xkcd of the day

https://xkcd.com/2169/

Linear Algebra Check

Linear algebra isn't a formal pre-requisite for this class but is used (in a very basic way!) in this topic. Let's review...

Question:

If $\pi_1 = [0\ \ 0.5\ \ 0.5]$ and $P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$, what is $\pi_2 = \pi_1 P$?

How do we compute this vector-matrix product?

Linear Algebra Query

Linear algebra isn't a formal pre-requisite for this class but is used (in a very basic way!) in this topic. Let's review...

Question:

If $\pi_1 = [0 \;\; 0.5 \;\; 0.5]$ and $P = \begin{bmatrix} \mathbf{0} & 0.5 & 0.5 \\ \mathbf{1} & 0 & 0 \\ \mathbf{0.5} & 0.5 & 0 \end{bmatrix}$, what is $\pi_2 = \pi_1 P$?

$\pi_2 = [a \qquad] = [0.75 \qquad\qquad]$, where:

$$a = 0 \times \mathbf{0} + 0.5 \times \mathbf{1} + 0.5 \times \mathbf{0.5} = 0.75$$

Linear Algebra Query

Linear algebra isn't a formal pre-requisite for this class but is used (in a very basic way!) in this topic. Let's review...

Question:

If $\pi_1 = [0 \;\; 0.5 \;\; 0.5]$ and $P = \begin{bmatrix} 0 & \mathbf{0.5} & 0.5 \\ 1 & \mathbf{0} & 0 \\ 0.5 & \mathbf{0.5} & 0 \end{bmatrix}$, what is $\pi_2 = \pi_1 P$?

$\pi_2 = [a \;\; b \quad] = [0.75 \;\; 0.25 \quad]$, where:

$$a = 0 \times 0 + 0.5 \times 1 + 0.5 \times 0.5 = 0.75$$

$$b = 0 \times \mathbf{0.5} + 0.5 \times \mathbf{0} + 0.5 \times \mathbf{0.5} = 0.25$$

Linear Algebra Query

Linear algebra isn't a formal pre-requisite for this class but is used (in a very basic way!) in this topic. Let's review...

Question:

If $\pi_1 = [0 \;\; 0.5 \;\; 0.5]$ and $P = \begin{bmatrix} 0 & 0.5 & \mathbf{0.5} \\ 1 & 0 & \mathbf{0} \\ 0.5 & 0.5 & \mathbf{0} \end{bmatrix}$, what is $\pi_2 = \pi_1 P$?

$\pi_2 = [a \;\; b \;\; c] = [0.75 \;\; 0.25 \;\; 0]$, where:

$$a = 0 \times 0 + 0.5 \times 1 + 0.5 \times 0.5 = 0.75$$

$$b = 0 \times 0.5 + 0.5 \times 0 + 0.5 \times 0.5 = 0.25$$

$$c = 0 \times \mathbf{0.5} + 0.5 \times \mathbf{0} + 0.5 \times \mathbf{0} = 0$$

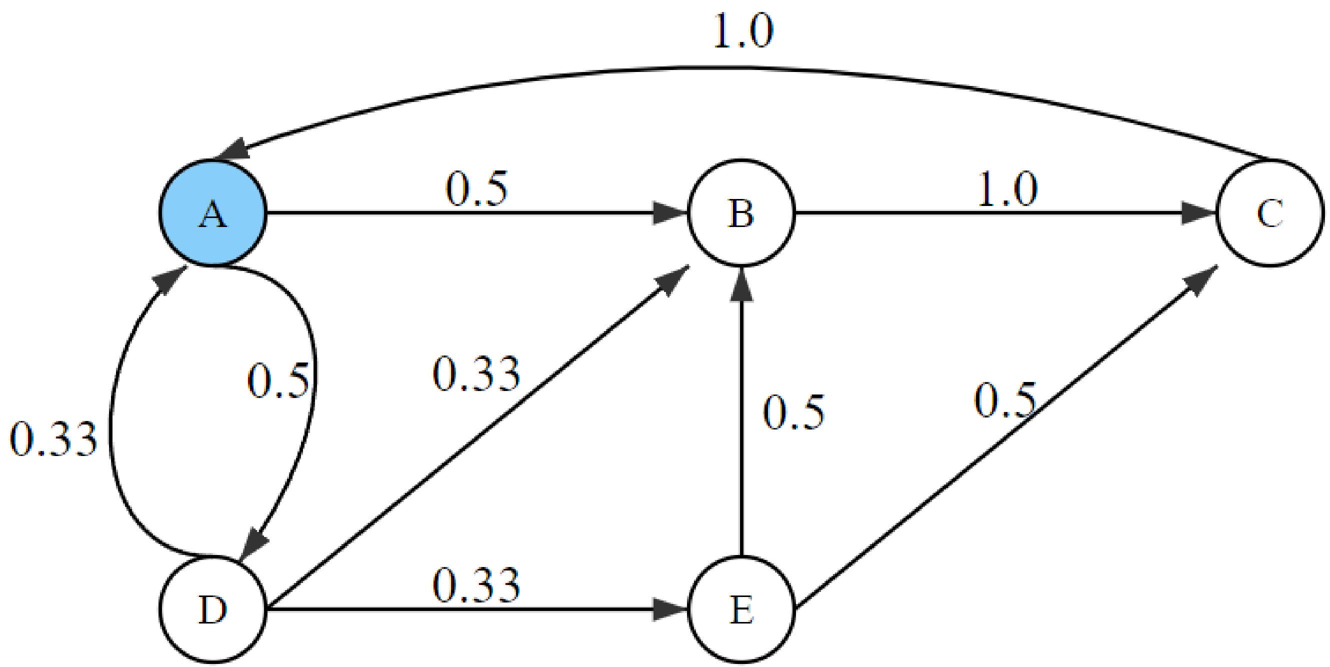
Markov Chain Introduction

Lecture 26, CS70 Summer 2025

Example Markov Chain Samples

Let's see an example of samples generated by a Markov Chain.

- This is the same Markov Chain in page 3 of the notes.



System is initially in state A.

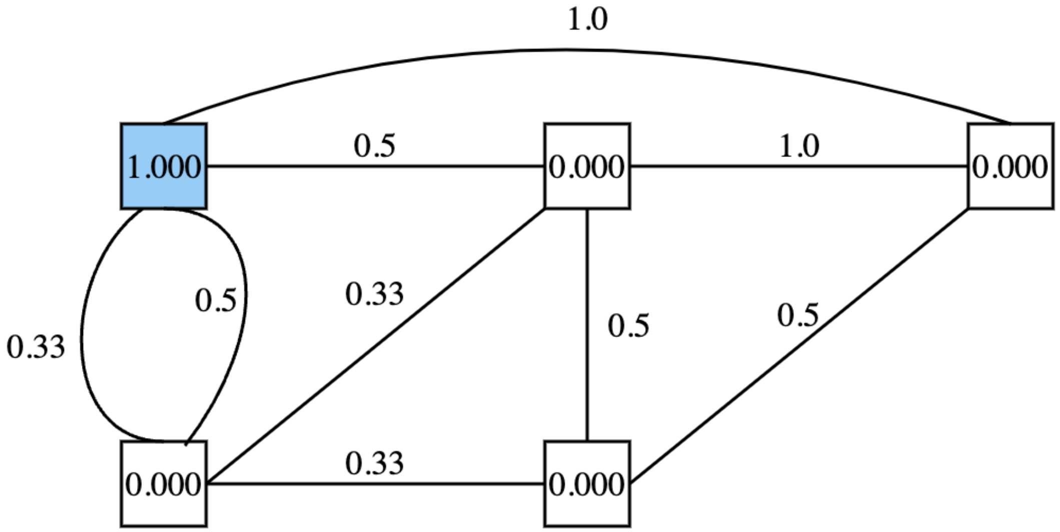
- From A it can go to B with probability 0.5, and to D with probability 0.5.
- From B it can only go to C (with probability 1).
- ...

Simulator at https://joshh.ug/cs70/markov_simulator_g5.html

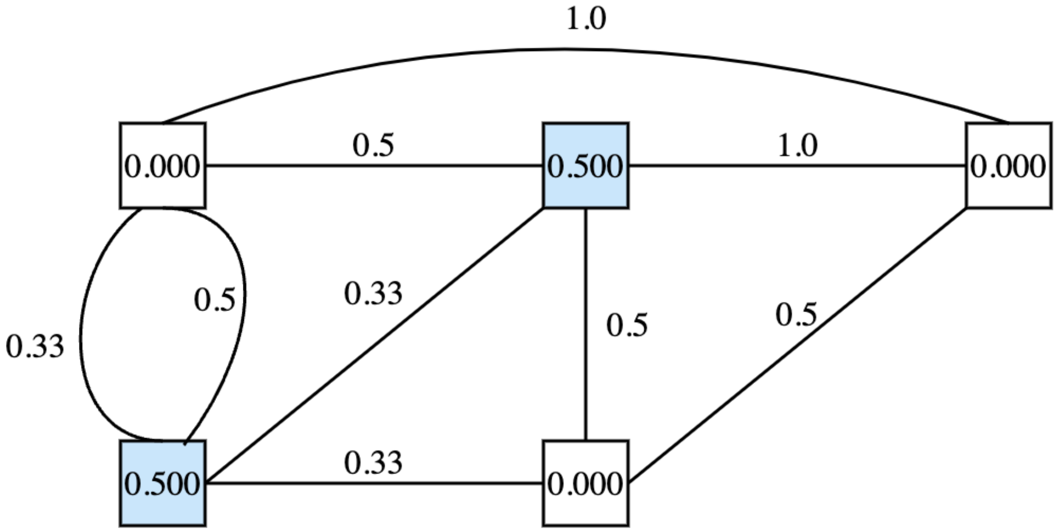
Markov Chain: Informal Definition

Before we saw samples generated by a Markov Chain.

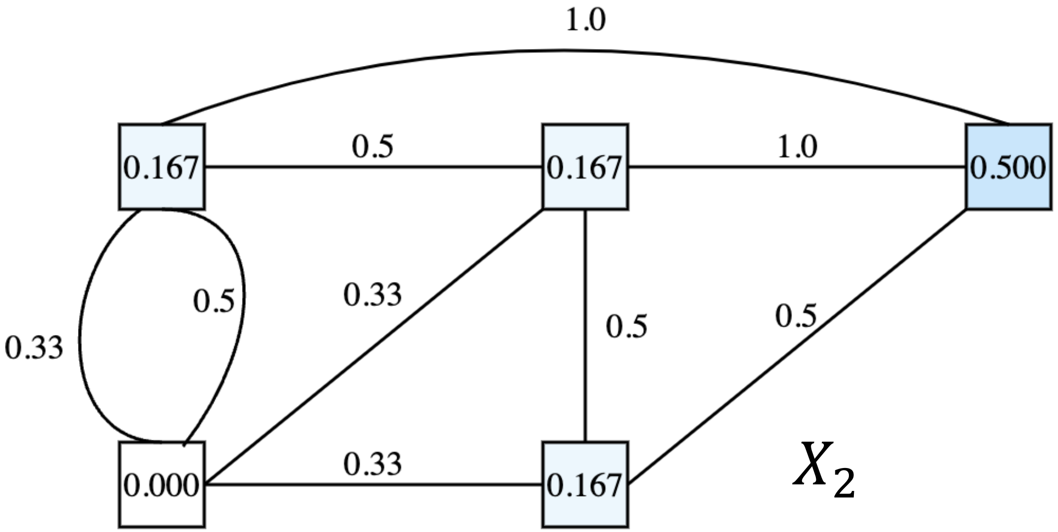
- A Markov Chain is a sequence of random variables – X_t = state at time t:



X_0



X_1



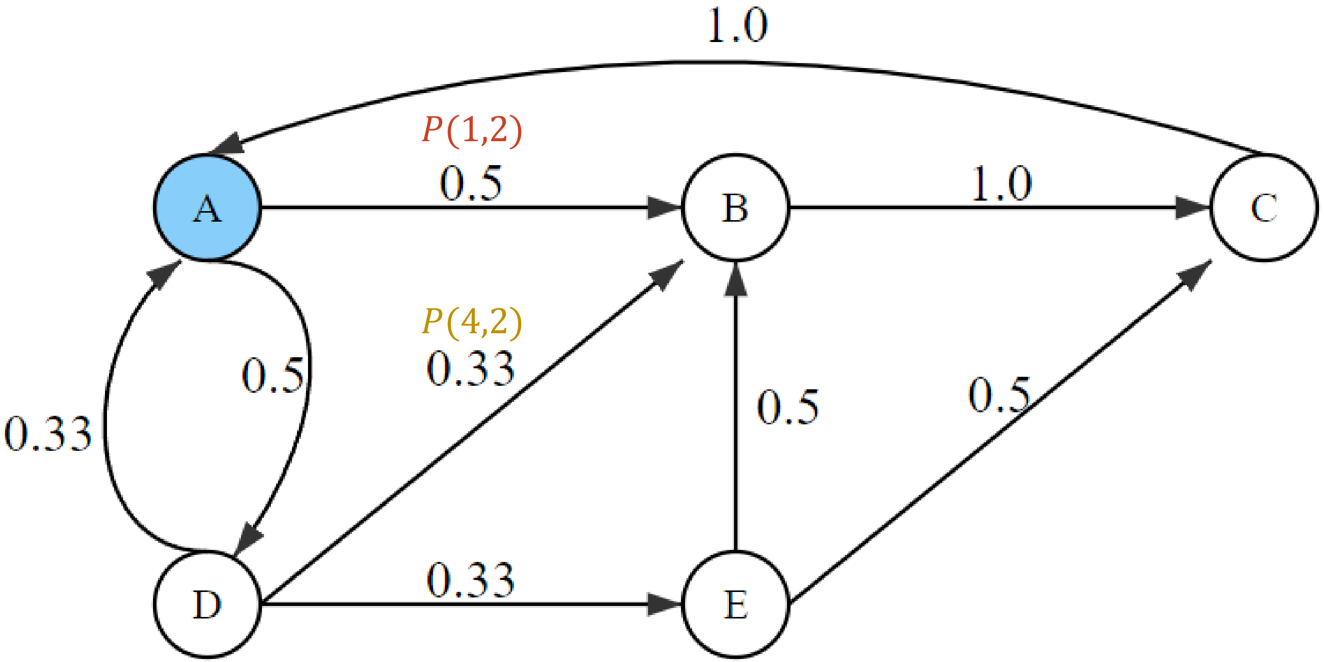
X_2

[Click this link to run a simulation](#).

Markov Chain: Formal Definition

A Markov Chain is a sequence of random variables $X_0, X_1, X_2, \ldots, X_n, \ldots$

- Each random variable takes on some value from $\mathfrak{X} = \{1, 2, \ldots, K\}$ for some finite K. X_i represents state of Markov chain at time step i.
- X_0 is given by the distribution π_0, i.e., $P(X_o = i) = \pi_0(i)$
- $P(X_{n+1} = j | X_n = i, X_{n-1} = x_{n-1}, \ldots, X_o = x_0) = P(X_{n+1} = j | X_n = i) = P(i, j)$



Examples for our simulation:

- $\mathfrak{X} = \{1, 2, 3, 4, 5\}$ (one for each state)
- $\pi_0 = [1\ 0\ 0\ 0\ 0]$ (always start in A)
- $P(X_1 = 2 | X_0 = 1) = 1/2$
- $P(1, 2) = 1/2$
- $P(X_{100} = 2 | X_{99} = 4, X_{98} = 1) = P(4,2) = 1/3$

Markov Chain: State Space and Transition Probability Matrix

The state space of a Markov Chain is $\mathfrak{X} = \{1, 2, \dots, K\}$ for some finite K.

The transition probability matrix P is a $K \times K$ matrix such that:

$$P(i,j) \geq 0, \qquad \forall i,j \in \mathfrak{X}$$

and the sum of each row is 1, i.e.

$$\sum_{j=1}^{K} P(i,j) = 1, \qquad \forall i \in \mathfrak{X}$$

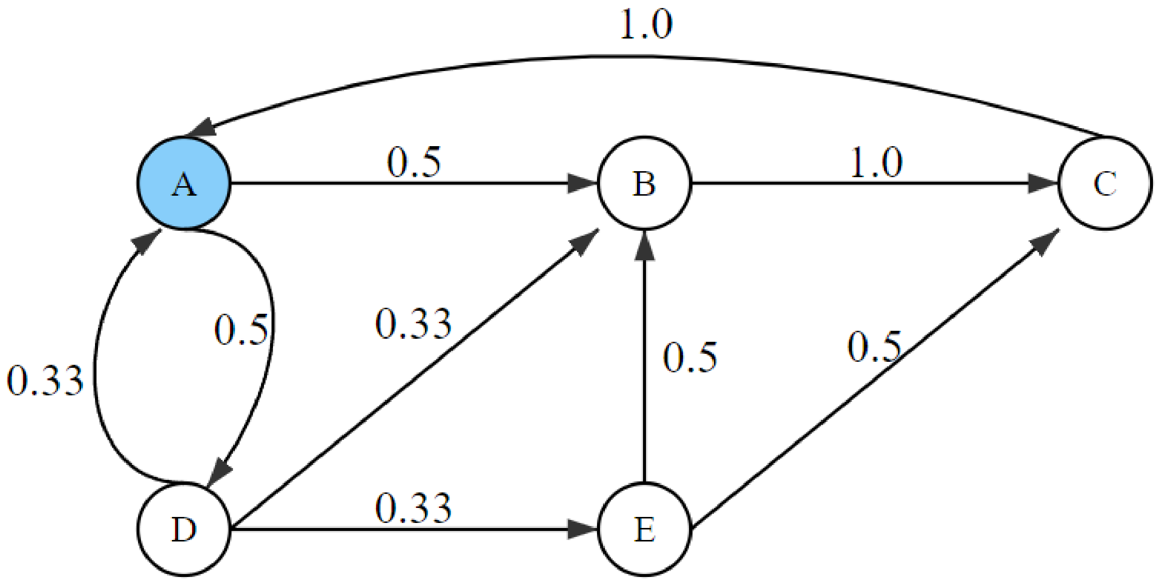
Markov Chain: Terminology

The state space of a Markov Chain is $\mathfrak{X} = \{1, 2, \dots, K\}$ for some finite K.

- The transition probability matrix P is a $K \times K$ matrix such that:

$$P(i,j) \geq 0, \qquad \forall i,j \in \mathfrak{X}$$

$$\sum_{j=1}^{K} P(i,j) = 1, \qquad \forall i \in \mathfrak{X}$$

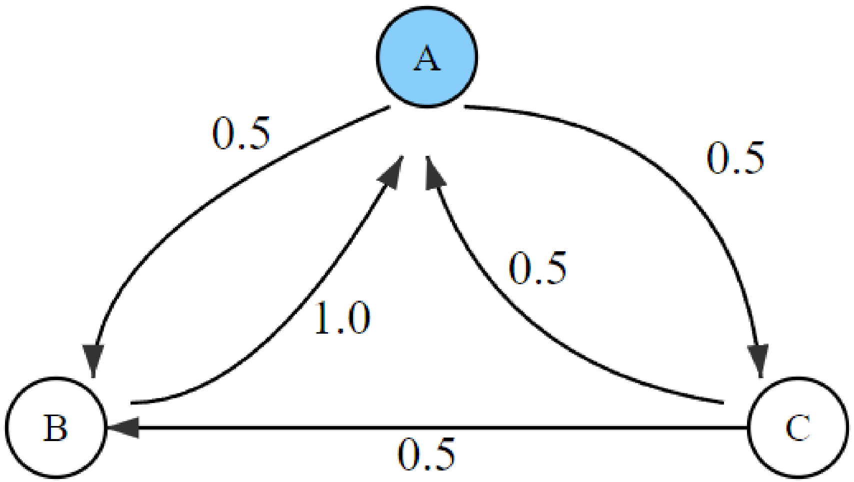


Probability of going from A to B and A to D

$$\begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0.33 & 0.33 & 0 & 0 & 0.33 \\ 0 & 0.5 & 0.5 & 0 & 0 \end{bmatrix}$$

Test Your Understanding

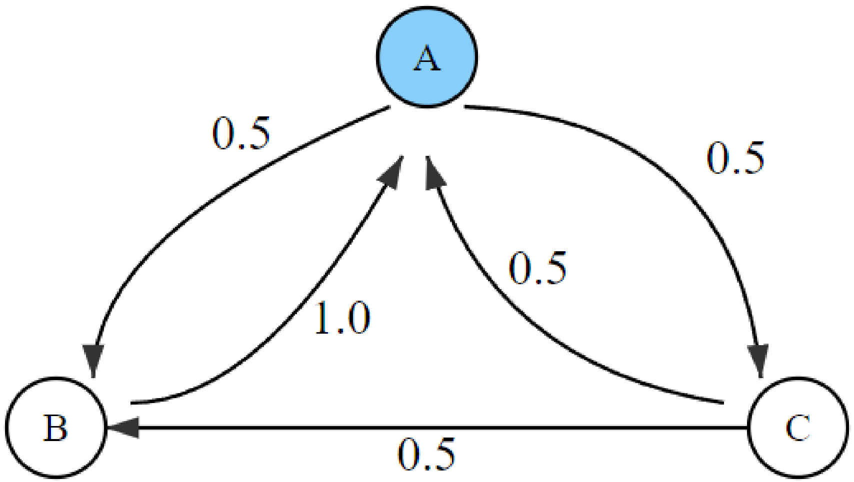
What is the first row of the transition probability matrix for the Markov Chain below?



Test Your Understanding

What is the first row of the transition probability matrix for the Markov Chain below?

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

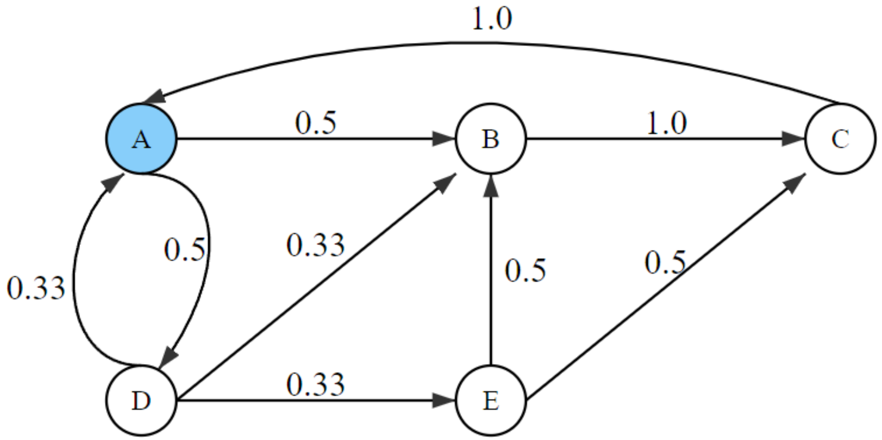


Markov Chain

Some questions we might ask:

- What is the probability that we're at a given node after a long time?
- In the long run, does the starting state matter?
- How long do we expect it to take before we reach E for the first time?
- What is the probability that we visit state E before state C?

We'll see that many questions can be framed in terms of questions about Markov Chains.



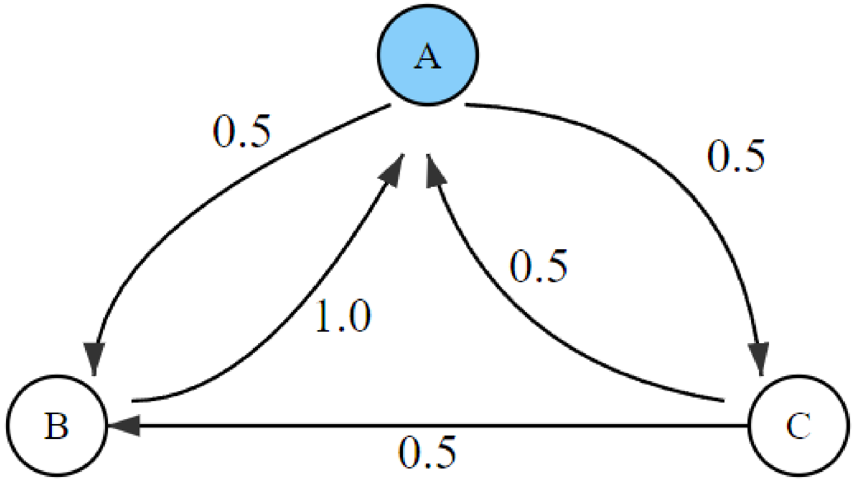
The Invariant Distribution

Lecture 26, CS70 Summer 2025

Long Term Behavior of Three State Markov Chain

As a smaller running example, let's consider the three state Markov Chain below. Simulation yields counts shown "Count" column below.

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$



Statistics

State	Count	Fraction
A	716	0.4420
B	545	0.3364
C	359	0.2216

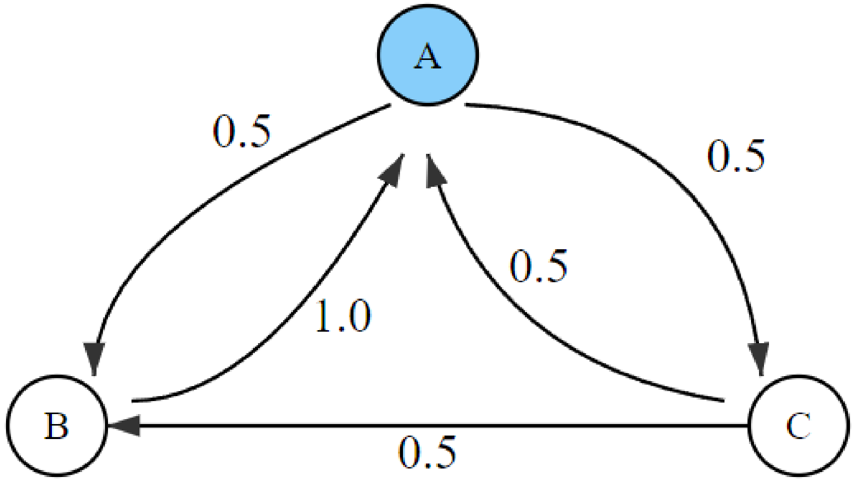
If we generate samples, we end up with around:

- 44% of the time in state A.
- 33% of the time in state B.
- 22% of the time in state C.

Long Term Behavior of Three State Markov Chain

We can use basic linear algebra to show why we always end up with this 44%/33%/22% pattern.

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$



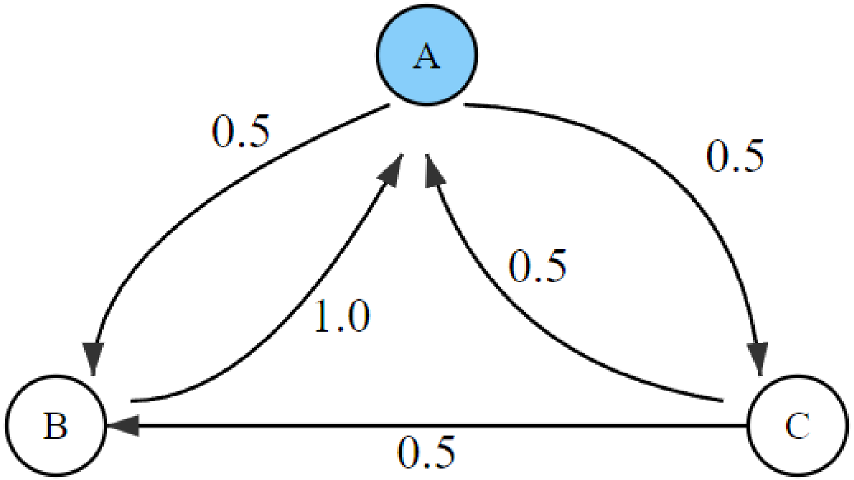
Let π_0 be a vector giving the probability that we're in any given state. For our simulation, that means $\pi_0 = [1 \;\; 0 \;\; 0]$, since we always start in state A.

What is the chance of being in each state at time step 1, i.e., what is π_1?

Long Term Behavior of Three State Markov Chain

We can use basic linear algebra to show why we always end up with this 44%/33%/22% pattern.

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$



Let π_0 be a vector giving the probability that we're in any given state. For our simulation, that means $\pi_0 = [1\ \ 0\ \ 0]$, then $\pi_1 = [0\ \ 0.5\ \ 0.5]$.

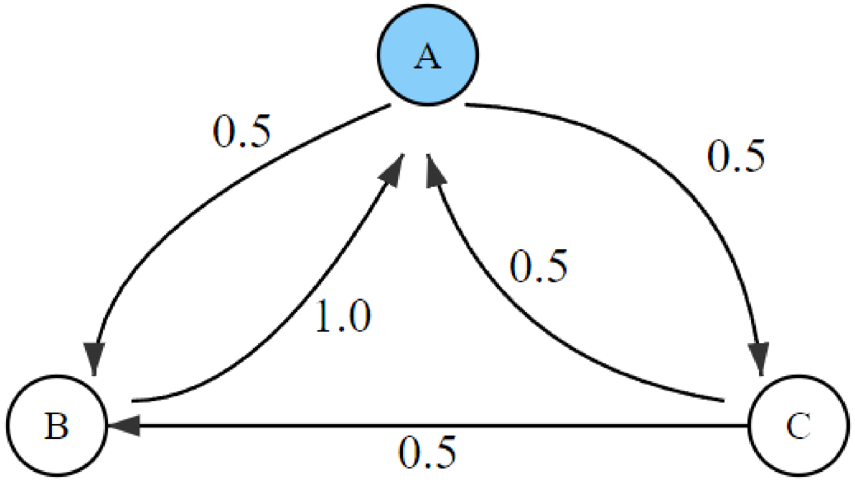
What is the chance of being in each state at time step 1, i.e., what is π_1?

- 50% chance of going into state B or state C. $\pi_1 = [0\ \ 0.5\ \ 0.5]$

Long Term Behavior of Three State Markov Chain

We can use basic linear algebra to show why we always end up with this 44%/33%/22% pattern.

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$



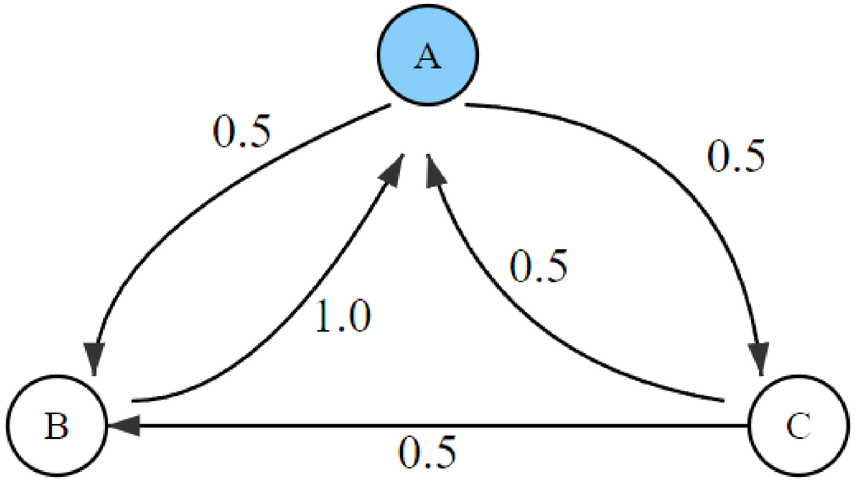
Let π_0 be a vector giving the probability that we're in any given state. For our simulation, that means $\pi_0 = [1 \;\; 0 \;\; 0]$, then $\pi_1 = [0 \;\; 0.5 \;\; 0.5]$.

What is the chance of being in each state at time step 2, i.e., what is π_2?

Long Term Behavior of Three State Markov Chain

We can use basic linear algebra to show why we always end up with this 44%/33%/22% pattern.

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$



Let π_0 be a vector giving the probability that we're in any given state. For our simulation, that means $\pi_0 = [1\ \ 0\ \ 0]$, then $\pi_1 = [0\ \ 0.5\ \ 0.5]$.

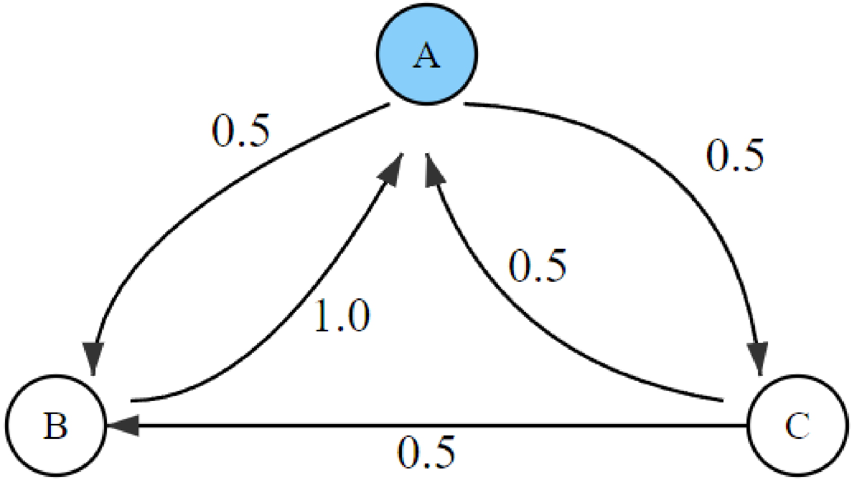
What is the chance of being in each state at time step 2, i.e., what is π_2?

- Can reason through the possibilities, or we can use linear algebra!

Long Term Behavior of Three State Markov Chain

We can use basic linear algebra to show why we always end up with this 44%/33%/22% pattern.

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$



If $\pi_1 = [0 \;\; 0.5 \;\; 0.5]$ and $P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$, what is $\pi_2 = \pi_1 P$?

We did this at the beginning of lecture today!

- $\pi_2 = \pi_1 P = [0.75 \;\; 0.25 \;\; 0]$

Long Term Behavior of Three State Markov Chain using Linear Algebra

We can use basic linear algebra to show why we always end up with this 44%/33%/22% pattern.

Let π_0 be a vector giving the probability that we're in any given state. For our simulation, that means $\pi_0 = [1 \;\; 0 \;\; 0]$, then $\pi_1 = [0 \;\; 0.5 \;\; 0.5]$.

- $\pi_1 = \pi_0 P = [0 \;\; 0.5 \;\; 0.5]$
- $\pi_2 = \pi_1 P = [0.75 \;\; 0.25 \;\; 0]$
- $\pi_3 = \pi_2 P = [0.25 \;\; 0.375 \;\; 0.375]$
- $\pi_4 = \pi_3 P = \pi_2 P^2 = \pi_1 P^3 = \pi_0 P^4 = [0.5625 \;\; 0.3125 \;\; 0.125]$
- $\pi_5 = \pi_0 P^5 = [0.375 \;\; 0.34375 \;\; 0.28125]$
- …
- $\pi_9 = \pi_0 P^9 = [0.4375 \quad 0.33398438 \quad 0.22851562]$

Long Term Behavior of Three State Markov Chain using Linear Algebra

We can use basic linear algebra to show why we always end up with this 44%/33%/22% pattern.

Let π_0 be a vector giving the probability that we're in any given state. For our simulation, that means $\pi_0 = [1 \;\; 0 \;\; 0]$, then $\pi_1 = [0 \;\; 0.5 \;\; 0.5]$.

- $\pi_n = [1 \;\; 0 \;\; 0]P^n$

Limit as $n \to \infty$ is $\left[\frac{4}{9} \;\; \frac{3}{9} \;\; \frac{2}{9}\right]$

In the next lecture, we'll show that this holds for any starting distribution, not just $\pi_0 = [1 \;\; 0 \;\; 0]$.

Long Term Behavior of Three State Markov Chain – using observations

Note: If we have information about the Markov Chain at some point other than time zero, we can update probabilities accordingly.

Example: Suppose we know $\pi_0 = [0.3\ \ 0.2\ \ 0.5]$, we know that $X_2 = 3$, and $X_5 = 1$, then:

$$\pi_0 = [0.3\ \ 0.2\ \ 0.5]$$

$$\pi_1 = [0.3\ \ 0.2\ \ 0.5]P$$

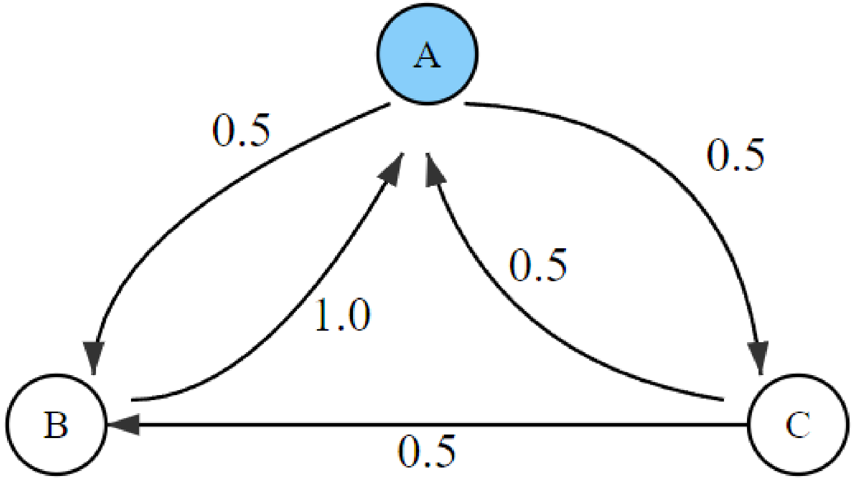
$X_2 = 3$ Or equivalently: $\pi_2 = [0\ 0\ 1]$

$$\pi_3 = [0\ \ 0\ \ 1]P$$

$$\pi_4 = [0\ \ 0\ \ 1]P^2$$

$X_5 = 1$ Or equivalently: $\pi_5 = [1\ 0\ 0]$

$$X_6 = [1\ 0\ 0]P$$



Alternate View: Distribution of X_1

Denote the distribution of X_1 by π_1

$$\pi_1(j) = P(X_1 = j) = \sum_{i=1}^{k} P(X_0 = i, X_1 = j)$$

$$= \sum_{i=1}^{k} P(X_0 = i) \cdot P(X_1 = j | X_0 = i)$$

$$= \sum_{i=1}^{k} \pi_0(i) \cdot P(i, j)$$

Or in linear algebra notation: π_0 and π_1 are row vectors, and P is a matrix of transition probabilities. We have that $\pi_1 = \pi_0 P$.

Alternate View: The Distribution of X_n

Denote the distribution of X_n by π_n

$$\pi_n(j) = P(X_n = j) = \sum_{i=1}^{k} P(X_{n-1} = i, X_n = j)$$

$$= \sum_{i=1}^{k} P(X_{n-1} = i) \cdot P(X_n = j | X_{n-1} = i)$$

$$= \sum_{i=1}^{k} \pi_{n-1}(i) \cdot P(i, j)$$

In vector-matrix form, $\pi_n = \pi_{n-1} P$

... and $\pi_{n-1} = \pi_{n-2} P$ so $\pi_n = \pi_{n-1} P = (\pi_{n-2} P) P = \pi_{n-2} P^2$

Continuing, we have $\pi_n = \pi_{n-1} P = \pi_{n-2} P^2 = \cdots = \cdots = \pi_0 P^n$

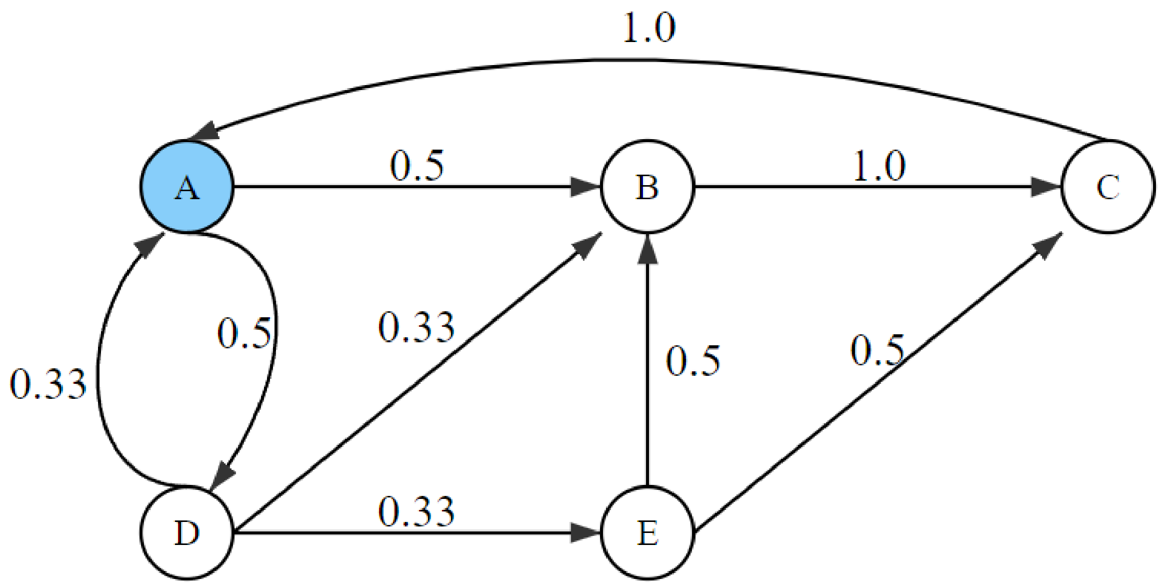
Hitting Time

Lecture 26, CS70 Summer 2025

Hitting Time

Another question we might ask ourselves: If we start in state i, how many time steps $\beta(i)$ do we expect it to take before we hit end state $\mathfrak{E}$?

For example, if our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.



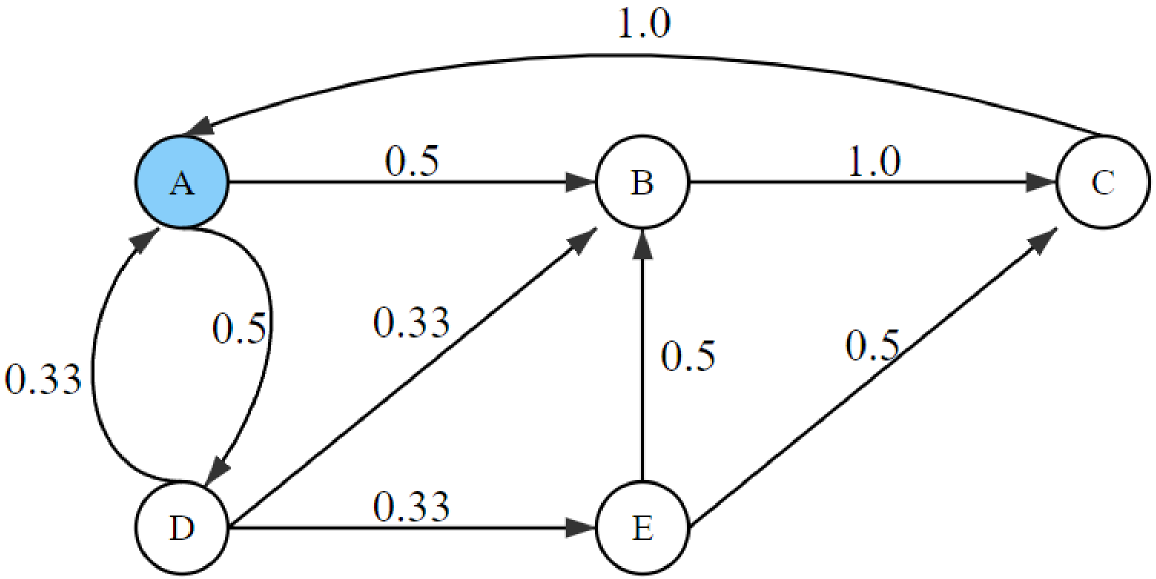
Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- To do this, let's write the so-called first step equations for this Markov Chain.

"The journey of a thousand miles begins with a single step." – Lao Tzu.

First, we'll observe $\beta(E) = 0$. This is trivial: If we're at E already, we have to wait 0 time steps to get to E.



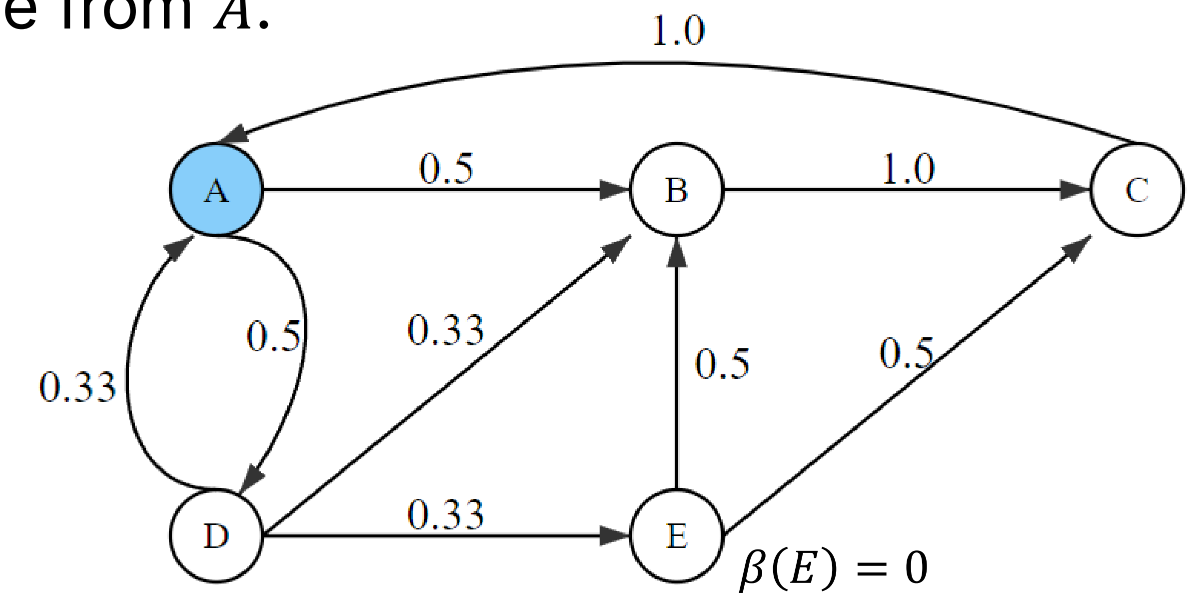
Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- To do this, let's write the so-called first step equations for this Markov Chain.

Next, let's consider $\beta(C)$.

- The only thing that can happen next is that we go to state A.
- Thus, expected wait time is $\beta(C) = 1 + \beta(A)$, where $\beta(A)$ is whatever the expected wait time from A.

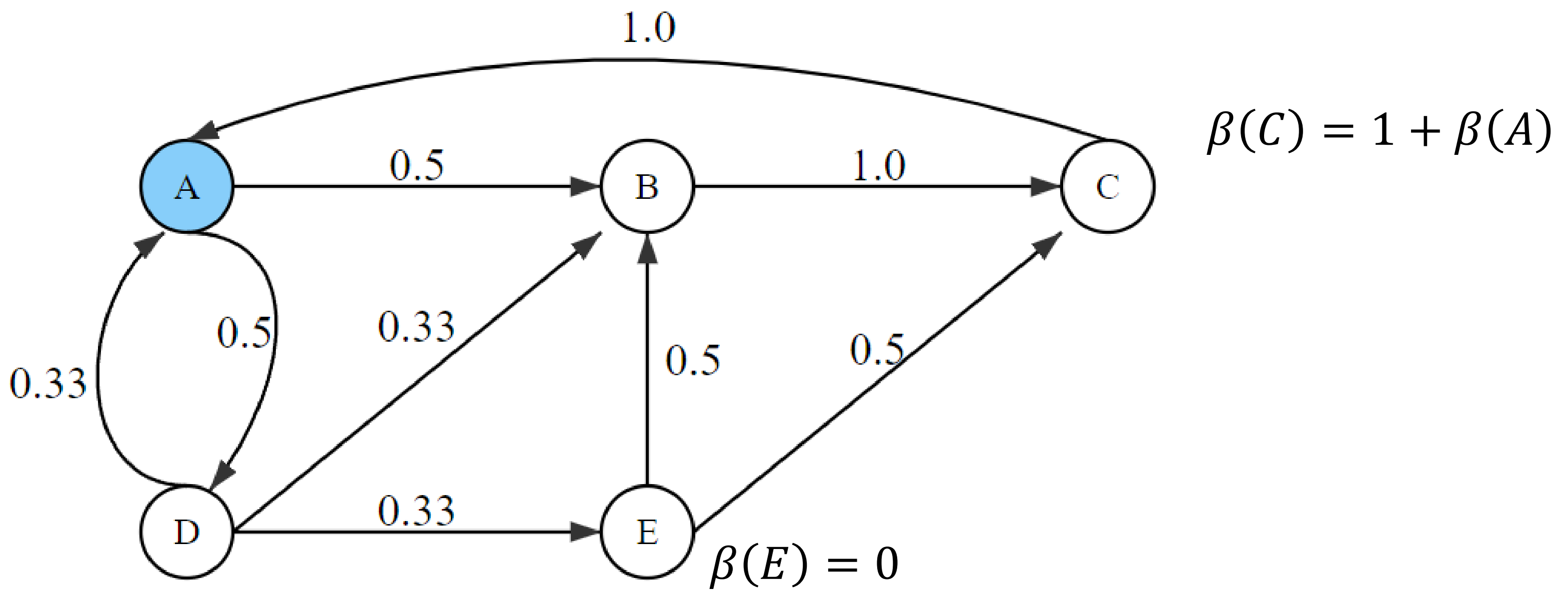


Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- To do this, let's write the so-called first step equations for this Markov Chain.

What is $\beta(B)$?



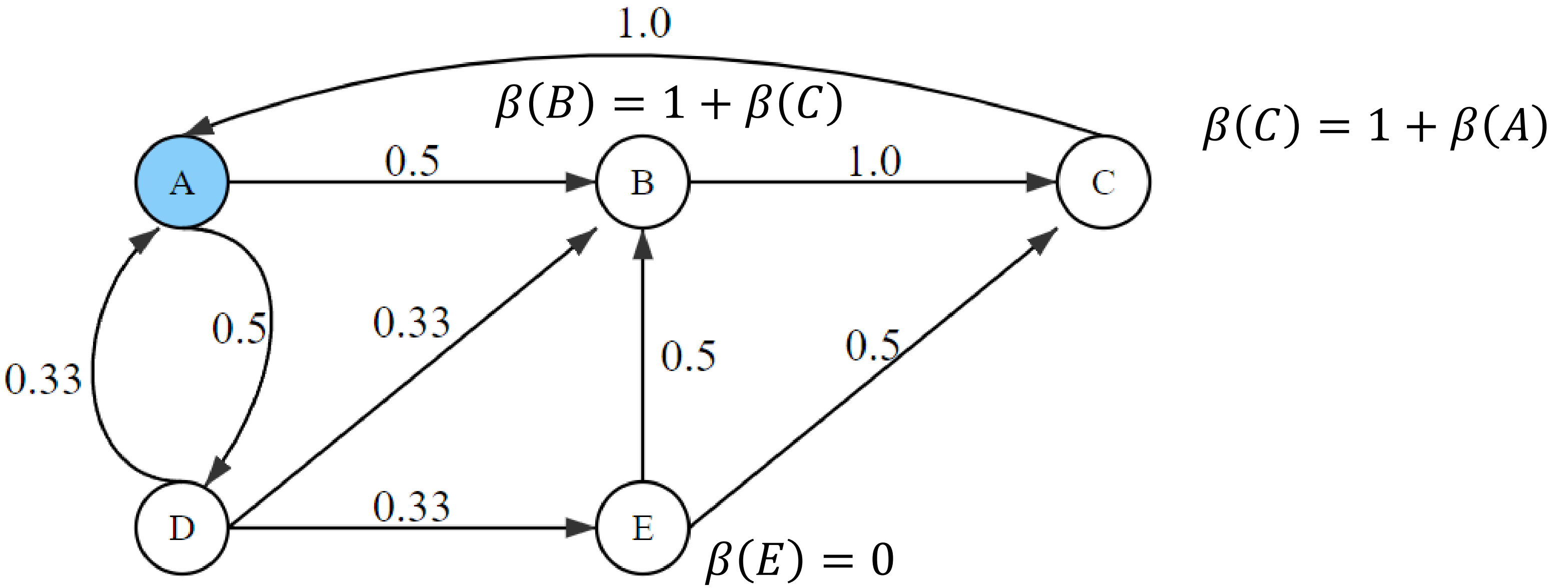
Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- To do this, let's write the so-called first step equations for this Markov Chain.

What is $\beta(B)$?

- $\beta(B) = 1 + \beta(C)$

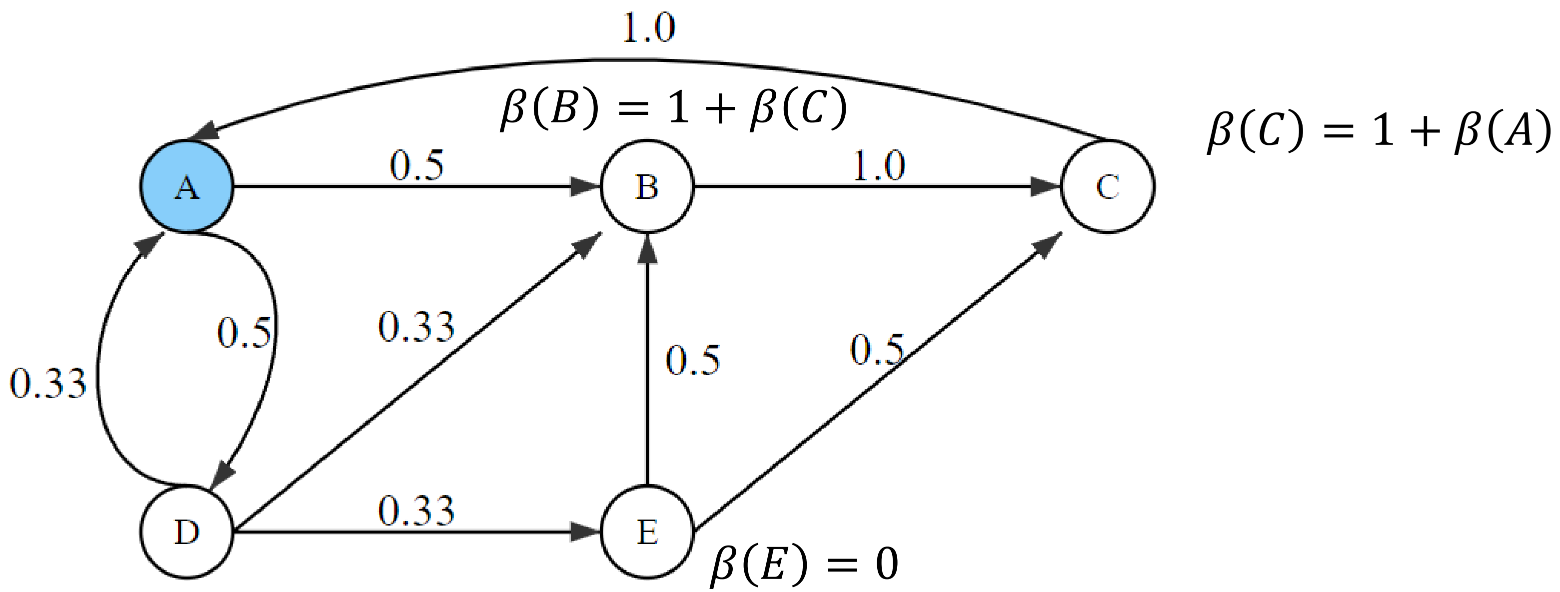


Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- To do this, let's write the so-called first step equations for this Markov Chain.

What is $\beta(A)$?



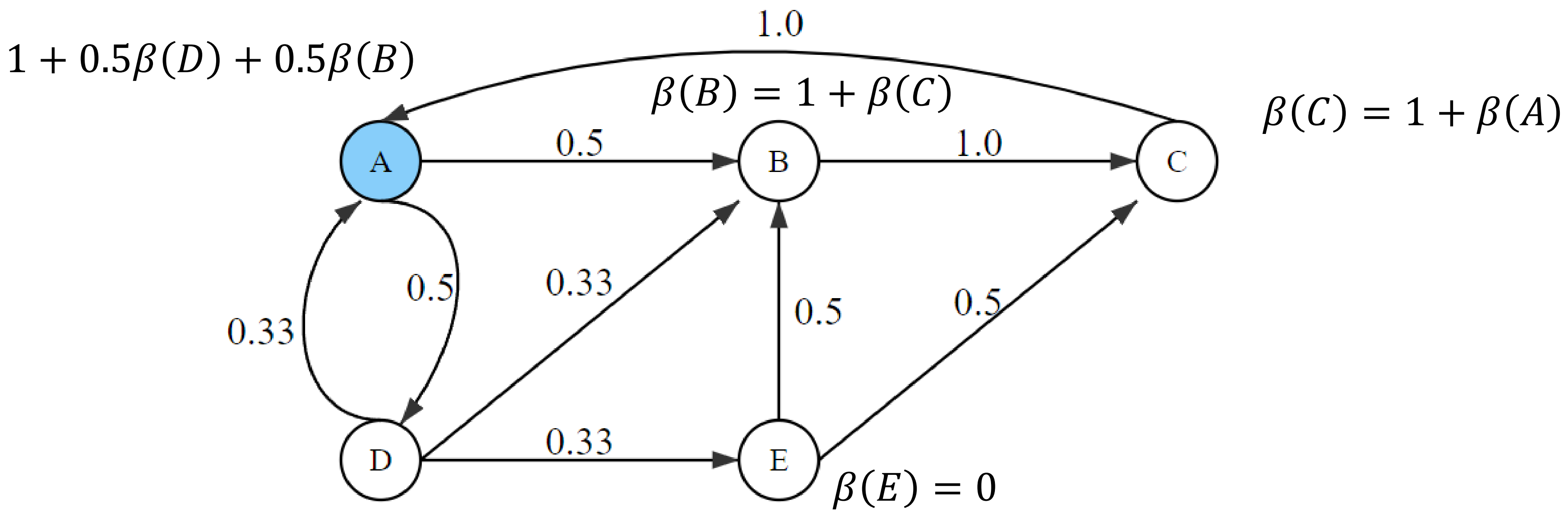
Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- To do this, let's write the so-called first step equations for this Markov Chain.

What is $\beta(A)$?

- $1 + 0.5\beta(D) + 0.5\beta(B)$ $\leftarrow$ *nothing new… just conditional expectation*

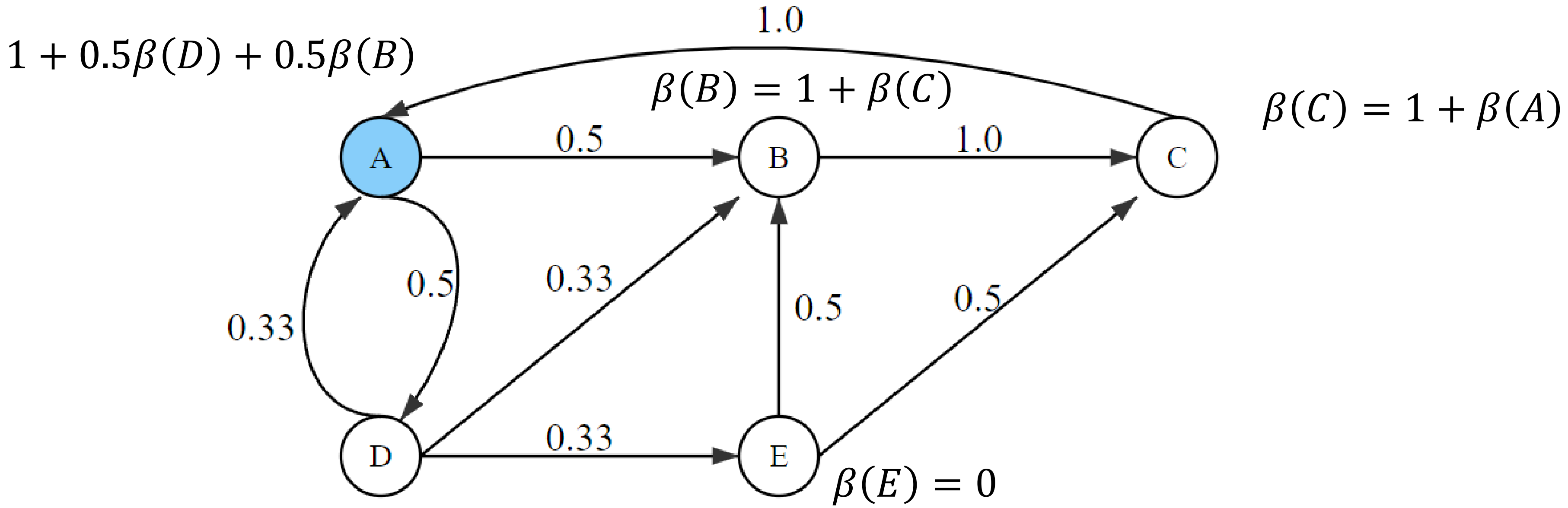


Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- To do this, let's write the so-called first step equations for this Markov Chain.

What is $\beta(D)$?



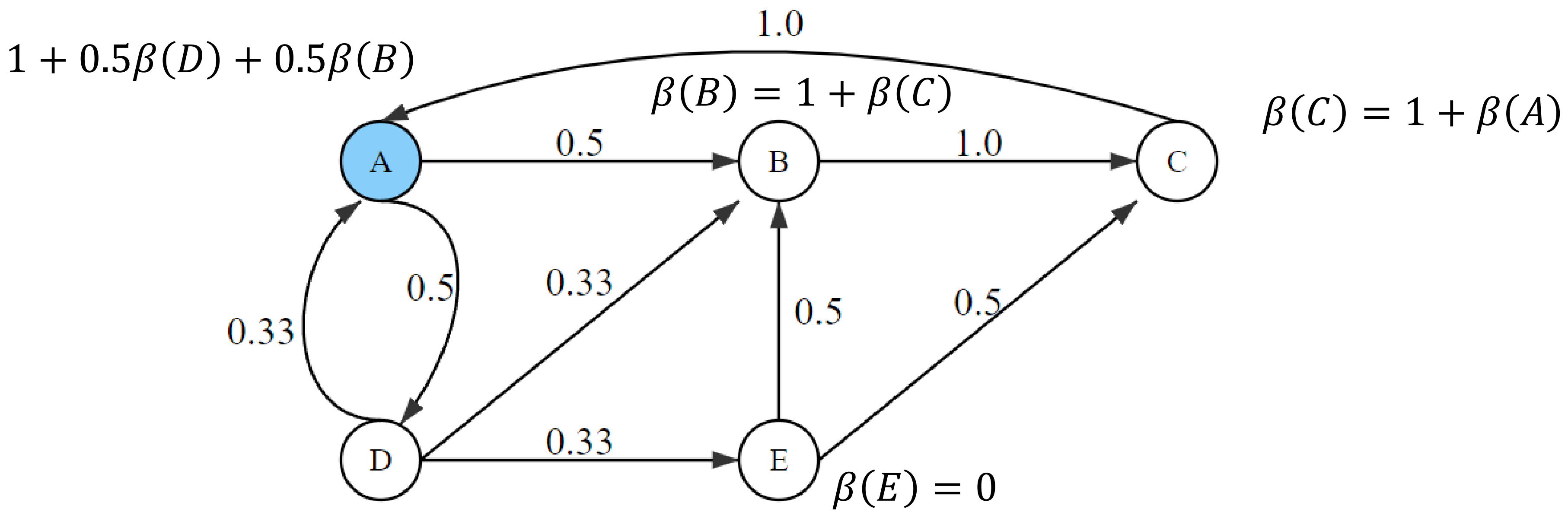
Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- To do this, let's write the so-called first step equations for this Markov Chain.

What is $\beta(D)$?

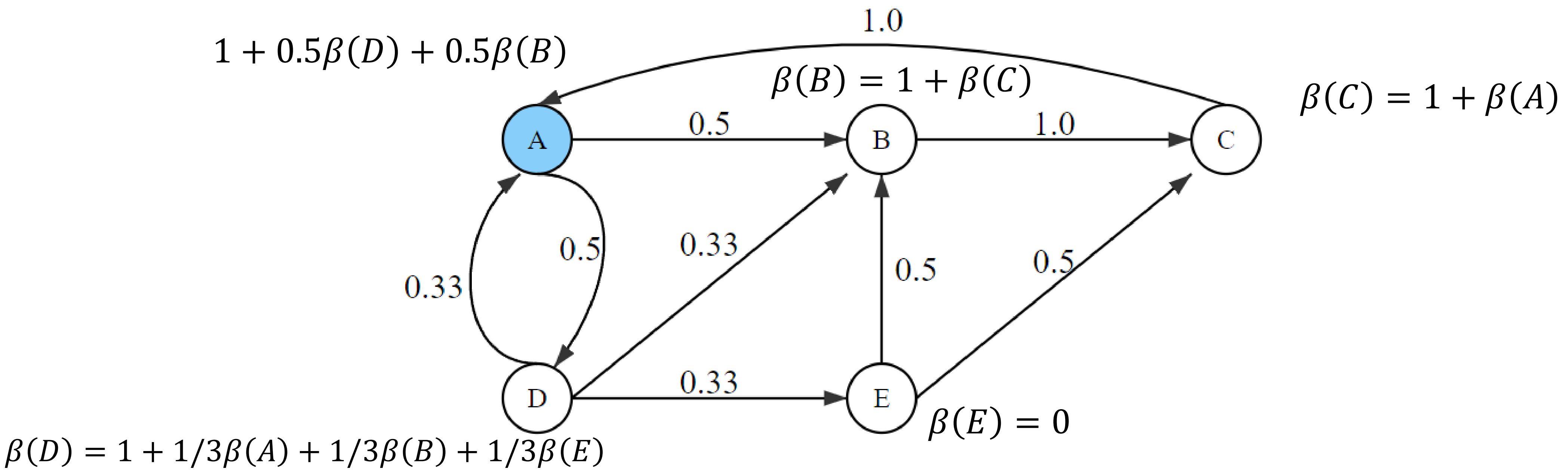
- $1 + 1/3\beta(A) + 1/3\beta(B) + 1/3\beta(E)$



Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- The first step equations for this Markov Chain are:
  - $\beta(A) = 1 + 0.5\beta(D) + 0.5\beta(B)$
  - $\beta(B) = 1 + \beta(C)$
  - $\beta(C) = 1 + \beta(A)$
  - $\beta(D) = 1 + 1/3\beta(A) + 1/3\beta(B) + 1/3\beta(E)$
  - $\beta(E) = 0$



Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- The first step equations for this Markov Chain are:
  - $\beta(A) = 1 + 0.5\beta(D) + 0.5\beta(B)$
  - $\beta(B) = 1 + \beta(C)$
  - $\beta(C) = 1 + \beta(A)$
  - $\beta(D) = 1 + 1/3\beta(A) + 1/3\beta(B) + 1/3\beta(E)$
  - $\beta(E) = 0$

This is just a system of 5 linear equations in five unknowns. Straightforward to solve (through substitution, gaussian elimination, computer solver, etc).

- (Or you could say it's a system of four equations in four unknowns since one of them is just zero)

Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- The first step equations for this Markov Chain are:
  - $\beta(A) = 1 + 0.5\beta(D) + 0.5\beta(B)$
  - $\beta(B) = 1 + \beta(C)$
  - $\beta(C) = 1 + \beta(A)$
  - $\beta(D) = 1 + 1/3\beta(A) + 1/3\beta(B) + 1/3\beta(E)$
  - $\beta(E) = 0$

Ad-hoc solve – first, easy eliminations are $\beta(E)$, $\beta(C)$, and $\beta(B)$ - leaving:

- $\beta(A) = 1 + 0.5\beta(D) + 0.5\big(2 + \beta(A)\big) = 2 + 0.5\beta(D) + 0.5\beta(A) = 4 + \beta(D)$
- $\beta(D) = 1 + 1/3\beta(A) + 1/3\beta(B) + 1/3\beta(E)$

  $= 1 + 1/3\beta(A) + 1/3\big(2 + \beta(A)\big) = 5/3 + 2/3\beta(A) = 5/3 + 2/3(4 + \beta(D))$

  $= 13/3 + 2/3\beta(D) = 13$

Then: backsolve for others…. $\beta(A) = 17$, $\beta(B) = 19$, $\beta(C) = 18$

Hitting Time

Our desired end state is $\mathfrak{E} = E$, we want to find $\beta(A)$, $\beta(B)$, $\beta(C)$, $\beta(D)$, and $\beta(E)$.

- The first step equations for this Markov Chain are:
  - $\beta(A) = 1 + 0.5\beta(D) + 0.5\beta(B)$
  - $\beta(B) = 1 + \beta(C)$
  - $\beta(C) = 1 + \beta(A)$
  - $\beta(D) = 1 + 1/3\beta(A) + 1/3\beta(B) + 1/3\beta(E)$
  - $\beta(E) = 0$

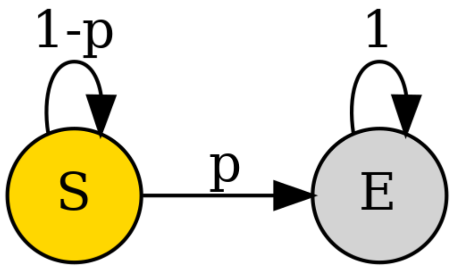
... or ... solve with linear system solution software...

Result: $\beta(A) = 17$, $\beta(B) = 19$, $\beta(C) = 18$, $\beta(D) = 13$

Example 2: Expectation of a Geometric Random Variable

For the fourth time, let's compute the expectation of a geometric random variable.

- We can model a geometric random variable as a Markov chain with two states. One is the state where we have not yet gotten our first heads, the other is where we've gotten our first heads.



$\beta(S)$ is average time a Markov Chain starting at S takes to reach E.

- First step equation is just $\beta(S) = 1 + (1-p) \cdot \beta(S) + p \cdot \underbrace{\beta(E)}$ This is just 0.

$$\beta(S) = 1 + \beta(S) - p\beta(S)$$

$$\beta(S) = 1/p$$

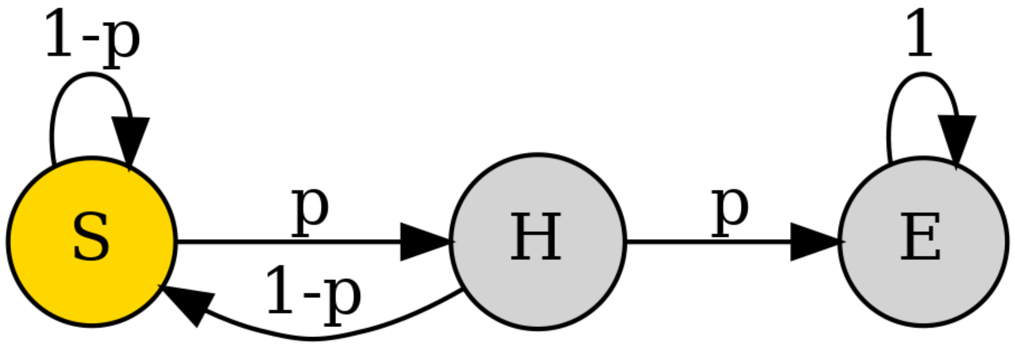
Example 3: Flipping Until Two Consecutive Heads.

Suppose we now want to model the process of flipping a coin until we get two consecutive heads.

- Flips at times 2 and 3 not independent of flips at times 1 and 2 – not Bernoulli!
- How many flips on average do you think it will take if coin is fair, i.e., $p = 0.5$?
- We can model with a Markov chain though - what does it look like?

Example 3: Flipping Until Two Consecutive Heads.

Suppose we now want to model the process of flipping a coin until we get two consecutive heads. What does the equivalent Markov Chain look like?



First step equations:

- $\beta(S) = 1 + (1-p)\beta(S) + p\beta(H)$
- $\beta(H) = 1 + (1-p)\beta(S) + p\beta(E)$
- $\beta(E) = 0$

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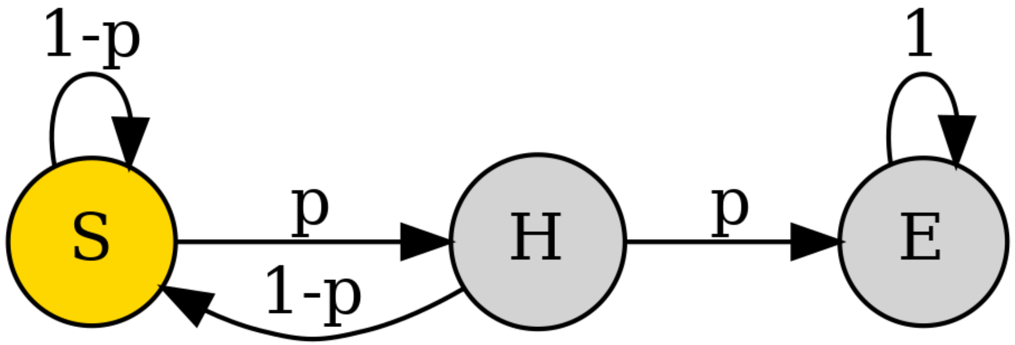


First step equations:

- $\beta(S) = 1 + (1-p)\beta(S) + p\beta(H) = 1 + (1-p)\beta(S) + p(1 + (1-p)\beta(S))$
- $\beta(H) = 1 + (1-p)\beta(S)$

$$\beta(S) = 1 + \beta(S) - p\beta(S) + p + p\beta(S) - p^2\beta(S)$$

$$p^2\beta(S) = 1 + p$$

$$\beta(S) = \frac{1+p}{p^2}$$

If $p = 0.5$: $\beta(S) = \dfrac{3/2}{1/4} = 6$

Probability of A Before B

Lecture 26, CS70 Summer 2025

Probability of A Before B

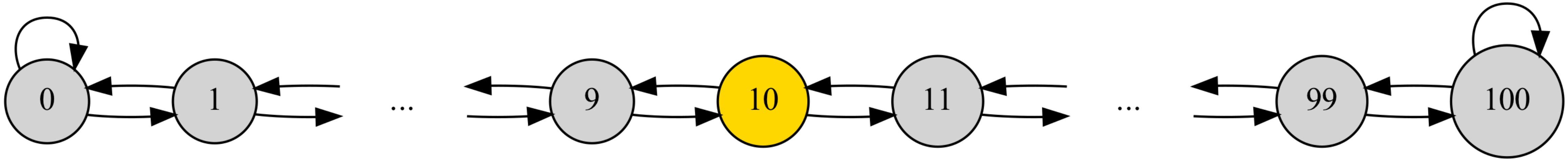
Another problem we might ask about Markov Chains: What is the probability that if we start at state i, that we reach state A before state B?

Example: You're gambling, have a 50/50 chance of winning. Every round:

- 50% chance you win $1
- 50% chance you lose $1

Your plan is to keep playing until you make $100.

- If you start with $10 dollars, what's the chance you get to $100 before you get to $0?

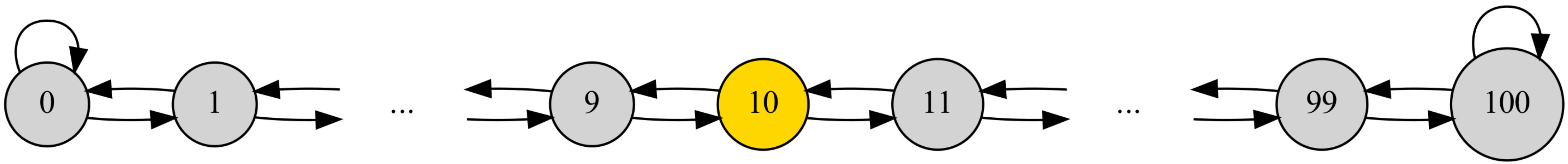


Probability of A Before B

Another problem we might ask about Markov Chains: What is the probability that if we start at state i, that we reach state $A = 100$ before state $B = 0$?

For $i \in \{0, 1, \ldots, 100\}$. Let $\alpha(i)$ be the probability of reaching 100 before 0 starting at i. Which of these are true?

- $\alpha(0) = 1$
- $\alpha(0) = 0$
- $\alpha(100) = 1$
- $\alpha(100) = 0$



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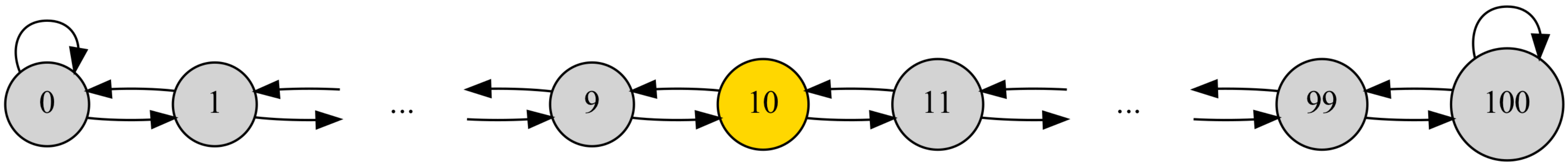
- $\boldsymbol{\alpha(0) = 0}$
- $\boldsymbol{\alpha(100) = 1}$
- ~~$\alpha(100) = 0$~~



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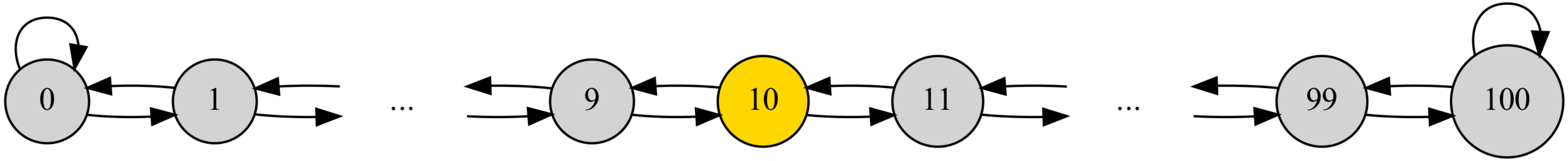
For $i \in \{0, 1, \ldots, 100\}$. Let $\alpha(i)$ be the probability of reaching 100 before 0 starting at i. We know that $\alpha(0) = 1$, $\alpha(100) = 0$. Which of the two statements below are true?

- $\alpha(i) = 1 + 0.5\alpha(i-1) + 0.5\alpha(i+1)$
- $\alpha(i) = 0.5\alpha(i-1) + 0.5\alpha(i+1)$



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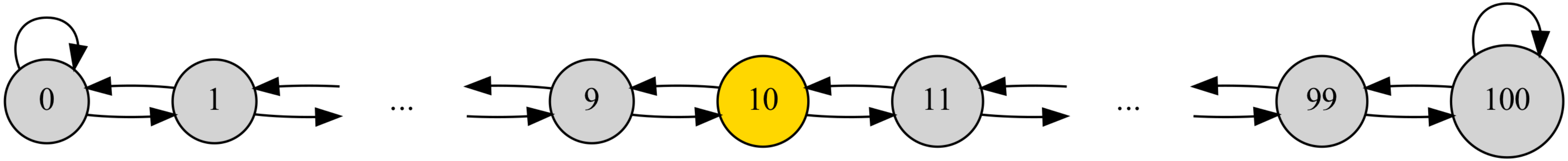
- ~~$\alpha(i) = 1 + 0.5\alpha(i-1) + 0.5\alpha(i+1)$~~ *remember: these are probabilities!*
- $\alpha(i) = 0.5\alpha(i-1) + 0.5\alpha(i+1)$



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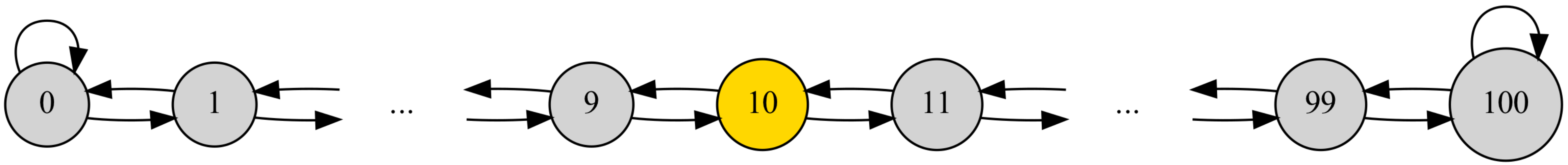
Why is this true? The event that the Markov chain gets to 100 before 0 is partitioned into two events:

- Go to $i-1$, then later get to 100: $P(\text{go left}) \cdot P(100 \text{ before } 0 \mid \text{go left})$
- Go to $i+1$, then later get to 100: $P(\text{go right}) \cdot P(100 \text{ before } 0 \mid \text{go right})$



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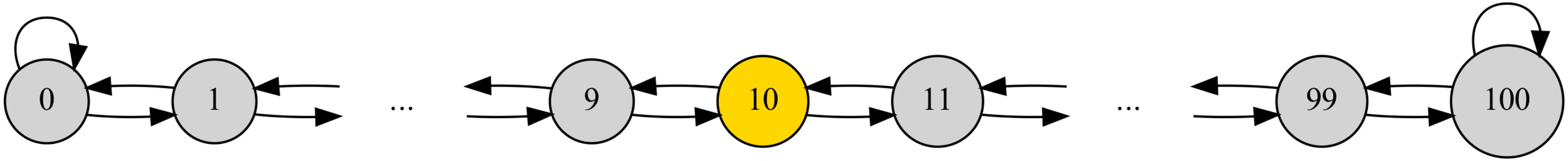
Now we have a system of 99 linear equations in 99 unknowns.

- Could solve with a computer.
- Or we can be clever!



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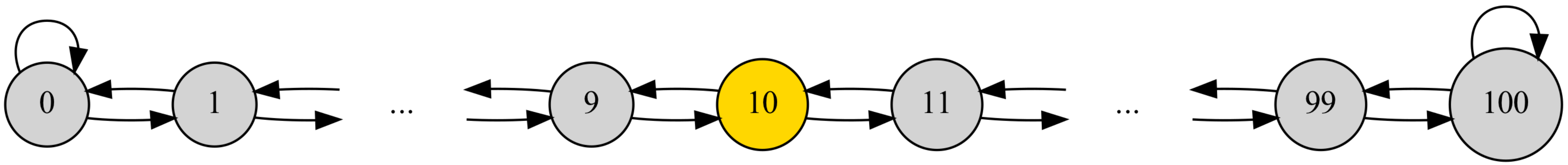
Now we have a system of 99 linear equations in 99 unknowns.

- Every $\alpha(i)$ is the average of its left and right neighbor, except leftmost node is 0 and rightmost node is 1. So what is $\alpha(i)$?

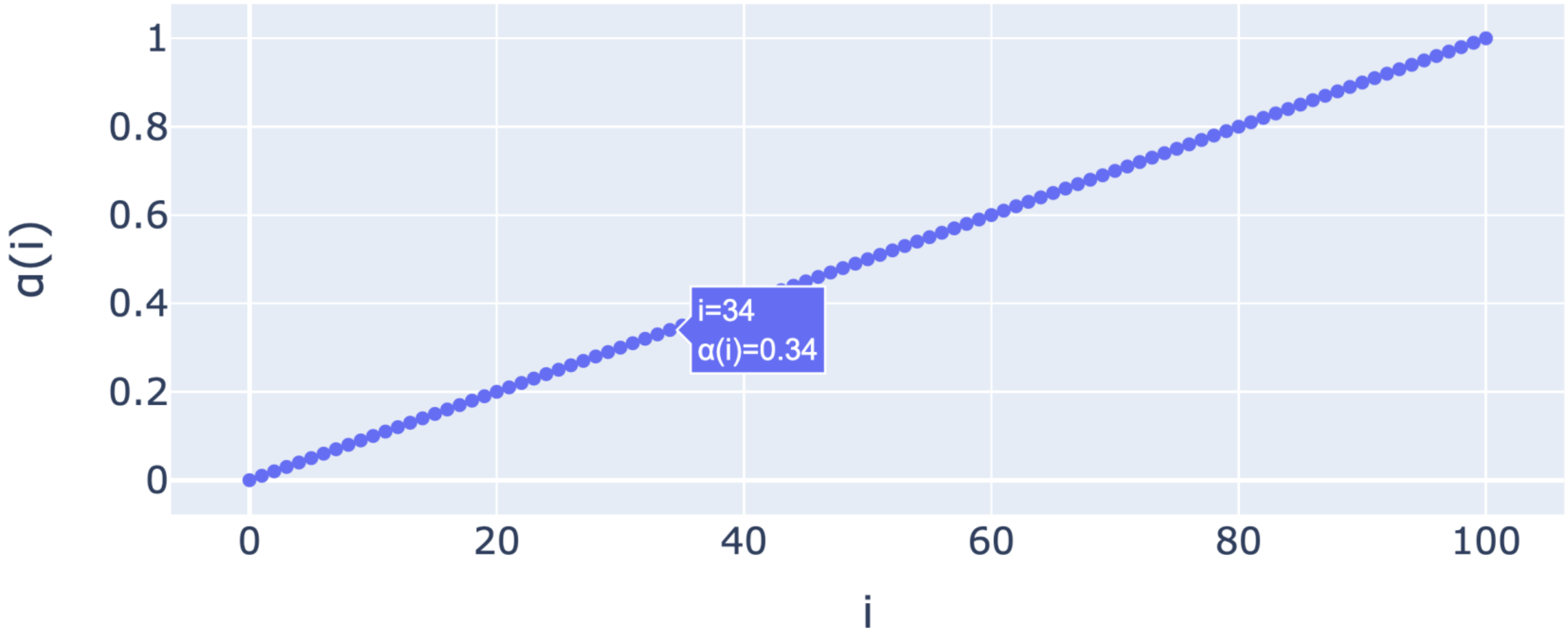


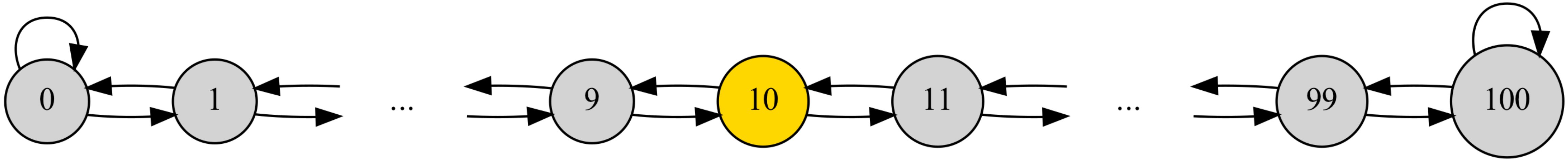
Probability of A Before B

Now we have a system of 99 linear equations in 99 unknowns.

- Every $\alpha(i)$ is the average of its left and right neighbor, except leftmost node is 0 and rightmost node is 1. So what is $\alpha(i)$? Must be a straight line starting at $\alpha(0) = 0$ and ending $\alpha(100) = 1$. *Intuition is great: but verify afterwards!*

$$\alpha(i) = \frac{i}{100}$$



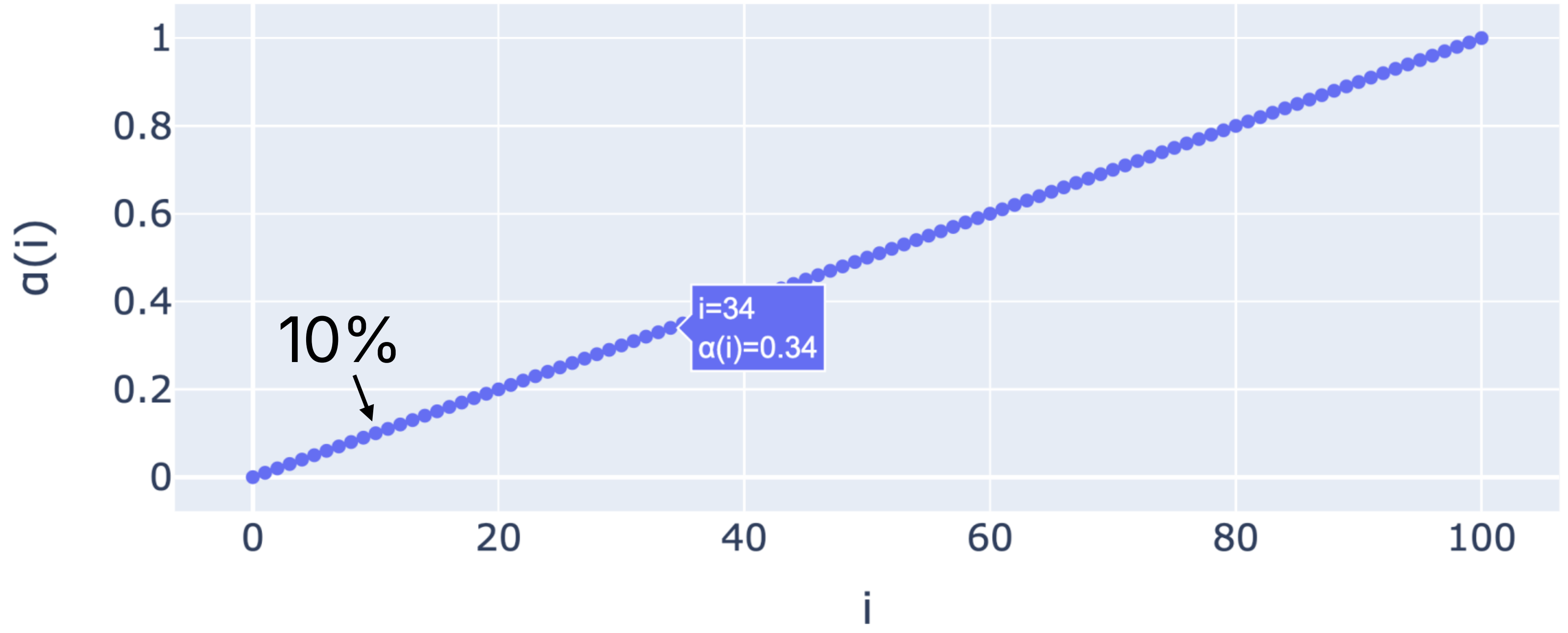


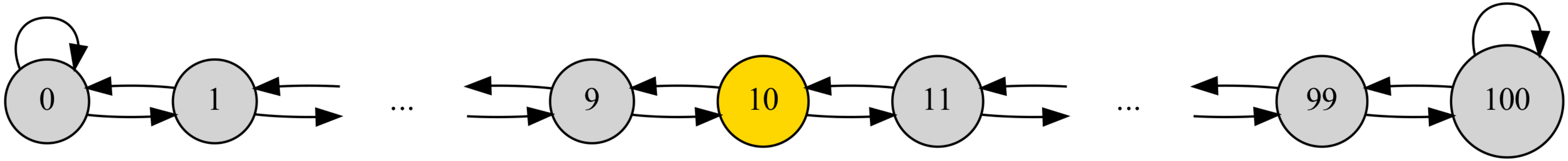
Probability of A Before B

Another problem we might ask about Markov Chains: What is the probability that if we start at state i, that we reach state $A = 100$ before state $B = 0$?

- Example: If we start with \$10, there is a 10/100=10% chance that we get to \$100 before we get to \$0.

$$\alpha(i) = \frac{i}{100}$$





Example 2: Casino with House Edge

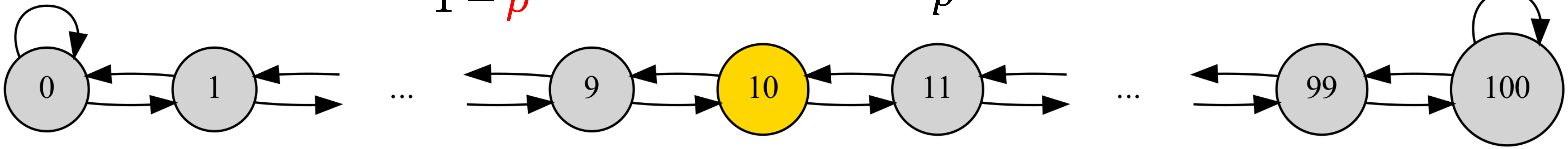
Example: You're gambling, have a p=48% chance of winning. Every round:

- 48% chance you win $1.
- 52% chance you lose $1.

Your plan is to keep playing until you make M=$100.

- If you start with $10 dollars, what's the chance you get to M=$100 before you get to $0?
  - $\alpha(i) = 0.48 \cdot \alpha(i+1) + 0.52 \cdot \alpha(i-1)$
  - 99 equations with 99 unknowns. Appendix of the notes gives solution.

$$\alpha(i) = \frac{1-\rho^i}{1-\rho^M} \qquad \rho = \frac{1-p}{p} \qquad \text{If } p \neq 0.5$$



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- 48% chance you win $1.
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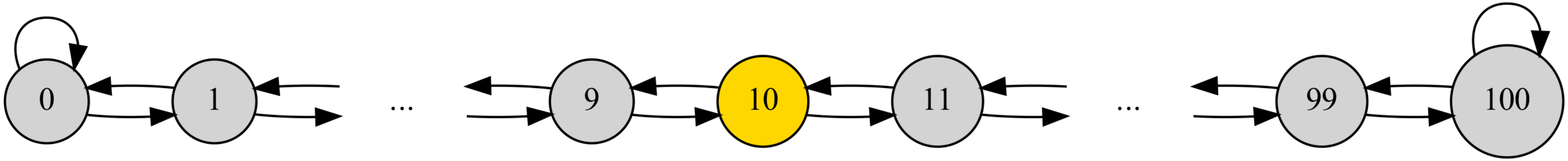
Your plan is to keep playing until you make M=$100.

- If you start with $10 dollars, what's the chance you get to M=$100 before you get to $0?

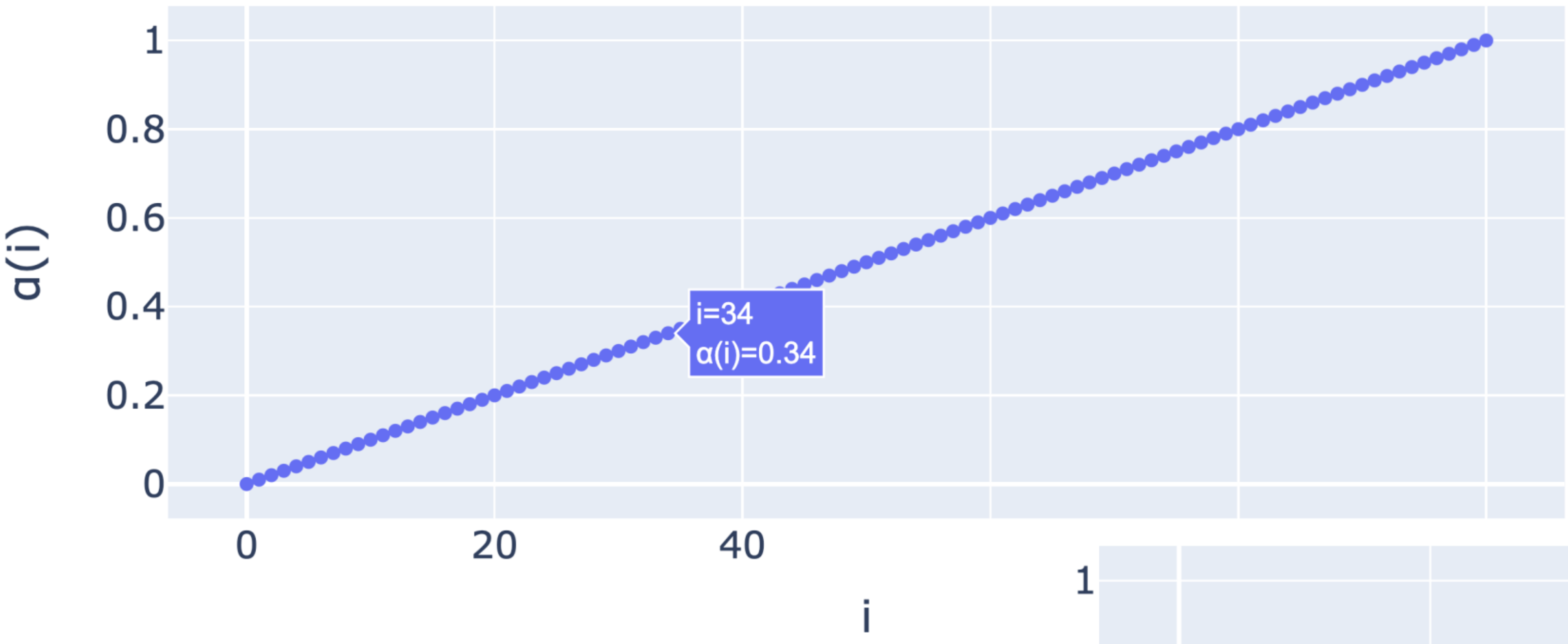
$$\alpha(i) = \frac{1-\rho^i}{1-\rho^{100}} \approx \frac{1}{2440} \qquad \rho = \frac{1-p}{p} = 0.52/0.48$$

i=10

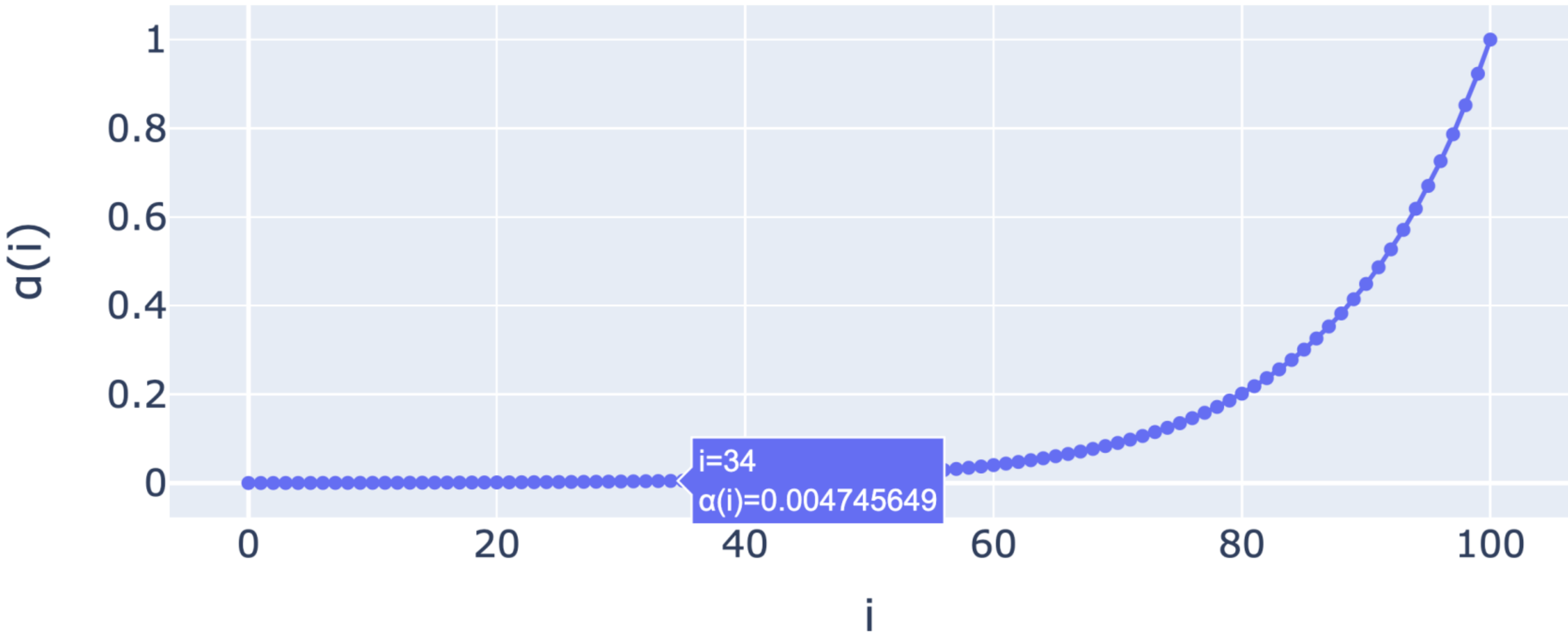


50/50 Odds vs. 48/52 Odds

Visually, we can compare the two situations below.



The tiny house edge makes it extremely difficult to walk away as a winner.



Summary

Lecture 26, CS70 Summer 2025

First Step Equations

A Markov Chain is a series of random variables, such that

- We know the distribution of X_0, denoted as π_0.
- We know the transition probability matrix P.
- $P\left(X_{next} = j \middle| X_{prev} = i\right) = P(i,j)$
- The distribution of kth random variable in the chain is $\pi_k = \pi_0 P^k$

By modeling a problem as a Markov Chain, can solve using "first step analysis".

- Computations are often easier than other "lower level" techniques.
- Examples:
  - Finding the expected time to reach a given state (getting two tails in a row).
  - Finding the probability of reaching one state before another (making $100 before running out of money).