A_i)$$

The notes give an even more awkward definition (definition 14.5) in terms of each event and its complement. We won't cover this. It represents the **exact** same idea.

Intersections and The Product Rule

Lecture 17, CS70 Summer 2025

Why Care About Intersections of Events?

In engineering, we often care about the probability of the intersection (or unions) of events. Example, if $A_1, \ldots A_n$ are all the things that have to go right, chance of success is $P(\bigcap_{i=1}^{n} A_i)$.

What if Events Aren't Independent?

Suppose we want to compute $P(\bigcap_{i=1}^{n} A_i)$, but the events are not independent?

- We can't just multiply $P(A)$ by $P(B)$.
- Example: A: Both heads, B: First coin heads. $P(A \cap B) \neq 1/4 \times 1/2$

If there are only two events, we can take advantage of our definition of conditional probability:

$$P(A \cap B) = P(A) \times P(B|A)$$

Coin example above:

- $P(A \cap B) = 1/4 \times 1$
- This is because $P(B|A) = 1$.

What if Events Aren't Independent?

Suppose we want to compute $P(\bigcap_{i=1}^{n} A_i)$, but the events are not independent?

- For two events: $P(A \cap B) = P(A) \times P(B|A)$

For events $A_1, \dots, A_n$:

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \times P(A_2|A_1) \times P(???) \times$$

What do you think is the next term?

What if Events Aren't Independent?

Suppose we want to compute $P(\bigcap_{i=1}^{n} A_i)$, but the events are not independent?

- For two events: $P(A \cap B) = P(A) \times P(B|A)$

For n events $A_1, \dots, A_n$, we have the Product Rule:

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots$$

Next term is $P(A_3|A_1 \cap A_2)$.

Why?

- $P(A_1) \times P(A_2|A_1) = P(A_1 \cap A_2)$
- $P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) = P(A_1 \cap A_2) \times P(A_3|A_1 \cap A_2) = P(A_1 \cap A_2 \cap A_3)$

What if Events Aren't Independent?

Suppose we want to compute $P(\bigcap_{i=1}^{n} A_i)$, but the events are not independent?

- For two events: $P(A \cap B) = P(A) \times P(B|A)$

For n events $A_1, \ldots, A_n$, we have the Product Rule:

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P\left(A_{n-1}\middle| \cap_{i=1}^{n-2} A_i\right) \times P\left(A_n\middle| \cap_{i=1}^{n-1} A_i\right)$$

If we keep going, we get the expression above.

This makes intuitive sense, but let's prove that it is true.

Product Rule Proof

For n events $A_1, \dots, A_n$, we have the Product Rule:

$$P\left(\bigcap_{i=1}^{n} A_i\right)$$
$$= P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_{n-1}| \cap_{i=1}^{n-2} A_i) \times P\left(A_n \middle| \cap_{i=1}^{n-1} A_i\right)$$

Proof by induction. Base Case: $n = 1$

- Trivially: $P(A_1) = P(A_1)$

Product Rule Proof

For n events $A_1, \ldots, A_n$, we have the Product Rule:

$$P\left(\bigcap_{i=1}^{n} A_i\right)$$
$$= P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_{n-1}| \cap_{i=1}^{n-2} A_i) \times P\left(A_n \middle| \cap_{i=1}^{n-1} A_i\right)$$

Proof by induction. Assume inductive hypothesis for $n - 1$ and prove it holds for n.

- Assume $P\left(\cap_{i=1}^{n-1} A_i\right) = P(A_1) \times P(A_2|A_1) \times \cdots \times P\left(A_{n-1}\middle| \cap_{i=1}^{n-2} A_i\right)$
- We know $P(\cap_{i=1}^{n} A_i) = P\left(A_n \cap \left(\cap_{i=1}^{n-1} A_i\right)\right)$
  - Why?

Product Rule Proof

For n events $A_1, \ldots, A_n$, we have the Product Rule:

$$P\left(\bigcap_{i=1}^{n} A_i\right)$$
$$= P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_{n-1}| \cap_{i=1}^{n-2} A_i) \times P\left(A_n \middle| \cap_{i=1}^{n-1} A_i\right)$$

Proof by induction. Assume inductive hypothesis for $n-1$ and prove it holds for n.

- Assume $P\left(\cap_{i=1}^{n-1} A_i\right) = P(A_1) \times P(A_2|A_1) \times \cdots \times P\left(A_{n-1} \middle| \cap_{i=1}^{n-2} A_i\right)$
- We know $P(\cap_{i=1}^{n} A_i) = P\left(A_n \cap \left(\cap_{i=1}^{n-1} A_i\right)\right)$

$$= P(A_n | \cap_{i=1}^{n-1} A_i) \times P\left(\cap_{i=1}^{n-1} A_i\right)$$

Why?

Product Rule Proof

For n events $A_1, \dots, A_n$, we have the Product Rule:

$$P\left(\bigcap_{i=1}^{n} A_i\right)$$

$$= P(A_1) \times P(A_2|A_1) \times P(A_3|A_1 \cap A_2) \times \cdots \times P(A_{n-1}| \cap_{i=1}^{n-2} A_i) \times P\left(A_n \middle| \cap_{i=1}^{n-1} A_i\right)$$

Proof by induction. Assume inductive hypothesis for $n-1$ and prove it holds for n.

- Assume $P\left(\cap_{i=1}^{n-1} A_i\right) = P(A_1) \times P(A_2|A_1) \times \cdots \times P\left(A_{n-1} \middle| \cap_{i=1}^{n-2} A_i\right)$
- We know $P(\cap_{i=1}^{n} A_i) = P\left(A_n \cap \left(\cap_{i=1}^{n-1} A_i\right)\right)$

$$= P(A_n| \cap_{i=1}^{n-1} A_i) \times P\left(\cap_{i=1}^{n-1} A_i\right)$$

$$= P(A_n| \cap_{i=1}^{n-1} A_i) \times P(A_1) \times P(A_2|A_1) \times \cdots \times P\left(A_{n-1} \middle| \cap_{i=1}^{n-2} A_i\right)$$

QED

Product Rule Applications

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Example One: Chance of a Flush in Poker

The chain rule makes some probability calculations easier. Consider trying to calculate the probability of a heart flush. Recall that a flush is all 5 cards having the same suit.

Example Flushes: {A♥, 2♥, 5♥, K♥, Q♥}   {5♥, A♥, 2♥, Q♥, K♥}

Using our new framework, if A is the event where we get a heart flush, and A_i is the event where the ith draw is a heart:

$$P(A) = P\left(\bigcap_{i=1}^{5} A_i\right)$$

Example One: Chance of a Flush in Poker

If A is the event where we get a heart flush, and A_i is the event where the ith draw is a heart:

$$P(A) = \bigcap_{i=1}^{5} A_i$$

We can write this using the product rule as:

$$P(A) = P(A_1) \times P(A_2|A_1) \times \cdots \times P\left(A_5 \middle| \cap_i^4 A_i\right)$$

We have that $P(A_1) = \frac{13}{52} = \frac{1}{4}$.

What about $P(A_2|A_1)$?

Example Flushes: {A♥, 2♥, 5♥, K♥, Q♥}   {5♥, A♥, 2♥, Q♥, K♥}

Example One: Chance of a Flush in Poker

If A is the event where we get a heart flush, and A_i is the event where the i[th] draw is a heart:

$$P(A) = \bigcap_{i=1}^{5} A_i$$

We can write this using the product rule as:

$$P(A) = P(A_1) \times P(A_2|A_1) \times \cdots \times P\left(A_5 \middle| \cap_i^4 A_i\right)$$

We have that $P(A_1) = \frac{13}{52} = \frac{1}{4}$.

What about $P(A_2|A_1)$? $P(A_2|A_1) = \frac{12}{51}$

Example Flushes: {A❤, 2❤, 5❤, K❤, Q❤} {5❤, A❤, 2❤, Q❤, K❤}

Example One: Chance of a Flush in Poker

If A is the event where we get a heart flush, and A_i is the event where the i[th] draw is a heart:

$$P(A) = \bigcap_{i=1}^{5} A_i$$

We can write this using the product rule as:

$$P(A) = P(A_1) \times P(A_2|A_1) \times \cdots \times P\left(A_5 \middle| \cap_i^4 A_i\right)$$

$$P(A) = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} \times \frac{10}{49} \times \frac{9}{48} = 0.005$$

Easier than counting number of flushes and dividing by number of hands (?)

Example Flushes: {A♥, 2♥, 5♥, K♥, Q♥} {5♥, A♥, 2♥, Q♥, K♥}

Example Two: Monty Hall

Can think of Monty Hall in terms of the product rule.

Let:

- C_i be the event that contestant chooses door i.
- P_i be the event that the prize is behind door i.
- H_i be the event that the host shows the goat behind door i.

Example Two: Monty Hall

Can think of Monty Hall in terms of the product rule.

Let:

- C_i be the event that contestant chooses door i.
- P_i be the event that the prize is behind door i.
- H_i be the event that the host shows the goat behind door i.

Example: $P(C_1 \cap P_2 \cap H_3) = P(C_1) \times P(P_2|C_1) \times P(H_3|C_1 \cap P_2)$

- $P(C_1) = \frac{1}{3}$

Example Two: Monty Hall

Can think of Monty Hall in terms of the product rule.

Let:

- C_i be the event that contestant chooses door i.
- P_i be the event that the prize is behind door i.
- H_i be the event that the host shows the goat behind door i.

Example: $P(C_1 \cap P_2 \cap H_3) = P(C_1) \times P(P_2|C_1) \times P(H_3|C_1 \cap P_2)$

- $P(C_1) = \frac{1}{3}$, $P(P_2|C_1) = \frac{1}{3}$

Example Two: Monty Hall

Can think of Monty Hall in terms of the product rule.

Let:

- C_i be the event that contestant chooses door i.
- P_i be the event that the prize is behind door i.
- H_i be the event that the host shows the goat behind door i.

Example: $P(C_1 \cap P_2 \cap H_3) = P(C_1) \times P(P_2|C_1) \times P(H_3|C_1 \cap P_2)$

- $P(C_1) = \frac{1}{3}$, $P(P_2|C_1) = \frac{1}{3}$, $P(H_3|C_1 \cap P_2) = 1$
- Thus, $P(C_1 \cap P_2 \cap H_3) = 1/9$

See notes for more exploration of this idea.

Example Three: Coin Tosses

As an example of a degenerate case, let's consider 3 independent coin tosses using the product rule.

Let A be the event that all three tosses are heads.

$$P(A) = P(A_1) \times$$

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Example Three: Coin Tosses

As an example of a degenerate case, let's consider 3 independent coin tosses using the product rule.

Let A be the event that all three tosses are heads.

$$P(A) = P(A_1) \times P(A_2|A_1) \times P(A_3|A_2 \cap A_1)$$

$$= P(A_1) \times P(A_2) \times P(\mathrm{A}_3)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{1}{8}$$

Unions of Events

Lecture 17, CS70 Summer 2025

The Three Four-Sided Dice Game

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.

Claim: The probability of winning is 75%

- $P(A) = P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) = 3 \times \frac{1}{4} = \frac{3}{4}$

Is this claim correct?

The Three Four-Sided Dice Game

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- $P(A) = P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) = 3 \times \frac{1}{4} = \frac{3}{4}$

Is this claim correct? No!

- Imagine we roll four fair four-sided dice.
- This argument would imply the chance of winning is $4 \times \frac{1}{4} = 1$
- Or that the probability of winning with 6 dice is 1.5.

Overlapping Events

The issue is that the events are not disjoint.

- Two events are disjoint if they share no sample points.

Consider the sample space for three die four-sided die rolls:

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	441	442	443	444

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A_1 is the event where first die is 4.

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$$P(A_1 \cup A_2) \neq P(A_1) + P(A_2)$$

Some points are in both events, so adding probabilities double counts some of the points!

A_1 is the event where first die is 4.

A_2 is the event where second die is 4.

Solution: Use Inclusion-Exclusion Formula from Counting Lecture

We'll need to use the inclusion-exclusion formula from the counting lecture.

Reminder, this formula told us about the size of the union of arbitrary subsets of the same finite set A. Specifically:

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\}:|S|=k} \left| \bigcap_{i \in S} A_i \right|$$

Reminder: This big scary formula is very intuitive.

- Won't cover from scratch today – review counting lecture
  - Recall that for $n = 2$: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Using the Principle of Inclusion-Exclusion in the Context of Probability

The PIE formula lets us count the size of the intersection of several potentially overlapping subsets:

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\}:|S|=k} \left| \bigcap_{i \in S} A_i \right|$$

Naturally, we can use this to compute the probability of the intersection of events.

$$P(|A_1 \cup \cdots \cup A_n|) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\}:|S|=k} P\left(\left| \bigcap_{i \in S} A_i \right| \right)$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$$

$$-P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3)$$

$$+P(A_1 \cap A_2 \cap A_3)$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- What is $P(A_1 \cap A_2)$?

$$P(A_1 \cup A_2 \cup A_3) = 1/4 + 1/4 + 1/4$$

$$-P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3)$$

$$+P(A_1 \cap A_2 \cap A_3)$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- What is $P(A_1 \cap A_2)$? Probability of getting two fours on independent die rolls, so $1/4 \times 1/4 = 1/16$

$$P(A_1 \cup A_2 \cup A_3) = 1/4 + 1/4 + 1/4$$

$$-1/16 - 1/16 - 1/16$$

$$+P(A_1 \cap A_2 \cap A_3)$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- What is $P(A_1 \cap A_2 \cap A_3)$?

$$
\begin{aligned}
P(A_1 \cup A_2 \cup A_3) &= 1/4 + 1/4 + 1/4 \\
&\quad -1/16 - 1/16 - 1/16 \\
&\quad +P(A_1 \cap A_2 \cap A_3)
\end{aligned}
$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- What is $P(A_1 \cap A_2 \cap A_3)$? $1/64$

$$P(A_1 \cup A_2 \cup A_3) = 1/4 + 1/4 + 1/4$$

$$-1/16 - 1/16 - 1/16$$

$$+1/64$$

Using the PIE for Three Fair Four-Sided Dice

Suppose we roll three fair four-sided dice.

- We win if we get at least one 4.
- Overall probability is: $\frac{3}{4} - \frac{3}{16} + \frac{1}{64} = \frac{48}{64} - \frac{12}{64} + \frac{1}{64} = \frac{37}{64}$

$$P(A_1 \cup A_2 \cup A_3) = 1/4 + 1/4 + 1/4$$
$$-1/16 - 1/16 - 1/16$$
$$+1/64$$

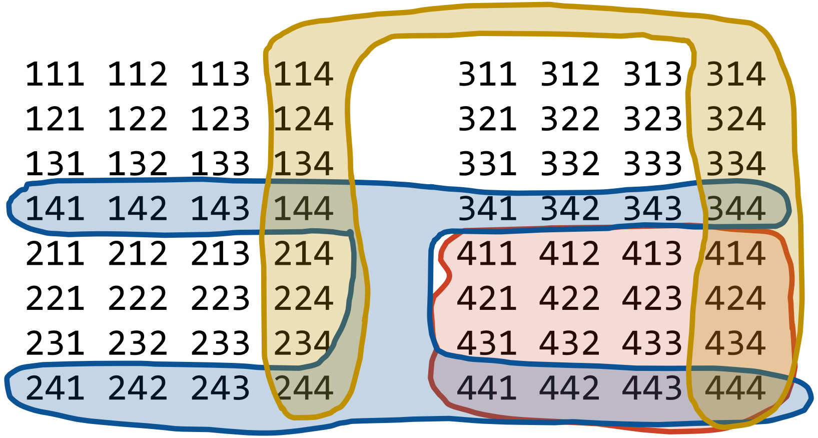
This all worked because $P(\bigcap_i A_i) = \prod_i P(A_i)$ for independent events, which let us easily compute values like $P(A_1 \cap A_2 \cap A_3)$.

Visualization of our Usage of the PIE

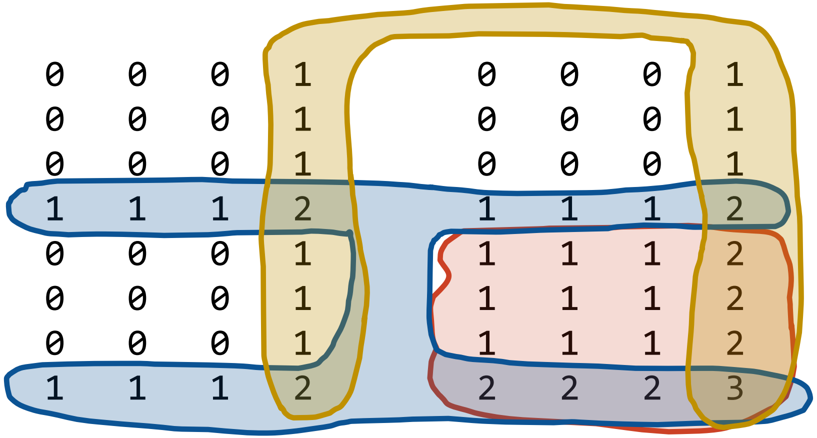
Let’s see a visual picture.

First, we added $P(A_1) + P(A_2) + P(A_3)$, giving us $48/64$.

- However, we’ve double counted and triple counted some points!



Sample points

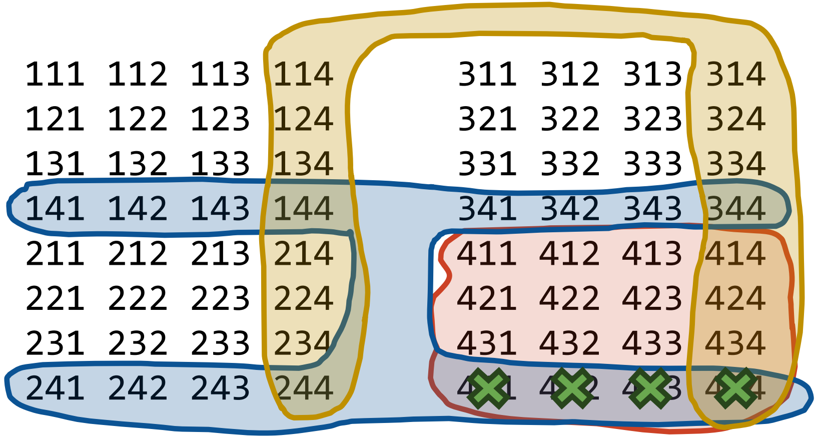


Times each point is counted

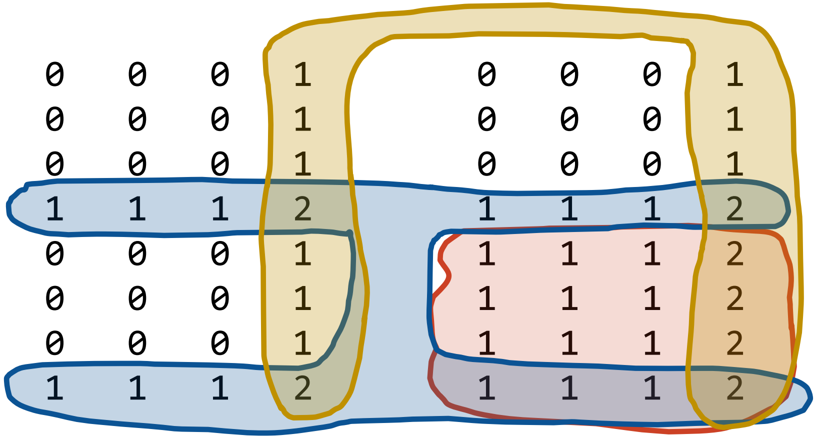
Visualization of our Usage of the PIE

First we added $P(A_1) + P(A_2) + P(A_3)$, giving us $48/64$.

- So then we subtracted $P(A_1 \cap A_2) = 4/64$ to avoid double counting $A_1 \cap A_2$.



Sample points



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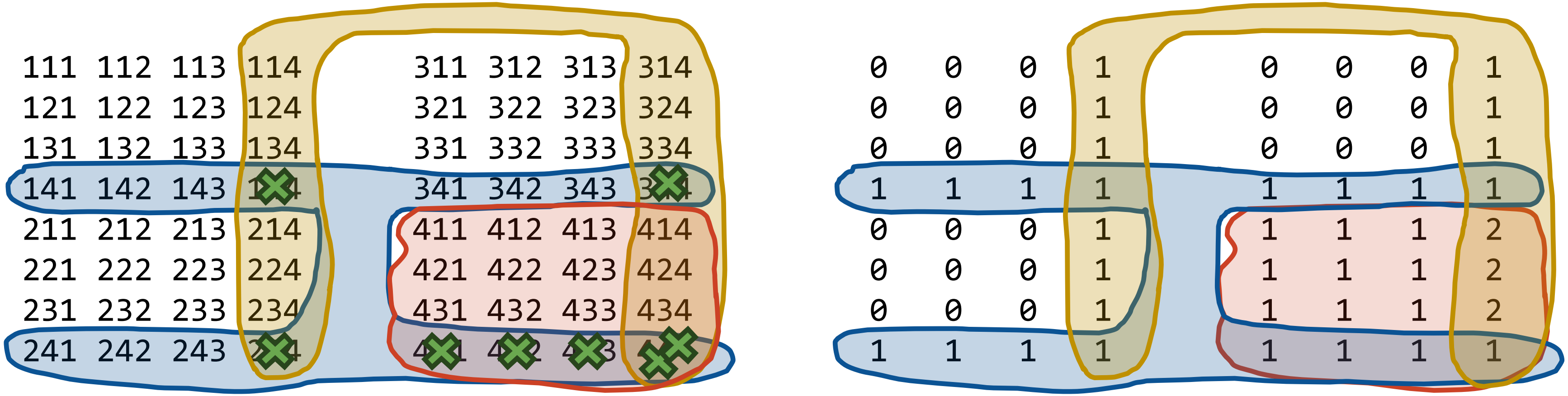
- Then we subtracted $P(A_2 \cap A_3) = 4/64$ to avoid double counting $A_2 \cap A_3$.



Sample points

Times each point is counted

Note the bottom right point is now added three times and subtracted two times.

Visualization of our Usage of the PIE

First we added $P(A_1) + P(A_2) + P(A_3)$, giving us $48/64$.

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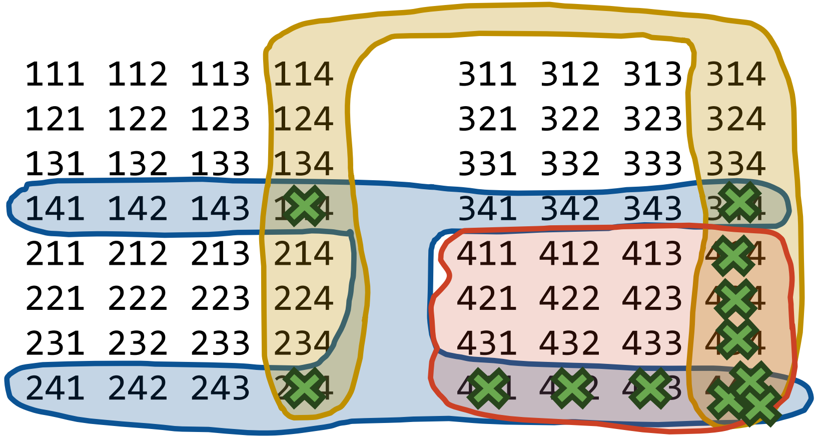
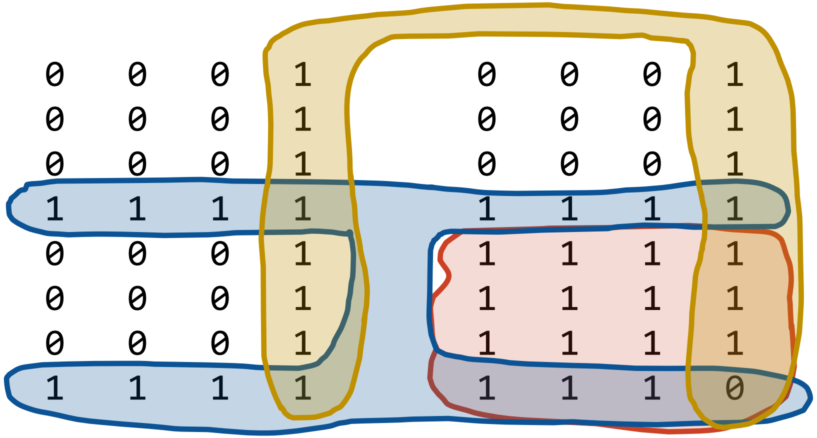


Sample points



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Visualization of our Usage of the PIE

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- So then we subtracted $P(A_1 \cap A_2) = 4/64$ to avoid double counting $A_1 \cap A_2$.
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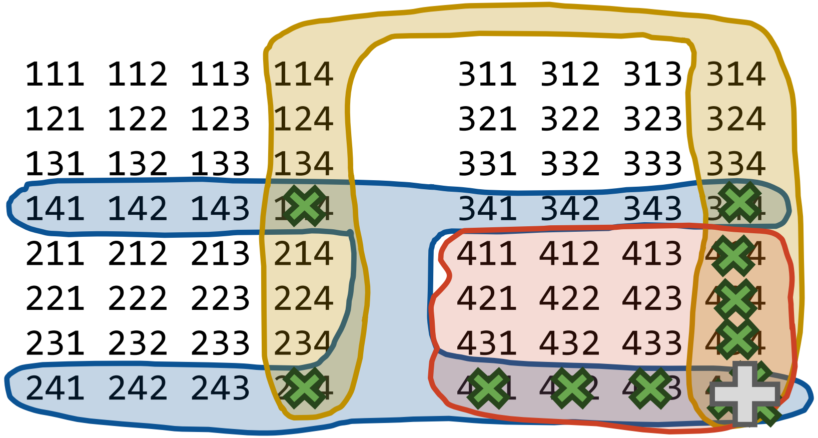

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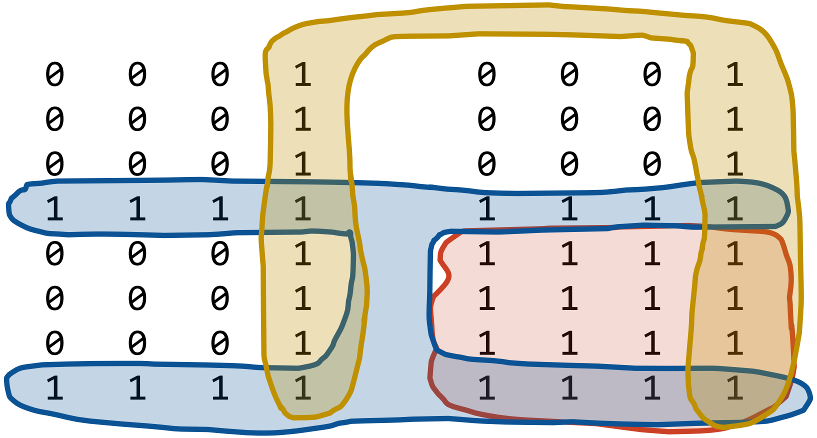

Times each point is counted

Lastly, we add back in $P(A_1 \cap A_2 \cap A_3)$, giving us $\frac{48-4-4-4+1}{64} = \frac{37}{64}$

Exercise: Union of Independent Events

Just like the notes, I've gone through and computed $P(A_1 \cup A_2 \cup A_3)$ using the PIE.

$$
\begin{aligned}
P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2) + P(A_3) \\
&\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\
&\quad + P(A_1 \cap A_2 \cap A_3)
\end{aligned}
$$

The notes don't mention it, but there is a much simpler way to compute this union that takes more direct advantage of the fact that these three events are independent.

- See if you can figure it out.

Unions of Events (Large N)

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The Principle of Inclusion/Exclusion

Mathematically, we can use PIE to compute the probability of intersections.

Practically, this becomes challenging even for modestly sized values of n.

Example: Suppose we have 100 events and want to compute the probability of their union.

- Have to add (or subtract) every pairwise, three-way, four-way, etc. probability.
- From the counting lecture, we know there are $2^n - 1$ such terms, because $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1$
- 2^{100} is too long to wait for an answer
  - Note: Age of the universe is $\approx 2^{59}$ seconds

Inclusion-Exclusion: Mutually Exclusive Events

This isn't a problem if our events are mutually exclusive, i.e., $P(A_i \cap A_j) = 0$ for all $i \neq j$, or equivalently $A_i \cap A_j = \emptyset$ for all $i \neq j$.

We've actually already used this fact before.

Example: Computing the probability of a flush.

- A_1: ♠ flush, A_2: ♥ flush, A_3: ♦ flush, A_4: ♣ flush
- Flushes in different suits are clearly disjoint!
  - $P(A_1 \cup A_2 \cup A_3 \cup A_4) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$

Inclusion-Exclusion: Approximation

For non-mutually exclusive events, we can also truncate the summation:

$$P(|A_1 \cup \cdots \cup A_n|) = \sum_{k=1}^{n_s} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\}:|S|=k} P\left(\left|\bigcap_{i \in S} A_i\right|\right)$$

Let $n_s < n$ be the maximum value of k through which we sum.

- If n_s is odd, the computed value is an overestimate.
- If n_s is even, the computed value is an underestimate.

Example: For our dice game with 3 four sided-dice:

- $n_s = 1$, $P(|A_1 \cup A_2 \cup A_3|) \approx P(A_1) + P(A_2) + P(A_3)$ [overestimate]
- $n_s = 2$, $P(|A_1 \cup A_2 \cup A_3|) \approx P(A_1) + P(A_2) + P(A_3)$
  $-P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_1 \cap A_3)$
  [underestimate]

Inclusion-Exclusion: Approximation

For non-mutually exclusive events, we can also truncate the summation:

$$P(|A_1 \cup \cdots \cup A_n|) = \sum_{k=1}^{n_s} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\}:|S|=k} P\left(\left|\bigcap_{i \in S} A_i\right|\right)$$

Let $n_s < n$ be the maximum value of k through which we sum.

- If n_s is odd, the computed value is an overestimate.
- If n_s is even, the computed value is an underestimate.

As n_s grows, the quality of our estimate gets better.

Inclusion-Exclusion: The Union Bound

The union bound (also called Boole's inequality) comes from a special case of our approximation, specifically the case where $n_s = 1$.

- Example, for the dice game, just summing $P(A_1) + P(A_2) + P(A_3)$.

The union bound states:

$$P(|A_1 \cup \cdots \cup A_n|) \leq \sum_{i=1}^{n} P(A_i)$$

Proving this is given as an exercise in the notes.

- Can also go further and prove that as you increase n_s, you switch between overestimates to underestimates, and they just keep getting better.

Summary

Lecture 17, CS70 Summer 2025

Summary of the Lecture 17

We covered three distinct topics:

- Defining the conditional probability of outcomes and events.
- Bayesian inference:
  - Especially important: Total Probability Rule and Bayes Rule
- Combinations of events
  - Easy when the events are independent.
  - Intersections of non-independent events: Use product rule.
  - Unions of non-independent events: Can use principle of inclusion/exclusion, though it blows up exponentially. Truncate for an approximation.
    - Unions of independent events: Didn't explicitly discuss, but these are easy to compute without the PIE. See the slide "Exercise: Union of Independent Events"