Induction

UC Berkeley – Summer 2025 – Steve Tate

Lecture 3

Induction!!!

Topics for today:

1. Inductions basics (simple induction)
2. Strengthening the induction hypothesis
3. Strong induction
4. How to mis-use induction

A Teacher's Plans Foiled: 7-year old Gauss.

Teacher: Hello class.
Teacher: [*Thinking: I sure could use a break from these kids*]

Teacher: Please add the numbers from 1 to 100.
Teacher: [*Settles in for a nice break while students do busywork*]

Gauss: It's 5050!

Narrator: That's $\frac{(100)(101)}{2} = 50 \times 101$

Induction

Child Gauss: $(\forall n \in \mathbb{N})\left(\sum_{i=0}^{n} i = \frac{n(n+1)}{2}\right)$ But is it always true? Proof?

Generic problem: For predicate $P(n)$, prove $(\forall n \in \mathbb{N})(P(n))$

Can test small values of n directly: $P(0)$? $P(1)$?
But.... what about $P(100)$?
Even worse: Impossible to directly verify for infinitely many $n \in \mathbb{N}$.

Another approach – take one isolated step in sequence of natural numbers
Specifically, prove $(\forall k \in \mathbb{N})\big(P(k) \implies P(k+1)\big)$

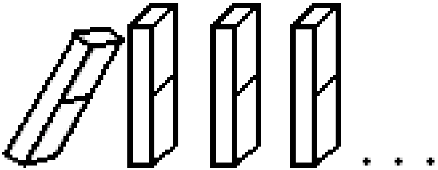
So: Verify $P(0)$ directly	so we know $P(0)$ is true
$P(0) \implies P(1)$	since $P(0)$ is true, $P(1)$ is true
$P(1) \implies P(2)$	since $P(1)$ is true, $P(2)$ is true
$P(2) \implies P(3)$	since $P(2)$ is true, $P(3)$ is true
$P(3) \implies P(4)$	since $P(3)$ is true, $P(4)$ is true $\cdots$

... goes on indefinitely!

Every n is reached in a finite number of steps, so $P(n)$ is true *for all* $n \in \mathbb{N}$!

Notes Visualization

Visualization: an infinite(?!) sequence of dominoes.



Prove they all fall down.

- $P(0)$ = "First domino falls"
- $(\forall k)\ \big(P(k) \implies P(k+1)\big)$:
  "kth domino falls implies that $k+1$st domino falls"

Induction: Proof Form

This is the form for what we call “simple induction” – to prove:

$$(\forall n \in \mathbb{N})\big(P(n)\big)$$

Directly prove $P(0)$ – this is called the base case.

Prove $(\forall k \in \mathbb{N})\big(P(k) \implies P(k+1)\big)$ – this is the induction step.

Just an implication, so do a direct proof, as described in the last lecture.

Assume $P(k)$ is true – this is called the induction hypothesis.

Prove that $P(k+1)$ is true.

This is the standard form and the pieces people expect in an induction proof.

Follow the form and label the pieces!

Back To Gauss!

Theorem: For all $n \in \mathbb{N}$, $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Proof: We proceed by induction on n.

Base Case ($n = 0$): $\sum_{i=0}^{n} i = 0$, and $\frac{0(0+1)}{2} = 0$, so the base case holds.

Induction Hypothesis: Assume the formula holds for $n = k$, so $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$

Inductive Step: We prove the formula holds at $n = k+1$: $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

We separate out the final term in the sum and then apply the induction hypothesis:

$$\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1).$$

Simplifying, we get

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+2)(k+1)}{2},$$

which is the RHS of what we were trying to prove, completing the induction step. By the principle of mathematical induction, the theorem follows. □

Another Induction Example

Theorem: For all $n \in \mathbb{N}$, $n^3 - n$ is divisible by 3.

Proof: We proceed by induction on n.

Base Case ($n = 0$): When $n = 0$, $n^3 - n = 0$, which is divisible by 3, so the base case holds.

Induction Hypothesis: Assume for $n = k$ that $(k^3 - k)$ is divisible by 3.

Inductive Step: We prove that when $n = k + 1$, $\left((k+1)^3 - (k+1)\right)$ is divisible by 3.

Start by expanding

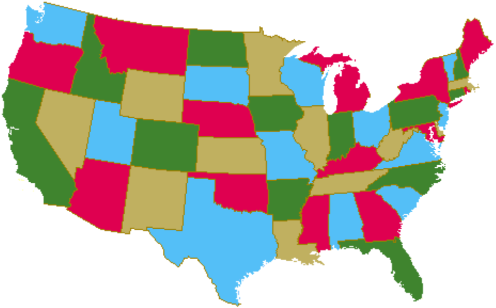
$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 + 3k^2 + 2k.$$

We adjust terms so that we can use the induction hypothesis, as $(k^3 - k) + (3k^2 + 3k) = (k^3 - k) + 3(k^2 + k)$.

By the induction hypothesis, $k^3 - k = 3q$ for some $q \in \mathbb{Z}$, so this becomes $3q + 3(k^2 + k) = 3(q + k^2 + k)$. This is 3 times an integer, so $(k+1)^3 - (k+1)$ is divisible by 3, completing the induction step. By the principle of mathematical induction, the theorem follows. $\square$

A Famous Theorem: The Four Color Theorem

Theorem: Any map can be colored so that those regions that share an edge have different colors.

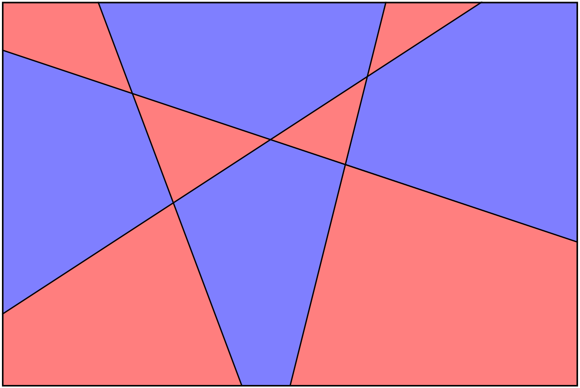


Fascinating history:

- Conjectured but unproven for over 100 years
- (One of the?) first major computer-assisted proof
- Proof by cases *(1,834 cases!)*

Simplification: Maps With Just Complete Lines

Simpler map: Only lines allowed (no line segments, curves, ...)



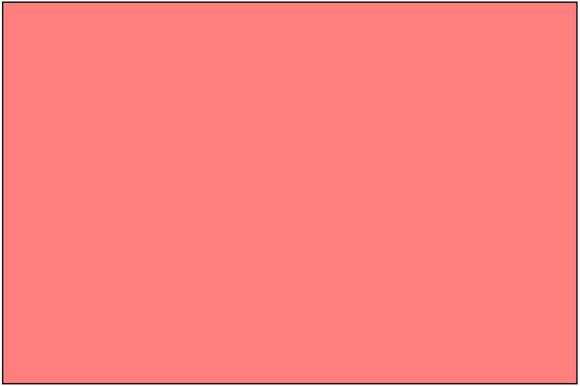
Claim: Any such map formed can be properly colored with at most two colors

We will prove this by induction, but visually – focus on the logic!

$\Rightarrow$ *"Visually" is not a proper proof – see notes for written*

Simplification: Maps With Just Complete Lines

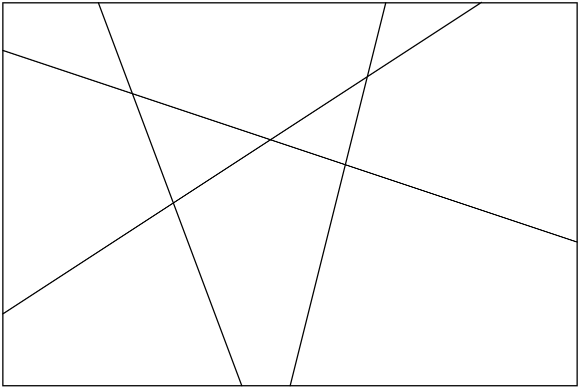
Base case (no lines): One color is sufficient



Induction Hypothesis: Assume true for $n = k$ lines...

Simplification: Maps With Just Complete Lines

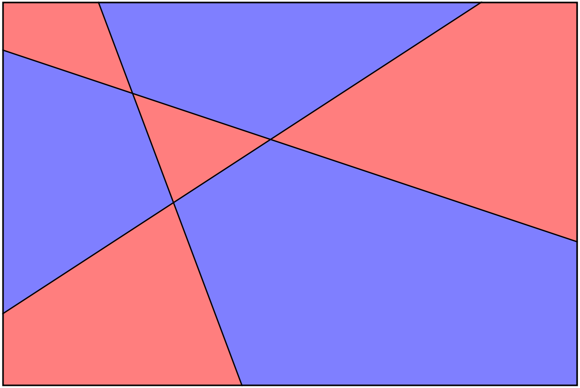
Inductive step: Consider a case with $n = k + 1$ lines (picture: $k = 3$)



Remove a line: Goes from $k + 1$ lines back to k lines

Simplification: Maps With Just Complete Lines

Inductive step: Consider a case with $n = k + 1$ lines (picture: $k = 3$)



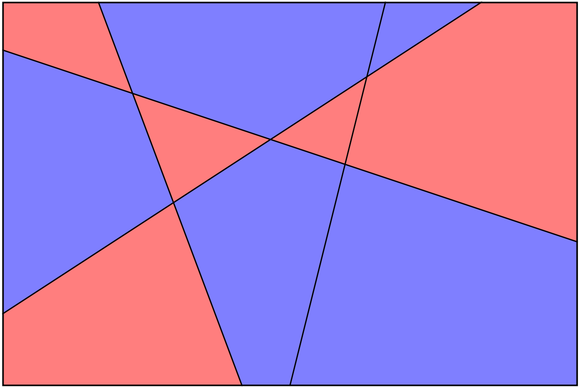
Remove a line: Goes from $k + 1$ lines back to k lines

Use induction hypothesis: We can color the map with k lines!

Add the $(k + 1)$st line back.

Simplification: Maps With Just Complete Lines

Inductive step: Consider a case with $n = k + 1$ lines (picture: $k = 3$)



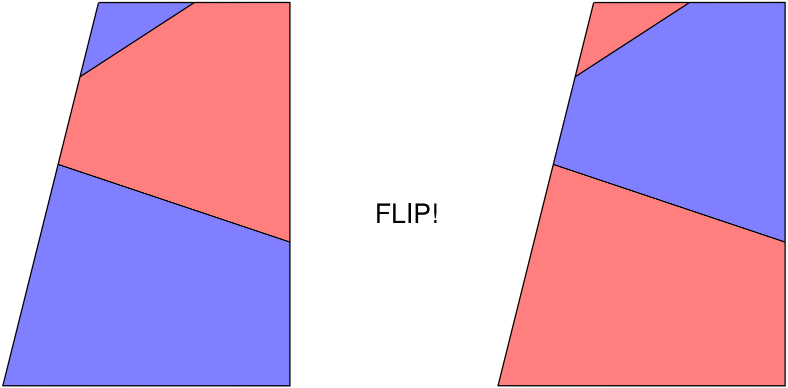
Remove a line: Goes from $k + 1$ lines back to k lines

Use induction hypothesis: We can color the map with k lines!

Add the $(k + 1)$st line back. Hmmm... not a valid coloring

Simplification: Maps With Just Complete Lines

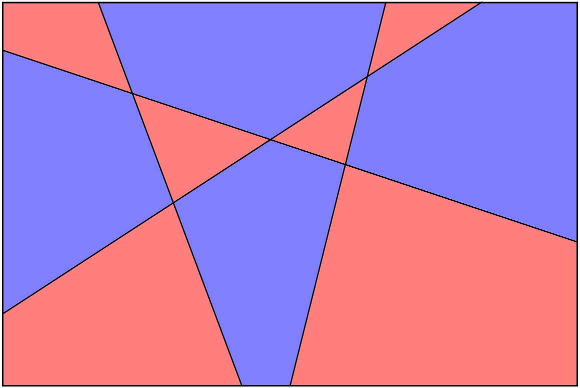
Observation: In any region with a valid color, can flip colors and it's still valid.



Back to the full map, with flipped colors to the right of line

Simplification: Maps With Just Complete Lines

Observation: In any region with a valid color, can flip colors and still valid.



Back to the full map, with flipped colors to the right of line

Now: Inside regions OK (I.H.), and across regions OK (flipped). □

Strengthening The Induction Hypothesis.

Theorem: The sum of the first n odd numbers is ~~a perfect square.~~ n^2.

Proof: Let s_n denote the sum of the first n odd numbers.

Base Case $(n = 0)$*:* s_0 is an empty sum, so is zero – a perfect square.

Induction Hypothesis: Assume for $n = k$, s_k is ~~a perfect square, say a^2.~~ k^2.

Induction Step: We prove that for $n = k+1$, s_{k+1} is ~~a perfect square.~~ $(k+1)^2$.

1. The $(k+1)$st odd number is $2k+1$, so $s_{k+1} = s_k + (2k+1)$
2. $\Longrightarrow$ The sum of the first $k+1$ odds is ~~$a^2 + (2k+1)$.~~ $k^2 + (2k+1)$.
3. ~~???~~ This is $k^2 + 2k + 1 = (k+1)^2$

This completes the induction step, and by the principle of mathematical induction, the theorem follows. □

It seems like proving something more specific should be harder than proving the looser statement. However, being more specific gave us a more powerful induction hypothesis to use!

A Surprisingly Subtle Example

Fibonacci numbers! A sequence where every value is the sum of the two preceding values. For $n \in \mathbb{N}$:

$$F_n = \begin{cases} F_{n-1} + F_{n-2} & \text{if } n \geq 2; \\ n & \text{if } n < 2. \end{cases}$$

Let's look at a few...

$F_0 = 0 \quad F_1 = 1 \quad F_2 = 1 \quad F_3 = 2 \quad F_4 = 3 \quad F_5 = 5 \quad F_6 = 8 \quad \cdots$

Hmmm... that starts to snowball – does it grow exponentially?

Theorem: For all $n \geq 1$, $F_n \geq (3/2)^{n-2}$.

For such a simple statement, this requires quite a few changes to our form:

- Starting induction at a value of $n > 0$
- Needing multiple values of n in the base case
- Strong induction

Induction: Exponential Growth of Fibonacci numbers

Issue 1: The base case

Theorem: For all $n \geq 1$, $F_n \geq (3/2)^{n-2}$.

Our high-level goal: lower bound F_n by an exponential function

Could we have used "for all $n \in \mathbb{N}$" and had base case $n = 0$ as before? No!

Problem: Exponential functions (like c^n) are always *strictly positive*, so impossible to lower-bound $F_0 = 0$.

Does it cause a *significant* problem? No! With $n = 1$ as the base case, we have $P(1) \implies P(2) \implies P(3) \implies \cdots$.

Bottom line: Just make sure you cover all cases in the theorem statement

- Can possibly adjust bounds in theorem statement!
- Generalizing induction: base case is *smallest n* covered by the theorem
- Warning! We'll see later that for *this problem*, just $n = 1$ doesn't work!

Induction: Exponential Growth of Fibonacci numbers

Issue 2: Available induction hypothesis

Let's sketch out the inductive step

Induction Hypothesis: Assume that for $n = k$ we have $F_k \geq (3/2)^{k-2}$.

Inductive Step: We will show that when $n = k+1$ we have $F_{k+1} \geq (3/2)^{k-1}$.

Start by using the definition to get $F_{k+1} = F_k + F_{k-1}$, and since F_k is bounded by the induction hypothesis, we have $F_{k+1} \geq (3/2)^{k-2} + F_{k-1}$.

??? What can we do with F_{k-1}???

We actually have what we need, we just didn't state it.

To get to $P(k)$ we proved $P(0) \implies P(1) \implies \cdots P(k-1) \implies P(k)$

Change induction hypothesis from assuming $P(k)$ to assuming $(\forall n \leq k) P(n)$.

- Now $F_{k-1} \geq (3/2)^{k-3}$ is covered by the induction hypothesis
- Need to be more careful about lower bound for n (base case)
- Introduces some subtle issues (on next slide)
- This is called strong induction

Induction: Exponential Growth of Fibonacci numbers

Issue 3: Base case, revisited

Strong induction introduces a subtle issue not present in simple induction.

Idea from before: Use $n = 1$ as the base case (smallest n in theorem)

To establish $P(k+1)$ we used both $P(k)$ and $P(k-1)$.

So to establish $P(2)$ we need $P(1)$ and... $P(0)$? Oops.

Fortunately, easy to fix:

Prove $P(1)$ and $P(2)$ directly, so first use of induction step is to prove $P(3)$
$\Rightarrow$ This only "reaches back" to $P(1)$ and $P(2)$, which we proved.

Why this is an issue for strong induction: Reaching back farther than previous step gives the possibility to skip *over* the base case.

Induction: Exponential Growth of Fibonacci numbers

Putting it all together

Theorem: For all $n \geq 1$, $F_n \geq (3/2)^{n-2}$.

Proof: We proceed by induction on n.

Base Case ($n = 1$ and $n = 2$):

When $n = 1$, $F_n = 1$ and $(3/2)^{n-2} = (3/2)^{-1} = (2/3)$, so $F_n \geq (3/2)^{n-2}$.

When $n = 2$, $F_n = 1$ and $(3/2)^{n-2} = (3/2)^{0} = 1$, so $F_n \geq (3/2)^{n-2}$.

Induction Hypothesis: Assume that $F_n \geq (3/2)^{n-2}$ for all $1 \leq n \leq k$.

Inductive Step: We prove when $n = k+1$ we have $F_{k+1} \geq (3/2)^{k-1}$.

By definition, $F_{k+1} = F_k + F_{k-1}$, and we use the induction hypothesis to bound

$$F_{k+1} \geq (3/2)^{k-2} + (3/2)^{k-3} = (3/2)^{k-2} + (2/3)(3/2)^{k-2} = (5/3)(3/2)^{k-2} \geq (3/2)^{k-1}$$

This is what we need for the induction step, so completes the proof. □

A Bad Proof

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: Assume for $n = k$ any k horses have the same color.

Induction step: Prove for $n = k + 1$

Given a set of $k + 1$ horses, remove one (Mr. Ed), leaving k horses. By the induction hypothesis, they must be the same color.

Put Mr. Ed back into the set and remove a different horse (Wilbur). Apply the induction hypothesis again to show these k horses have the same color.

BoJack was in both sets, and didn't change colors during this, so it follow that all $k + 1$ horses must be the same color as BoJack.

This proves the induction step, so the theorem follows. □

Obviously wrong, but *why*?

Summary

Basic principle of induction – proving $\forall n \in \mathbb{N}$ by simple induction

- Prove $P(0)$ directly (base case)
- Prove that $P(k) \implies P(k+1)$ for all $k \geq 0$ (inductive step)

What if it doesn't work? (almost but not quite)

- Do we need to change the base case?
- Would a stronger theorem (so a stronger induction hypothesis) work?
- Would it help to "reach back" farther than just the previous step (just $P(k)$ isn't sufficient to prove $P(k+1)$)?
  - Strong induction lets you use all $P(0)$ through $P(k)$
  - Make sure "reaching back farther than the previous step" doesn't skip over the base case