Definition of a Random Variable

Lecture 19, CS70 Summer 2025

Measure Experiment Outcomes

Sample spaces can be big – three 4-sided die rolls:

111	112	113	114	311	312	313	314
121	122	123	124	321	322	323	324
131	132	133	134	331	332	333	334
141	142	143	144	341	342	343	344
211	212	213	214	411	412	413	414
221	222	223	224	421	422	423	424
231	232	233	234	431	432	433	434
241	242	243	244	441	442	443	444

64 different outcomes

... and overly detailed. Often interested in some measurement on the outcome.

Example: **How many** 4s were rolled?

111 → 0   142 → 1

434 → 2   444 → 3

4 possible values of interest (0..3)

Measure Experiment Outcomes

Sample spaces can be big – handing papers back randomly to 100 students:

```
23, 58, 92, 37, 21, … , 42
72, 2, 28, 33, 91, … , 78
72, 2, 15, 4, 82, … , 99
1, 2, 3, 4, 5, 6, … , 100
…
```

$100! \approx 9.33 \times 10^{157}$ different outcomes

... and overly detailed. Often interested in some measurement on the outcome.

Example: **How many** students received their own paper?

```
23, 58, 92, 37, 21, … , 42 → 0
72, 2, 28, 33, 91, … , 78 → 1
72, 2, 15, 4, 82, … , 99 → 2
1, 2, 3, 4, 5, 6, … , 100 → 100
```

101 possible values of interest (0..100)

Random Variables: Informally

Idea: Define a variable to represent the measurement/value of interest.

Suppose we flip a fair coin 4 times. Let X be the number of heads we see.

What is X?

- Given a specific outcome: A value between 0 and 3.
- In general: Some indeterminate value between 0 and 3.

What is $X + 3$?

- Some indeterminate value between 3 and 7.

What is $X - X$?

- 0   (note: repeated use of X means measured on the *same outcome*)

What is X^2?

- Some indeterminate value from {0, 1, 4, 9, 16}.

Random Variables: Informally

Suppose we take homework from 100 students and randomly give each student one homework. Let X be the number of students who get their own homework.

What is X?

- Some indeterminate value between 0 and 100

Is X more likely to be 1 or 3?

- Much more likely to be 1

What is X on average? (we need to define what "on average" means....)

- This turns out to be 1

Random Variables: Informally

Suppose we flip a fair coin 4 times. Let X be the number of heads we see.

- Consider all outcomes – each results in a specific value for X:

ω	X
TTTT	0
HTTT, THTT, TTHT, TTTH	1
HHTT, HTHT, HTTH, THHT, THTH, TTHH	2
HHHT, HHTH, HTHH, THHH	3
HHHH	4

Random Variable: Formally

A Random Variable X on a sample space is a function that maps $\Omega \to \mathbb{R}$, i.e., $X(\omega)$ is a real number for every $\omega \in \Omega$.

$\omega \in \Omega$ $\quad$ $0 \in \mathbb{R}$

Example, for coin flips: $X(TTTT) = 0$, $\quad X(HHTH) = 3$, $\quad$ ...

ω	X
TTTT	0
HTTT, THTT, TTHT, TTTH	1
HHTT, HTHT, HTTH, THHT, THTH, TTHH	2
HHHT, HHTH, HTHH, THHH	3
HHHH	4

So despite the name, a random variable is really a function.

Formal Random Variable Example 2

Imagine rolling two six sided dice. Let X be their sum.

- $X\big((1, 1)\big) = 2$

6						
5						
4						
3						
2						
1	2					
	1	2	3	4	5	6

Formal Random Variable Example 2

Imagine rolling two six-sided dice. Let X be their sum.

- $X\big((1,2)\big) = 3$ and $X\big((2,1)\big) = 3$

6						
5						
4						
3						
2	3					
1	2	3				
	1	2	3	4	5	6

Formal Random Variable Example 2

Imagine rolling two six-sided dice. Let X be their sum.

- The domain of X is the set of all tuples of integers (i, j) where $1 \leq i, j \leq 6$.
- The range of X is $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

Formal Random Variable Example 3

Consider the sample space of all sequences of coin flips consisting of a run of tails followed by one heads.

Let X be the number of flips until we get our first heads.

- $X(H) = 1$
- $X(TH) = 2$
- $X(TTTTTH) = 6$

The domain of X is infinitely large, it's any number of tails followed by heads.

The range of X is $\mathbb{N}^+$.

Note engineering application: "Heads" means system failure – X is time to failure....

Random Variables and Events

Lecture 19, CS70 Summer 2025

Random Variables and Events

Let y be any number in the range of a random variable X. Consider the set:

$$A_y = \{\omega \in \Omega : X(\omega) = y\}$$

Observation: This set is an event in the sample space, because it a subset of Ω.

Sometimes called the “pre-image of y”

Random Variables and Events

Let y be any number in the range of a random variable X. Consider the set:

$$A_y = \{\omega \in \Omega : X(\omega) = y\}$$

6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

Observation: This set A_y is an event in the sample space, because it a subset of Ω.

Example for two dice:

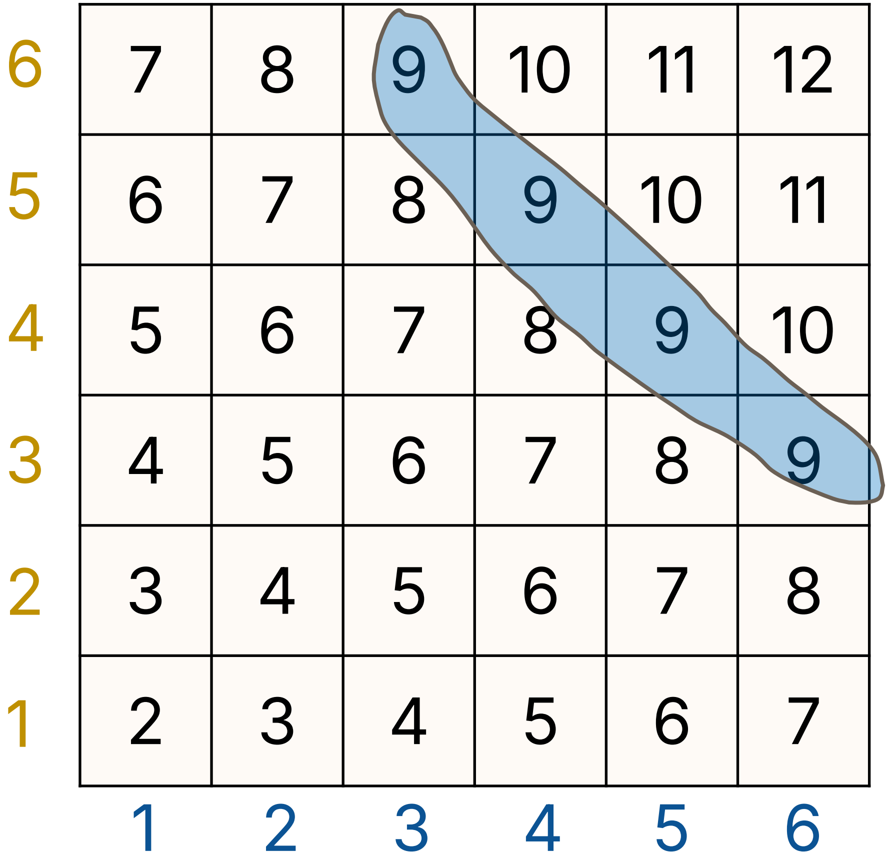
- $A_9 = \{\omega \in \Omega : X(\omega) = 9\}$

Samples in this event: (6, 3), (5, 4), (4, 5), (3, 6)

Random Variables and Events

Let a be any number in the range of a random variable X. Consider the set:

$$A_y = \{\omega \in \Omega : X(\omega) = y\}$$



Observation: This set A_y is an event in the sample space, because it a subset of Ω.

Example for two dice:

- $A_9 = \{\omega \in \Omega : X(\omega) = 9\}$

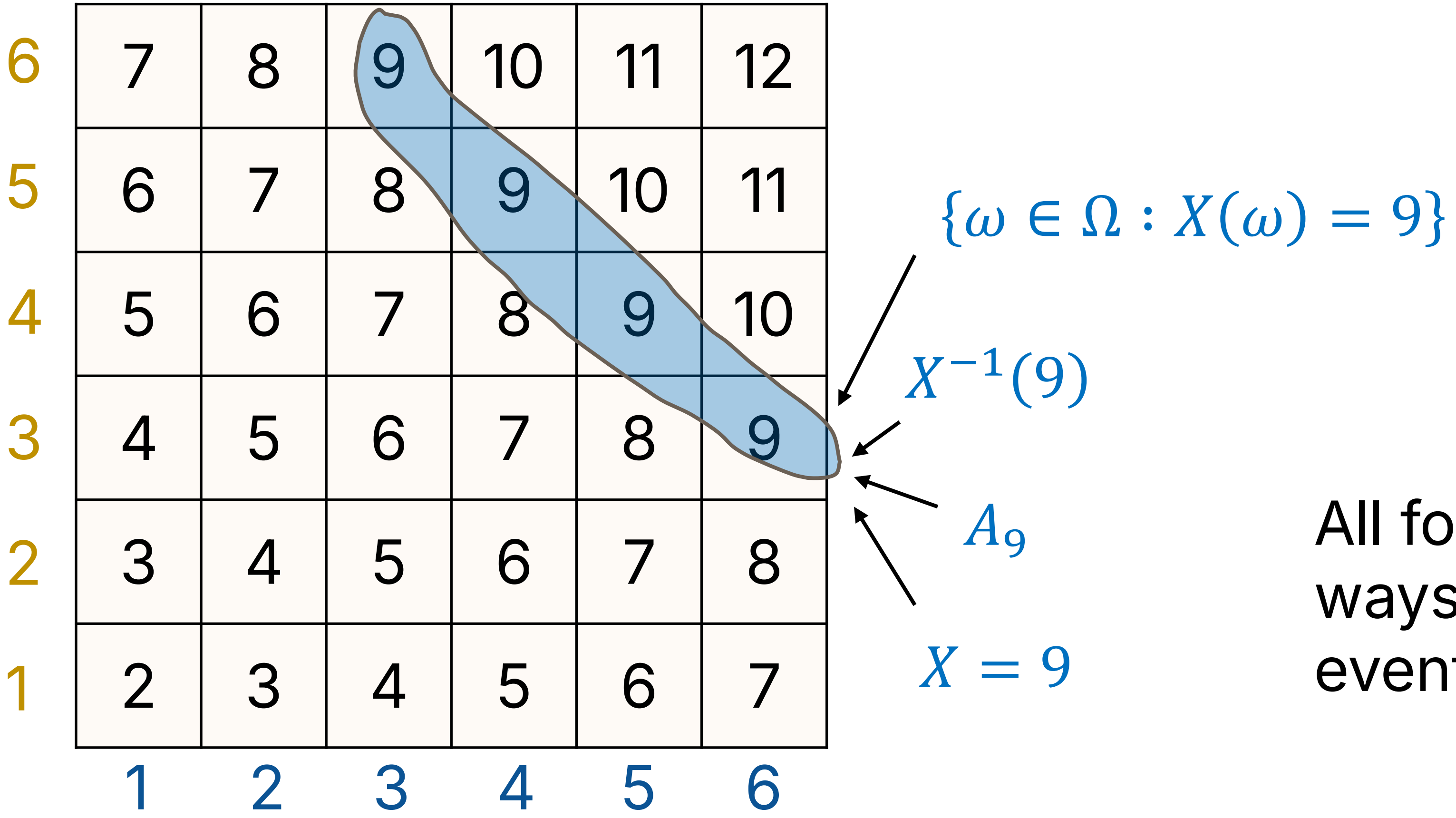
Other ways to "name" this event:

- $X^{-1}(9)$
- $X = 9$ (we'll use this in our class)

Random Variables and Events

Let a be any number in the range of a random variable X. Consider the set:

$$A_y = \{\omega \in \Omega : X(\omega) = y\}$$

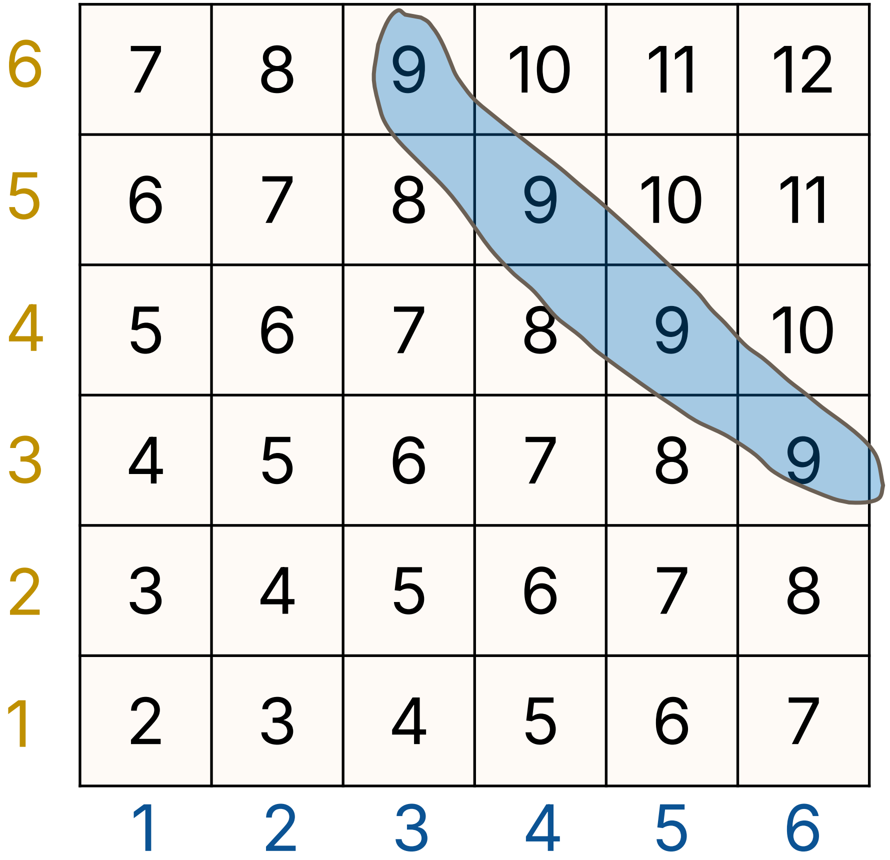


All four of these are different ways of referring to the same event.

Random Variables and Events

Let a be any number in the range of a random variable X. Consider the set:

$$A_y = \{\omega \in \Omega : X(\omega) = y\}$$



Since this is an event, we can also ask for the probability of this event, e.g. What is $P(X = a)$?

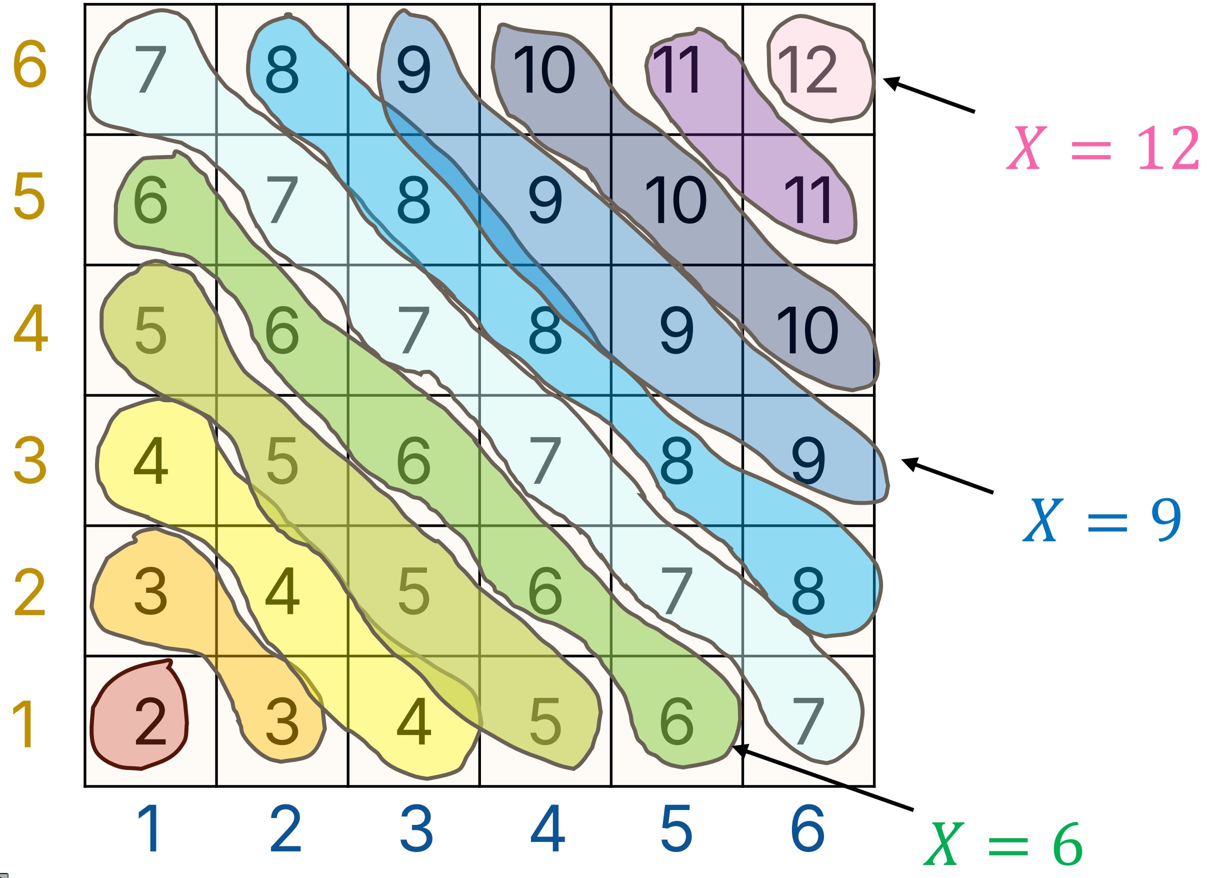
Example:

$$P(X = 9) = \sum_{\omega \in "X=9"} P(\omega) = 4/36$$

Random Variables and Partitions

A random variable X partitions the sample space into events. Why?

- The domain of X is the entire sample space.
- X is a function, so every sample is assigned to exactly one set.



Reminder: A partition of a set is defined as a set of sets whose union is the entire sample space, and whose pairwise intersections are $\emptyset$.

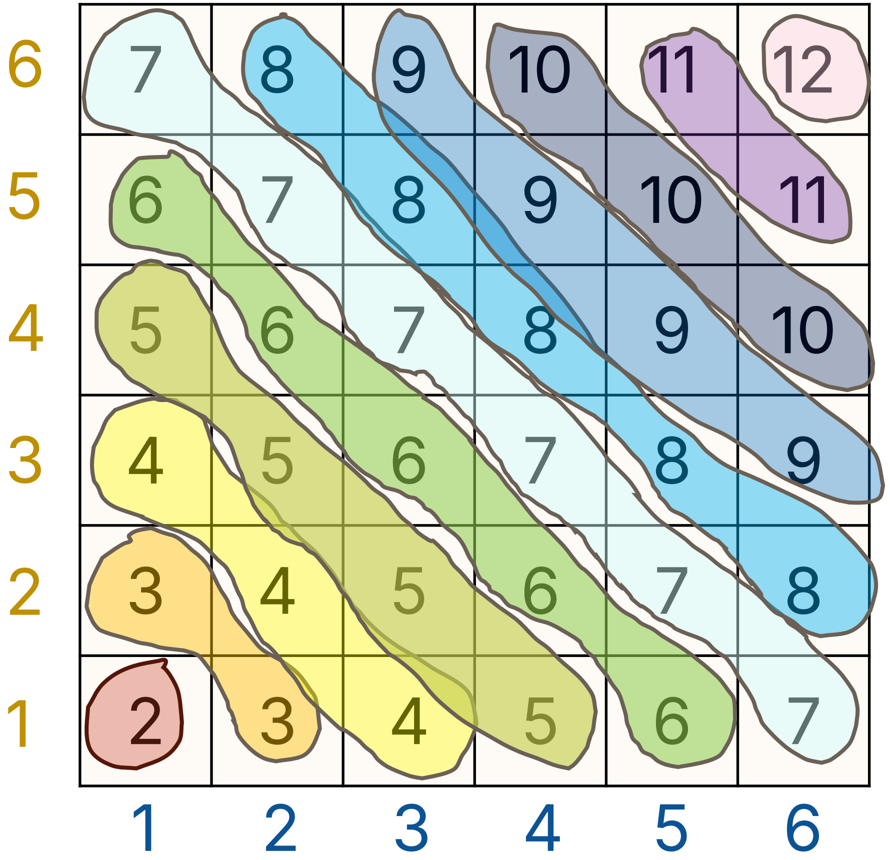
The Distribution of a Random Variable

Lecture 19, CS70 Summer 2025

Distribution of a Random Variable

The distribution of a random variable X is the collection of values

$$\{(a, P(X = a)) : a \in \text{range}(X)\}$$



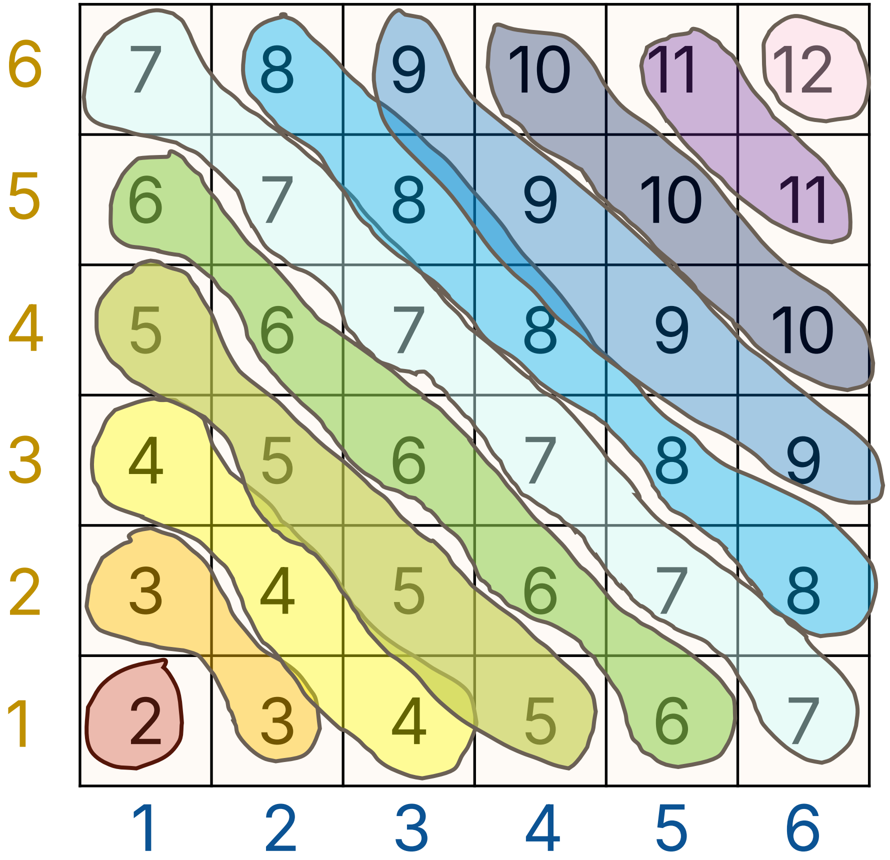
Example: If X is the sum of two six-sided dice, the distribution of X is:

$$\left\{ \begin{array}{c} \left(2, \frac{1}{36}\right), \left(3, \frac{2}{36}\right), \left(4, \frac{3}{36}\right), \left(5, \frac{4}{36}\right), \left(6, \frac{5}{36}\right), \left(7, \frac{6}{36}\right), \\ \left(8, \frac{5}{36}\right), \left(9, \frac{4}{36}\right), \left(10, \frac{3}{36}\right), \left(11, \frac{2}{36}\right), \left(12, \frac{1}{36}\right) \end{array} \right\}$$

Distribution of a Random Variable

The distribution of a random variable X is the collection of values

$$\{(a, P(X = a)) : a \in \text{range}(X)\}$$



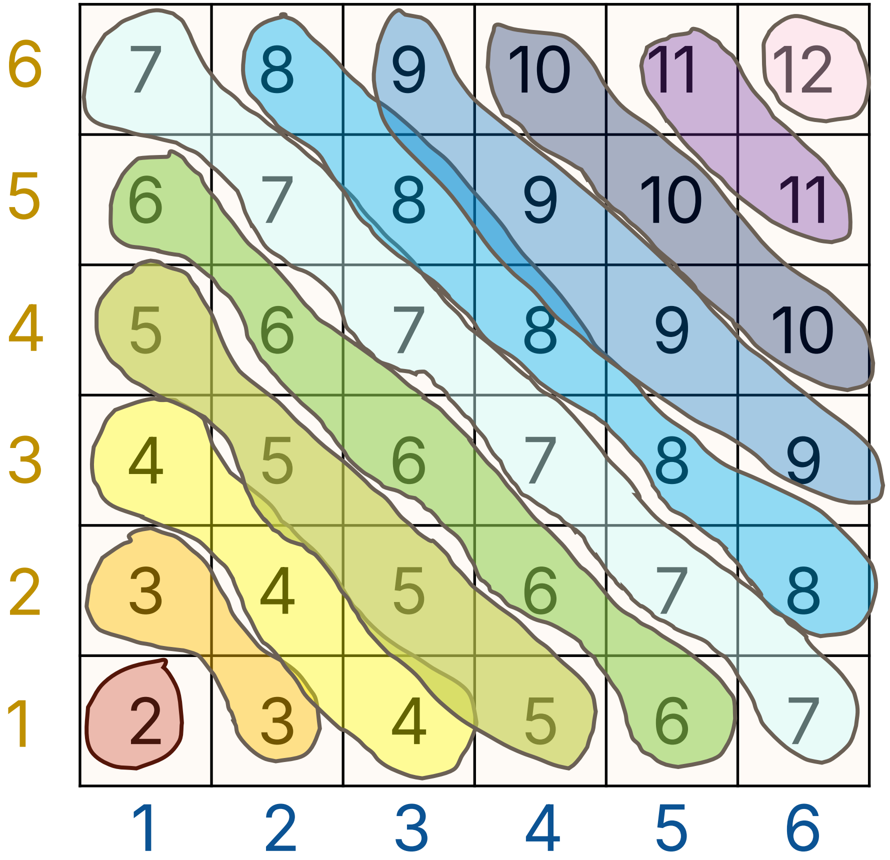
Can also show the distribution of X as a table:

a	$P(X = a)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

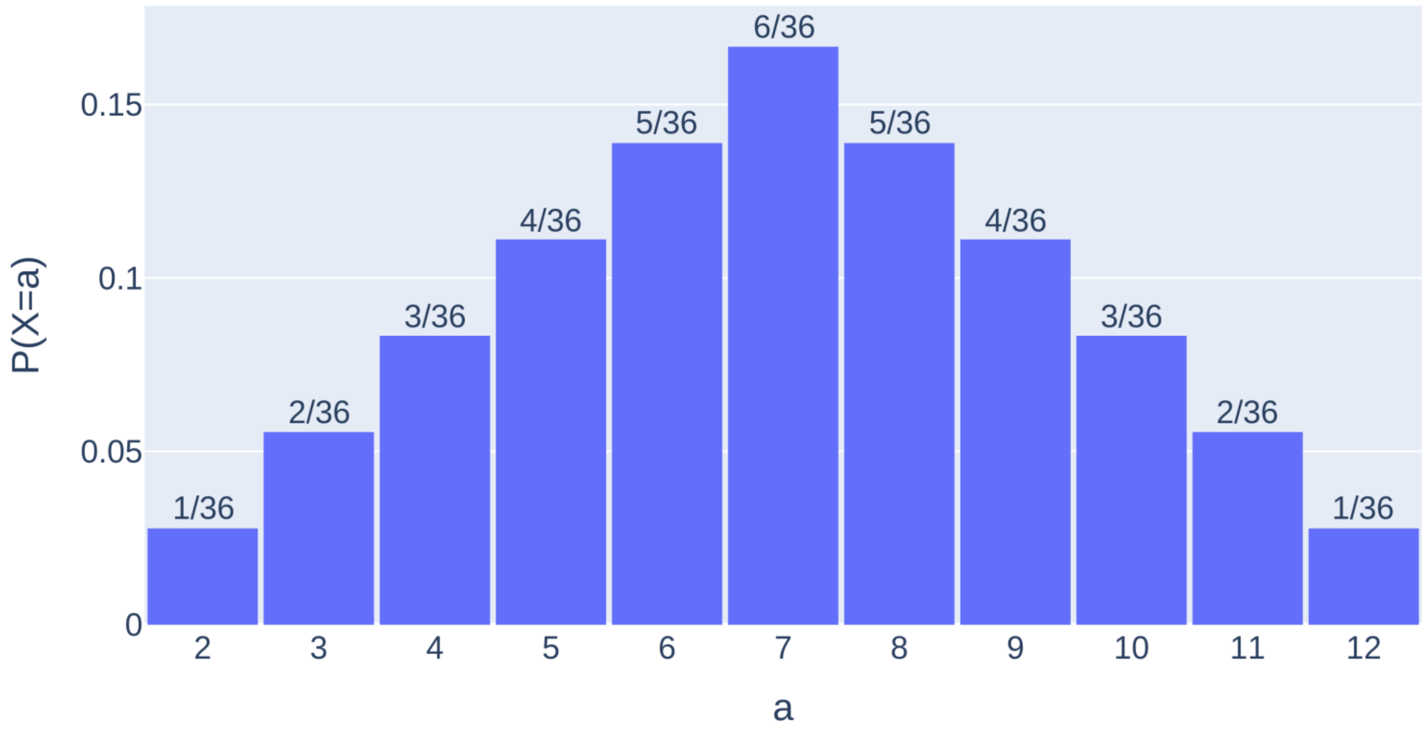
Distribution of a Random Variable

The distribution of a random variable X is the collection of values

$$\{(a, P(X=a)) : a \in \text{range}(X)\}$$



Can also show the distribution of X as a plot:

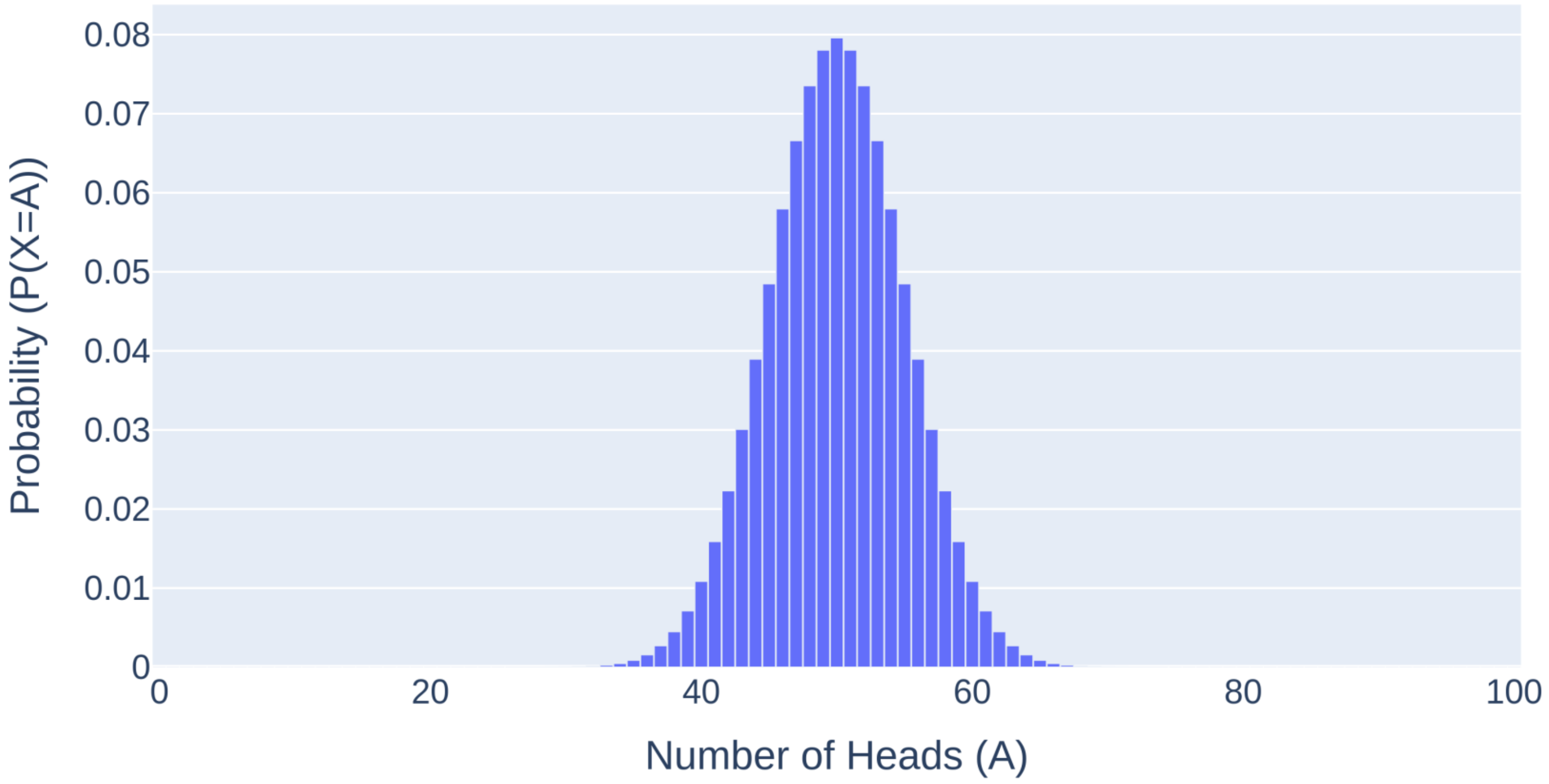


Distribution Example 2: Tossing 100 Coins

Toss 100 coins. Let X be the number of heads.

- Reminder: X is a function that maps samples to reals, e.g., $X(HHH \ldots H) = 100$

Distribution of X: $$P(X = a) = \frac{\text{\# outcomes with } a \text{ heads}}{|\Omega|} = \frac{\binom{100}{a}}{2^{100}}$$

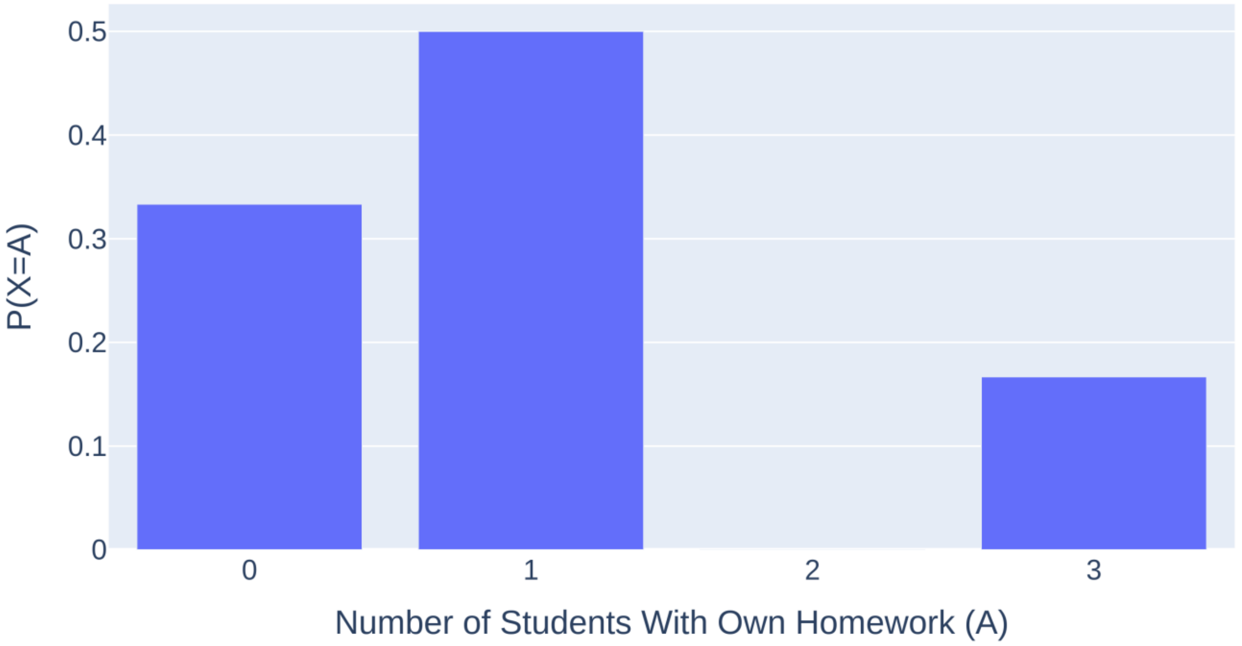


Distribution Example 3: Handing Back Assignments

Random Experiment: Suppose we take homework from 3 students, and randomly give each student one homework. Let X be the number of students who get their own homework.

X	3	1	1	0	0	1

$$\Omega = \{123, 132, 213, 231, 312, 321\}$$



The Expectation of a Random Variable

Lecture 19, CS70 Summer 2025

Distributions and Expectation

Random Experiment: Pick a random student from CS70. X is their grade on the first midterm.

What are some interesting questions we might ask about X?

- What is the distribution of X?
- What is the average of X? We'll call this the expectation of X.
- What is the standard deviation of X?

Expectation Informally

Let's do a little experiment and flip 4 coins. Let X be the number of heads. We'll count:

- How many times we get each value a from $\{0, 1, 2, 3, 4\}$.
- The average of the values we get.

https://joshh.ug/cs70/four_coin_flips_expectation_simulator.html

Simulation Results (Done Previously)

Out of 20 flips, the average number of heads per flip was 1.7.

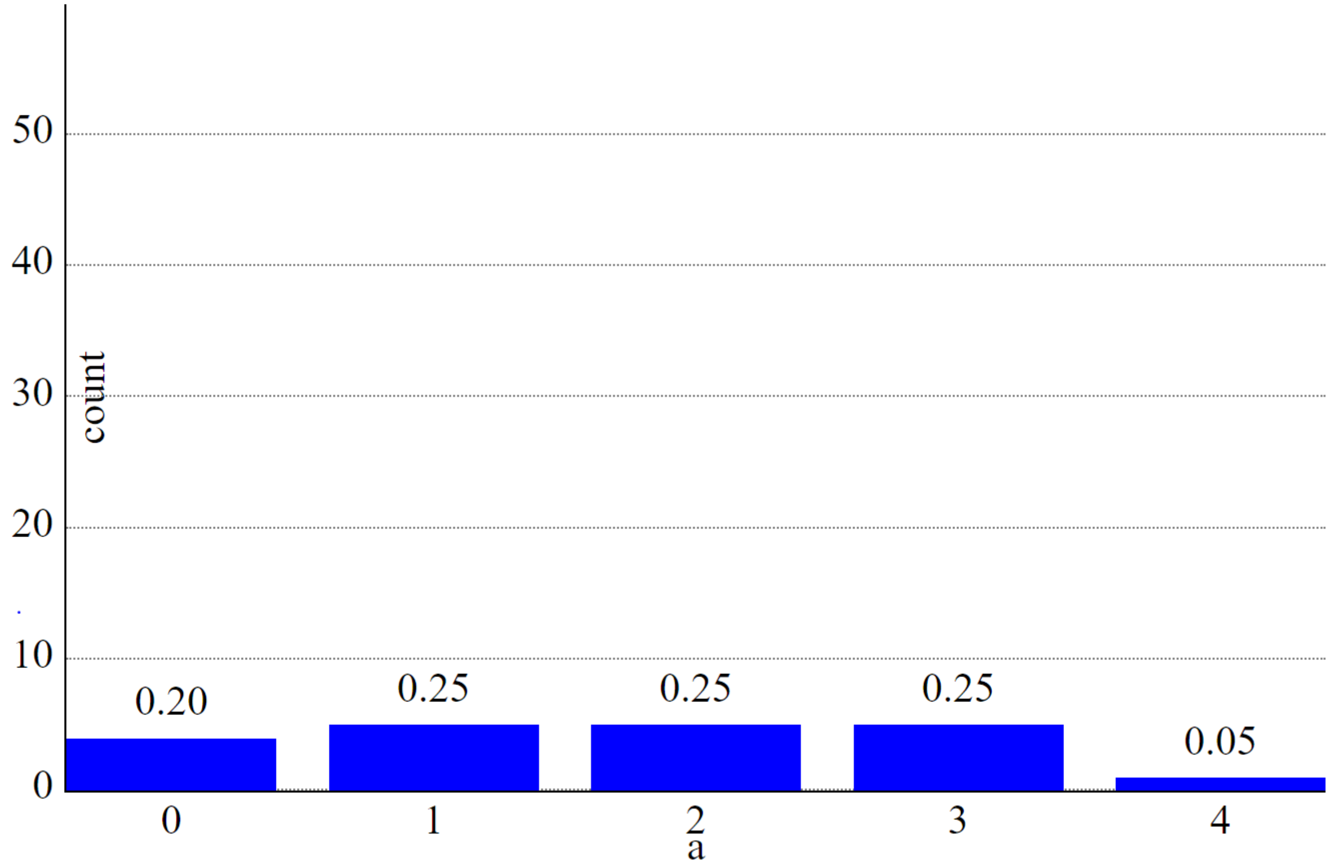
$$\frac{0+2+3+1+2+2+\cdots+2}{20} = 1.7$$

1, 2, and 3 were the most common value, occurring 25% of the time.

Total number of heads across all flips: 34

Total number of flips: 20

Average a: 34/20 = 1.70



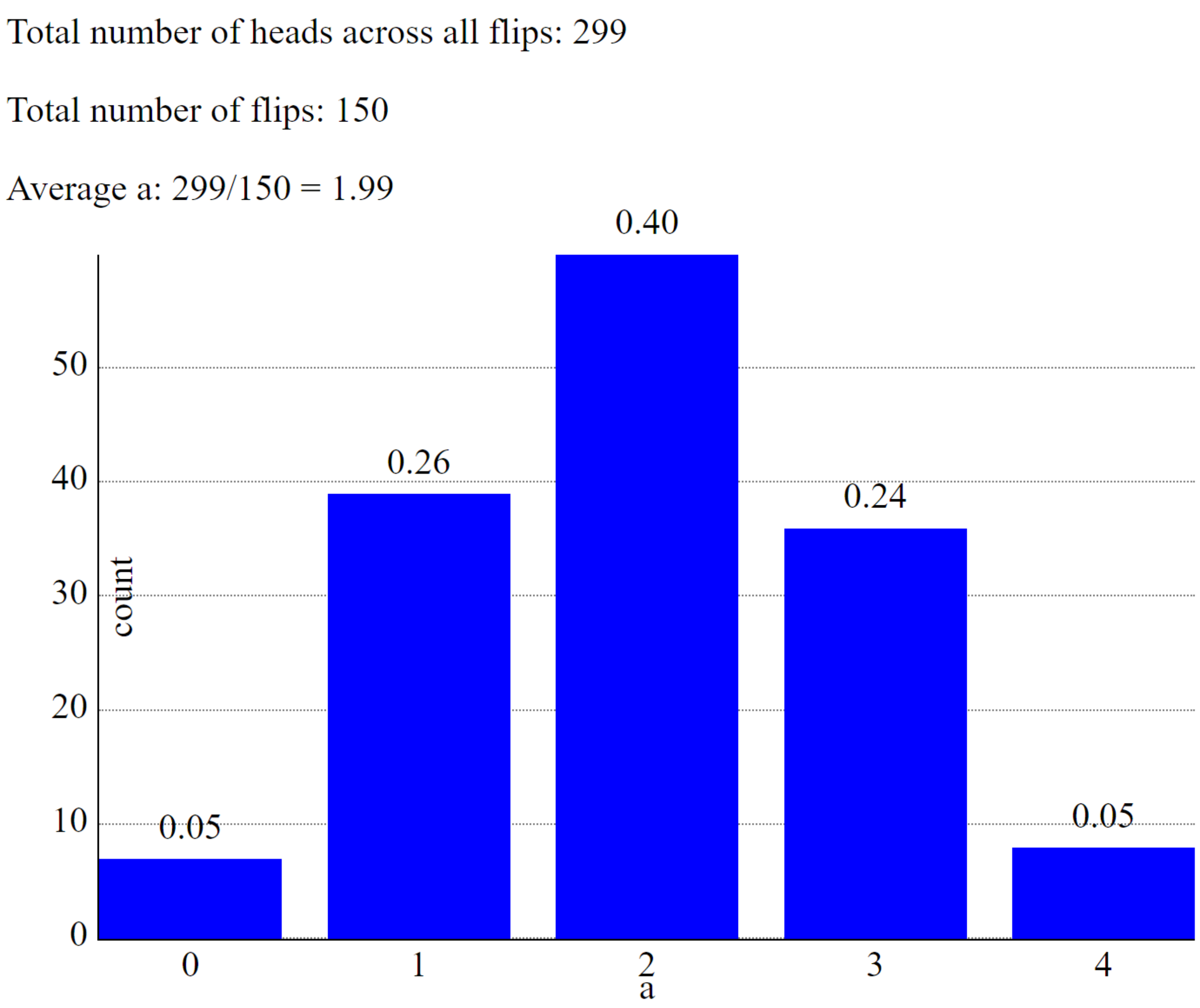
Simulation Results (Done Previously)

Out of 150 flips, the average number of heads per flip was 1.99.

$$\frac{0+2+3+1+2+2+\cdots+1}{150} = 1.99$$

2 was the most common value, occurring 40% of the time.

Note: As the number of trials grows, the histogram looks increasingly like the distribution for the random variable X.



Simulation Results (Done Previously)

Note: As the number of trials grows, the histogram looks increasingly like the distribution for the random variable X.

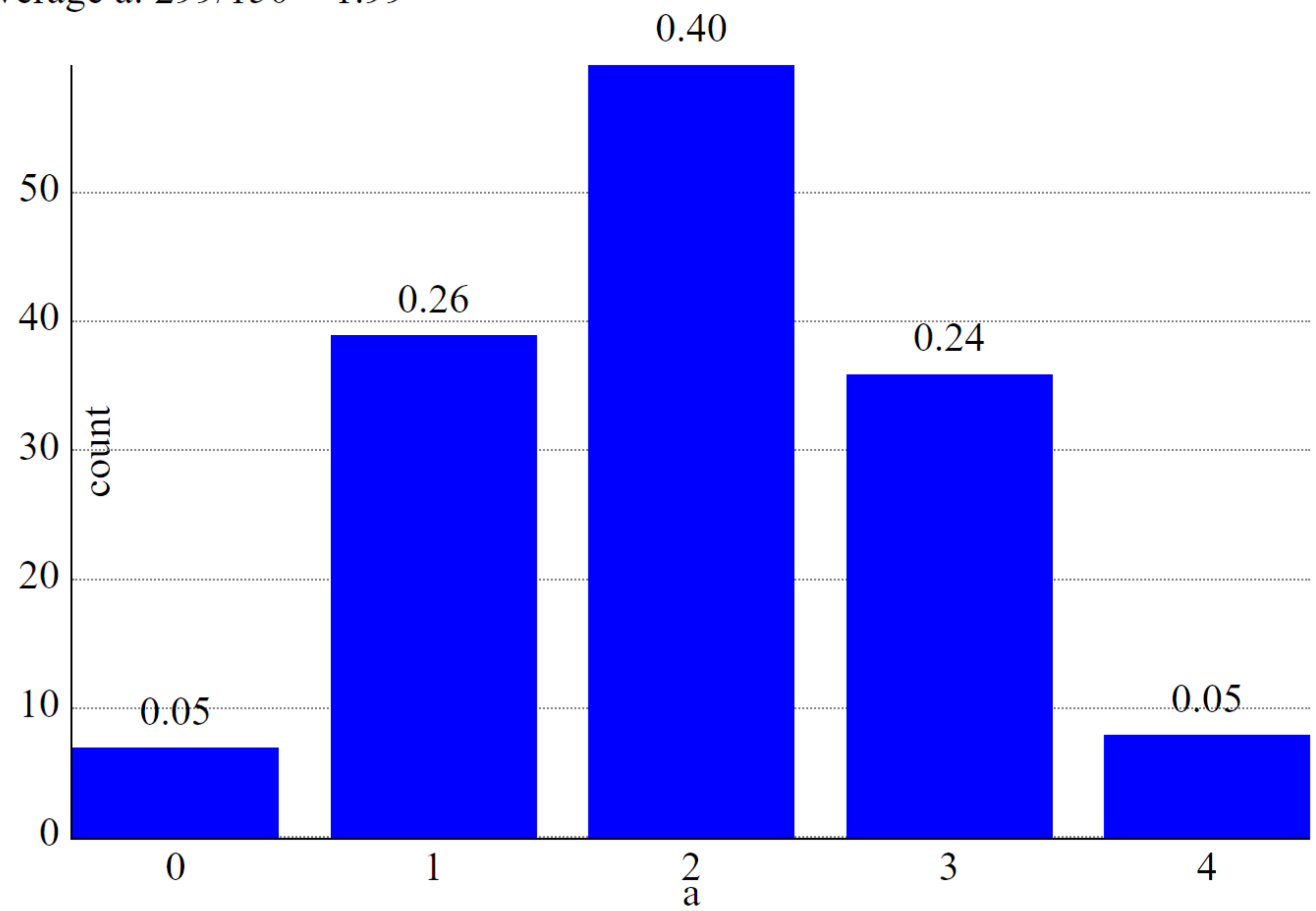
Question: If we know the exact distribution for X, how can we compute the average or expectation of X?

- This is the average value that the simulation will eventually converge to (given enough samples).

Total number of heads across all flips: 299

Total number of flips: 150

Average a: 299/150 = 1.99



Expectation (Formally)

The expectation of a random variable is defined as:

$$E[X] = \sum_{a \in \text{range}(X)} a \times P(X = a)$$

Example: Flipping four coins.

- $E[X] = 0 \times 1/16 + 1 \times 4/16 + 2 \times 6/16 + 3 \times 4/16 + 4 \times 1/16$

ω	a	$P(X = a)$
TTTT	0	$1/16$
HTTT, THTT, TTHT, TTTH	1	$4/16$
HHTT, HTHT, HTTH, THHT, THTH, TTHH	2	$6/16$
HHHT, HHTH, HTHH, THHH	3	$4/16$
HHHH	4	$1/16$

Expectation (Formally)

The expectation of a random variable is defined as:

$$E[X] = \sum_{a \in \text{range}(X)} a \times P(X = a)$$

Example: Flipping four coins.

- $E[X] = 0$ $+4/16$ $+12/16$ $+12/16$ $+4/16$

ω	a	$P(X = a)$
TTTT	0	$1/16$
HTTT, THTT, TTHT, TTTH	1	$4/16$
HHTT, HTHT, HTTH, THHT, THTH, TTHH	2	$6/16$
HHHT, HHTH, HTHH, THHH	3	$4/16$
HHHH	4	$1/16$

Expectation (Formally)

The expectation of a random variable is defined as:

$$E[X] = \sum_{a \in \mathrm{range}(X)} a \times P(X = a)$$

Example: Flipping four coins.

- $E[X] = 32/16 = 2$

ω	a	$P(X = a)$
TTTT	0	$1/16$
HTTT, THTT, TTHT, TTTH	1	$4/16$
HHTT, HTHT, HTTH, THHT, THTH, TTHH	2	$6/16$
HHHT, HHTH, HTHH, THHH	3	$4/16$
HHHH	4	$1/16$

Alternate Equivalent Definition

The expectation is:

$$E[X] = \sum_{a \in \text{range}(X)} a \times P(X = a)$$

We can also write this expression in terms of a sum over all outcomes $\omega \in \Omega$.

$$= \sum_{a \in \text{range}(X)} \left(a \times \sum_{\omega : X(\omega) = a} P(\omega) \right)$$

$$= \sum_{a \in \text{range}(X)} \sum_{\omega : X(\omega) = a} a \times P(\omega)$$

$$= \sum_{a \in \text{range}(X)} \sum_{\omega : X(\omega) = a} X(\omega) \times P(\omega)$$

$$= \sum_{\omega \in \Omega} X(\omega) \times P(\omega)$$

Bernoulli Random Variables

Lecture 19, CS70 Summer 2025

Bernoulli Random Variable

A Bernoulli random variable has a distribution given by the equation below, where $0 \leq p \leq 1$.

$$P(X = a) = \begin{cases} p, & \text{if } a = 1 \\ 1 - p, & \text{if } a = 0 \end{cases}$$

Example: Suppose we flip a coin that comes up heads 75% of the time. Let X be 1 if the coin comes up heads, and 0 if it comes up tails.

- We'd say "X is distributed as a Bernoulli random variable with $p = 0.75$"
- Or in writing we can say "$X \sim \text{Bernoulli}(0.75)$".
  - This is just shorthand for saying "X is distributed as a Bernoulli random variable with $p = 0.75$.

Bernoulli Random Variables and Expectation

Suppose $X \sim \text{Bernoulli}(0.5)$. What is $E[X]$?

$$E[X] = \sum_{a \in \text{range}(X)} a \times P(X = a)$$

$$P(X = a) = \begin{cases} p, & \text{if } a = 1 \\ 1 - p, & \text{if } a = 0 \end{cases}$$

We have that $E[X] = 0 \times P(X = 0) + 1 \times P(X = 1)$.

- $P(X = 0) = 1 - p = 0.5$
- $P(X = 1) = p = 0.5$

So: $E[X] = 0 \times 0.5 + 1 \times 0.5 = 0.5$

- Note: The expected value of X is a value that X can never equal

"Expected" is something we can't actually expect – English is different from math!

Bernoulli Random Variables and Expectation

Suppose $X \sim \text{Bernoulli}(p)$. What is $E[X]$?

$$E[X] = \sum_{a \in \text{range}(X)} a \times P(X = a) \qquad P(X = a) = \begin{cases} p, & \text{if } a = 1 \\ 1 - p, & \text{if } a = 0 \end{cases}$$

We have that $E[X] = 0 \times P(X = 0) + 1 \times P(X = 1)$.

- $P(X = 0) = 1 - p$
- $P(X = 1) = p$

So: $E[X] = 0 \times (1 - p) + 1 \times p = p$

- That is, the expected value of a Bernoulli random variable is just the probability that it is 1 (i.e., that the "coin flip" comes up "heads").

The Sum of Two Bernoulli Random Variables

Suppose $X_1 \sim \text{Bernoulli}(p)$ and $X_2 \sim \text{Bernoulli}(p)$. Suppose they are independent.

Let $Y = X_1 + X_2$.

- What is the distribution of Y?
- What is $E[Y]$?

The Sum of Two Bernoulli Random Variables

Suppose $X_1 \sim \text{Bernoulli}(p)$ and $X_2 \sim \text{Bernoulli}(p)$. Suppose they are independent.

Let $Y = X_1 + X_2$.

$$P(X = a) = \begin{cases} p^2, & \text{if } a = 2 \\ 2p(1-p), & \text{if } a = 1 \\ (1-p)^2, & \text{if } a = 0 \end{cases}$$

- What is the distribution of Y?
  - To get $a = 2$, both Bernoulli RVs must be 1.
  - To get $a = 1$, one Bernoulli RV is 1, and the other is 0.
  - To get $a = 0$, both Bernoulli RVs must be 0.
- What is $E[Y]$?

The Sum of Two Bernoulli Random Variables

Suppose $X_1 \sim \text{Bernoulli}(p)$ and $X_2 \sim \text{Bernoulli}(p)$. Suppose they are independent.

$$P(X = a) = \begin{cases} p^2, & \text{if } a = 2 \\ 2p(1-p), & \text{if } a = 1 \\ (1-p)^2, & \text{if } a = 0 \end{cases}$$

Let $Y = X_1 + X_2$.

- What is the distribution of Y?
  - To get $a = 2$, both Bernoulli RVs must be 1.
  - To get $a = 1$, one Bernoulli RV is 1, and the other is 0.
  - To get $a = 0$, both Bernoulli RVs must be 0.
- What is $E[Y]$?

$$E[Y] = 2 \times p^2 + 1 \times 2p(1-p) + 0 \times (1-p)^2$$

$$= 2p^2 + 2p - 2p^2$$

$$= 2p$$

Binomial Random Variables

Lecture 19, CS70 Summer 2025

Binomial Distribution

The Binomial distribution models counting the number of heads if we flip n coins that come up heads with probability p.

Let's work out the distribution by considering a specific example. Suppose that $X \sim \text{Binomial}(n = 5, p)$

- $P(HHTHT) = p \cdot p \cdot (1-p) \cdot p \cdot (1-p) = p^3(1-p)^2$
- $P(TTHHH) = (1-p) \cdot (1-p) \cdot p \cdot p \cdot p = p^3(1-p)^2$

Binomial Distribution

The Binomial distribution models counting the number of heads if we flip n coins that come up heads with probability p.

Let's work out the distribution by considering a specific example. Suppose that $X \sim \text{Binomial}(n = 5, p)$

- $P(HHTHT) = p \cdot p \cdot (1-p) \cdot p \cdot (1-p) = p^3(1-p)^2$
- $P(TTHHH) = (1-p) \cdot (1-p) \cdot p \cdot p \cdot p = p^3(1-p)^2$
- $P(X = 3) = (\text{\# of sequences with 3 heads})p^3(1-p)^2$

$$= \binom{5}{3} p^3(1-p)^2$$

Binomial Distribution

The Binomial distribution models counting the number of heads if we flip n coins that come up heads with probability p.

Let's work out the distribution by considering a specific example. Suppose that $X \sim \text{Binomial}(n = 5, p)$

- $P(X = i) = (\text{\# of sequences with i heads}) p^i (1-p)^{5-i}$

$$= \binom{5}{i} p^i (1-p)^{5-i} \quad \text{for } i = 0, 1, 2, 3, 4, 5$$

Binomial Distribution

The Binomial distribution models counting the number of heads if we flip n coins that come up heads with probability p.

Let's work out the distribution by considering a specific example. Suppose that $X \sim \text{Binomial}(n, p)$

- $P(X = i) = (\text{\# of sequences with i heads}) p^i (1-p)^{n-i}$

$$= \binom{n}{i} p^i (1-p)^{n-i} \quad \text{for } i = 0, 1, 2, \ldots, n$$

Binomial Distribution

The Binomial distribution models counting the number of heads if we flip n coins that come up heads with probability p.

Suppose that $X \sim \text{Binomial}(n, p)$

- $P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}$ for $i = 0, 1, 2, \ldots, n$

Exercise: Verify that $\sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1$.

We'll cover the expectation of a Binomial RV next lecture.

- You can do it using the usual sum, but that's much messier than the nicer approach from the next lecture.

Geometric Random Variables

Lecture 19, CS70 Summer 2025

Geometric Distribution

Imagine a random experiment where we repeatedly flip a coin with a probability of heads equal to p until we get heads. The flips are independent.

- Let X be the number of flips.

This distribution is called the geometric distribution, i.e., $X \sim \text{Geometric}(p)$

In the coin flipping example, what are the domain and range of X?

- Domain: Any sequence of 0 or more tails, followed by a heads.
- Range: Any integer greater than or equal to 1, i.e., $\mathbb{N}^+$

What is $P(X = 1)$?

- $P(X = 1) = p$

Geometric Distribution

Imagine a random experiment where we repeatedly flip a coin with a probability of heads equal to p until we get heads. The flips are independent.

- Let X be the number of flips.

This distribution is called the geometric distribution, i.e., $X \sim \text{Geometric}(p)$

What is $P(X = i)$?

- $P(X = 2) = (1 - p) \cdot p$
- $P(X = 3) = (1 - p)^2 \cdot p$
- $P(X = i) = (1 - p)^{i-1} \cdot p$

Verifying that the Probabilities Sum to 1

Imagine a random experiment where we repeatedly flip a coin with a probability of heads equal to p until we get heads. The flips are independent.

- Let X be the number of flips.

This distribution is called the geometric distribution, i.e., $X \sim \mathrm{Geometric}(p)$

- $P(X = i) = (1-p)^{i-1}p$ for $i = 1,2,\ldots$

Let's verify that the probabilities sum to 1.

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$\sum_{i=1}^{\infty} P(X = i) = \sum_{i=1}^{\infty} (1-p)^{i-1}p = p\sum_{i=1}^{\infty}(1-p)^{i-1} = \frac{p}{1-(1-p)} = \frac{p}{p} = 1$$

Expectation of a Geometric Random Variable

If $X \sim \text{Geometric}(p)$, what is $E[X]$? This is a fun one.

- $P(X = i) = (1-p)^{i-1}p$ for $i = 1,2,\ldots$

$$E[X] = p + 2(1-p)p + 3(1-p)^2p + \cdots$$

First, multiply both sides by $(1-p)$

$$(1-p)E[X] = (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots$$

Then, subtract the bottom equation from the top:

$$E[X] - (1-p)E[X] = p + 2(1-p)p + 3(1-p)^2p + \cdots$$

$$-(1-p)p \quad - 2(1-p)^2p - \cdots$$

This is just $\sum_{i=1}^{\infty} P(X = i)$

$$pE[X] = \underbrace{p + (1-p)p + (1-p)^2p + \cdots}$$

$$pE[X] = 1 \qquad E[X] = 1/p$$

Expectation of a Geometric Random Variable

If $X \sim \text{Geometric}(p)$, what is $E[X]$? This is a fun one.

- $P(X = i) = (1-p)^{i-1}p$ for $i = 1,2,\ldots$
- $E[X] = 1/p$

In this week's discussion, you'll get a chance to build some deeper intuition with geometric random variables, including the "memoryless" property.

Poisson Random Variables

Lecture 19, CS70 Summer 2025

Random Variables

- *The Idea and Definition*
- *Random Variables and Events*
- *The Distribution of a Random Variable*
- *Expectation*

Common Distributions

- *Bernoulli Distribution*
- *Binomial Distribution*
- *Geometric* Distribution
- *Poisson Distribution*

Summary

Poisson Distribution

Suppose we want to know how many alpha particles will be emitted by a radiation source. Let X be the number of emissions per unit time.

Assumptions:

- The average number of emissions is λ alpha particles per unit time.
- Emissions in disjoint time intervals are independent, e.g. if 2 particles were emitted in the interval [0, 2], this tells us absolutely nothing about how many were emitted in the interval [2.1, 4].

If assumptions hold, $X \sim \text{Poisson}(\lambda)$.

$$P(X = i) = \frac{\lambda^i}{i!} e^{-\lambda}, \qquad \text{for } i = 0, 1, 2, \ldots$$

Poisson Distribution Example

Suppose we want to know how many alpha particles will be emitted by a radiation source. Let X be the number of emissions per unit time.

Suppose we model this radiation source as $X \sim \text{Poisson}(\lambda)$, where $\lambda = 2$ emissions per minute.

- What is the probability of seeing 3 emissions in any given minute?

$$P(X = i) = \frac{\lambda^i}{i!} e^{-\lambda}, \qquad \text{for } i = 0, 1, 2, \ldots$$

$$P(X = 3) = \frac{2^3}{3!} e^{-2} = \frac{8}{6} e^{-2} \approx 18.04\%$$

Useful Poisson Distribution Facts

The Poisson distribution comes from the large n limit for Binomial RVs.

- That is, can think of $\frac{\lambda^i}{i!}e^{-\lambda}$ as arising from an infinite number of Bernoulli trials.
- We'll cover this in the next lecture. For today it's just magic.

The expected value of a Poisson random variable is λ.

- This is by design! λ was defined as the average number of events per unit time.

If we add independent $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$, and $Z = X + Y$, then $Z \sim \text{Poisson}(\lambda + \mu)$.

- We're not equipped to prove this yet. This will come next lecture.

Poisson Distribution Example 2

Suppose we have two radiation sources. We model one as $X_1 \sim \text{Poisson}(\lambda_1)$, and the other as $X_2 \sim \text{Poisson}(\lambda_2)$.

- If $\lambda_1 = 2$ emissions per minute, and $\lambda_2 = 5$ emissions per minute, what is the chance that the total number of emissions in a given minute is 4 emissions?

Earlier, we said that the sum of two RVs is also Poisson. So $S = X_1 + X_2$ is a random variable where $S \sim \text{Poisson}(7)$.

$$P(S = i) = \frac{\lambda^i}{i!} e^{-\lambda}, \qquad \text{for } i = 0, 1, 2, \ldots$$

$$P(S = 4) = \frac{7^4}{4!} e^{-7} \approx 9.1\%$$

Poisson Distribution Example 3

Suppose we want to know how many alpha particles will be emitted by a radiation source. Let X be the number of emissions per unit time.

Suppose we model this radiation source as $X \sim \mathrm{Poisson}(\lambda)$, where $\lambda = 2$ emissions per minute.

- What is the probability that we get 100 emissions in an hour?

Poisson Distribution Example 3

Suppose we want to know how many alpha particles will be emitted by a radiation source. Let X be the number of emissions per unit time.

Suppose we model this radiation source as $X \sim \mathrm{Poisson}(\lambda)$, where $\lambda = 2$ emissions per minute.

- What is the probability that we get 100 emissions in an hour?
- Need to convert our units. 2 emissions per minute is 120 emissions per hour.
  - *Can we really just multiply like this? Yes! Reason: Next lecture.*
- Let Y be X, but in units of emissions per hour. Then $\mathrm{Y} \sim \mathrm{Poisson}(120)$.

$$P(Y = i) = \frac{\lambda^i}{i!} e^{-\lambda} \qquad P(Y = 100) = \frac{120^{100}}{100!} e^{-120} \approx 0.6\%$$

Poisson Distribution Example 3 (Alternate Solution)

Suppose we want to know how many alpha particles will be emitted by a radiation source. Let X be the number of emissions per unit time.

Suppose we model this radiation source as $X \sim \text{Poisson}(\lambda)$, where $\lambda = 2$ emissions per minute.

- What is the probability that we get 100 emissions in an hour?
- The number of emissions per hour can be thought of as the sum of 60 independent RVs that are $\sim\text{Poisson}(\lambda)$.
- Let S be this sum. Then $S \sim \text{Poisson}(120)$.

$$P(S = i) = \frac{\lambda^i}{i!} e^{-\lambda} \qquad P(S = 100) = \frac{120^{100}}{100!} e^{-120} \approx 0.6\%$$

Summary

Lecture 19, CS70 Summer 2025

Summary

Today we introduced the concept of a random variable.

- A function that maps outcomes to real numbers, e.g., $X(\mathrm{HHHH})=4$.
- A random variable partitions a sample space. Each event in the partition is just the set of samples that yield a specific value.
- The distribution of a random variable X is the set of all values in the range of X, along with the probability of those values.
- The expectation of a random variable is the weighted (by probability) sum of values of X
  - This is also the limit of the average value if we average over a larger and larger number of random experiments with that RV.

Summary

We saw four important RVs:

- Bernoulli: Biased coin flips.
- Binomial: The sum of n biased coin flips.
- Geometric: The number of times we flip a coin before we get the first heads.
- Poisson: A model for the number of events when:
  - We know the average number of events per unit time.
  - Knowing the number of events in some time interval tells us absolutely nothing about the number of events in a disjoint time interval.
  - We'll relate this back to Binomial RVs in the next lecture.