Graphs – Part 1

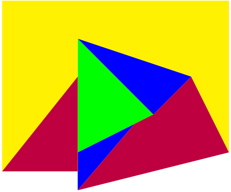
CS70: Discrete Mathematics and Probability Theory

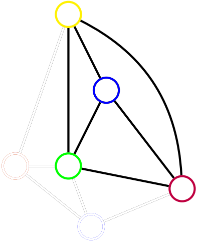
UC Berkeley – Summer 2025

Lecture 5

Ref: Note 5

Graph Idea: Map Coloring





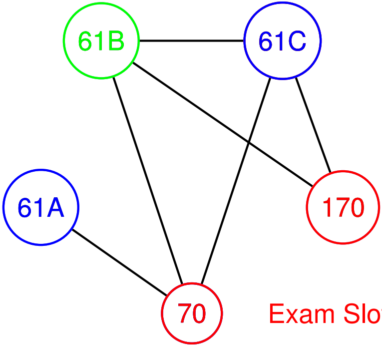
What is the essence of the map coloring problem?
Regions ... connected by borders
No two regions connected by a border can use the same color
Four colors used here – can we do better?
Yes! Three colors.
Now add this – three colors? Yes!
Now this? Connect... Three colors? No! Need four.

Remember: More than four never needed for a map (in the plane).

Scheduling: Coloring

Problem: Scheduling Exams

⇒ What courses are students simultaneously enrolled in?

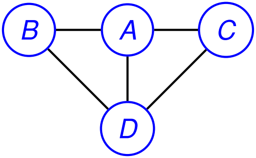


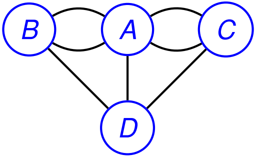
Exam Slot 1.

Exam Slot 2.

Exam Slot 3.

Graphs: Definitions





Graph: $G = (V, E)$

$V =$ set of vertices

$\{A, B, C, D\}$

$E \subseteq V \times V$: set of edges

$\{\{A, B\}, \{A, C\}, \{B, D\}, \{A, D\}, \{C, D\}\}$

Simple graph

No "parallel edges"

No self-loops (i.e., edge $\{A, A\}$)

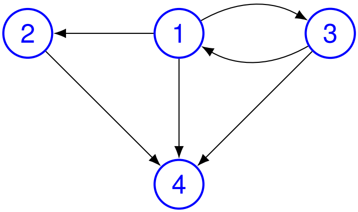
If not stated, a graph is a simple graph

Variant: Multi-graph

Edges are a *multiset*
Duplicates are allowed

CS 70: (usually) simple graphs

Directed Graphs



$G = (V, E)$

$V =$ set of vertices

$\{1, 2, 3, 4\}$

$E =$ *ordered pairs* of vertices

$\{(1,2),(1,3),(3,1),(1,4),(2,4),(3,4)\}$

Can't have duplicates: No $(1,2)$ and $(1,2)$

Can have both directions: $(1,3)$ and $(3,1)$

One way streets

Tournament: 1 beats 2, ...

Precedence: 1 is before 2, ...

Social Network: Directed? Undirected?

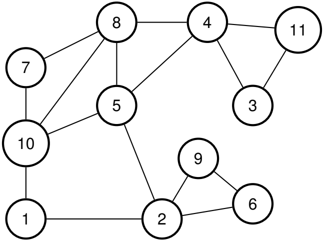
Friends: undirected

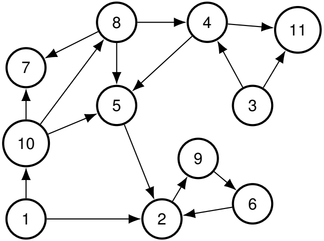
Likes: directed

Graph Concepts and Definitions

Graph: $G = (V, E)$

Terminology: neighbors, adjacent, incident, degree, in-degree, out-degree





u is neighbor of v if $\{u, v\} \in E$

Neighbors of 10? 1, 5, 7, 8

Vertex v is adjacent to each neighbor

Edge $\{u, v\}$ is incident to u and v

Edge $\{10, 5\}$ is incident to: vertices 10 and 5

Degree of vertex u is number of incident edges

Degree of vertex 10? 4

Directed graph:

In-degree is # of edges *to*

Out-degree is # of edges *from*

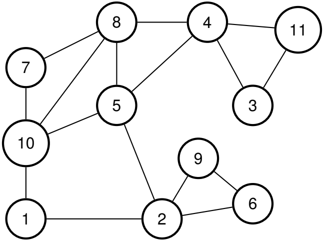
In-degree of 10? 1

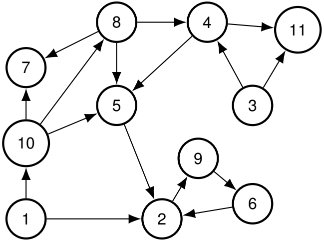
Out-degree of 10? 3

Graph Concepts and Definitions - Questions

Graph: $G = (V, E)$

Terminology: neighbors, adjacent, incident, degree, in-degree, out-degree





Edge $\{8,5\}$ is incident to:

(A) Vertex 8
(B) Vertex 5
(C) Edge $\{8,5\}$
(D) Edge $\{8,4\}$
(E) Vertex 10

Ans: Both (A) and (B)

The degree of a vertex is:

(A) The number of edges incident to it
(B) The number of neighbors of v
(C) The number of vertices in its connected component

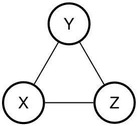
Ans: Both (A) and (B)

Properties of Graphs: Sum of Degrees

The sum of the vertex degrees is equal to

(A) The total number of vertices, $|V|$
(B) The total number of edges, $|E|$
(C) What?

Consider:



Degree of X? 2
Degree of Y? 2
Degree of Z? 2

Sum of degrees? 6

Answer above: Not (A) or (B)
(C) is fine for a poll with no correct answers!

Could sum always be...

(A) $2|E|$?
(B) $2|V|$?

Let's see...

The Degree-Sum Formula

The sum of the vertex degrees is equal to ??

Back to definitions:

The degree of u is the number of edges incident to u

Edge $\{u, v\}$ is incident to its endpoints, u and v
$\Rightarrow$ Call each endpoint an edge-vertex incidence

Let's count edge-vertex incidences in two ways:

How many incidences does each edge contribute? 2

Total Incidences in entire graph? $|E|$ edges, 2 each $\rightarrow$ $2|E|$

What is the degree of v? Incidences corresponding to v!

Total Incidences? The sum over vertices of degrees!

Theorem: In any graph $G = (V, E)$, the sum of vertex degrees is $2|E|$, or

$$\sum_{v \in V} \deg(v) = 2|E|.$$

This is called the "degree-sum formula."

Concept Check: Degree Sum

Which of the following are true?

(A) Number of edge-vertex incidences for an edge e is 2.

(B) Total number of edge-vertex incidences is $|V|$.

(C) Total number of edge-vertex incidences is $2|E|$.

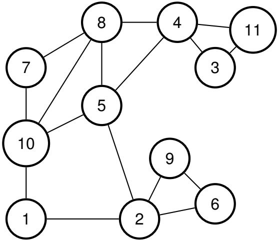
(D) Number of edge-vertex incidences for a vertex v is its degree.

(E) Sum of degrees is $2|E|$.

(F) Total number of edge-vertex incidences is the sum of the degrees.

Answer: All but (B)!

More Terminology: Paths, Walks, Cycles, and Tour



A path is a sequence of connected edges: $\{v_1, v_2\}, \{v_2, v_3\}, \ldots \{v_{k-1}, v_k\}$.

Path? $\{1, 10\}$, $\{8, 5\}$, $\{4, 5\}$? No! Each edge must connect to next

Path? $\{1, 10\}$, $\{10, 5\}$, $\{5, 4\}$, $\{4, 11\}$? Yes!

A simple path has no repeated vertices ("path" usually is simple)

The length of path is the number of "steps" — number of edges (*not* vertices!)

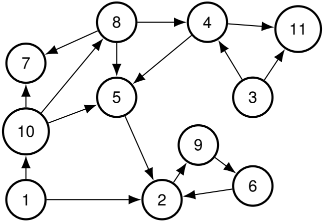
A cycle is a closed path: Path from v_1 to v_{k-1}, + edge $\{v_{k-1}, v_1\}$

Walk is sequence of edges with possible repeated vertex or edge.

Tour is walk that starts and ends at the same node.

Quick Check: Path is to Walk as Cycle is to ?? Tour!

Paths in Directed Graphs



Path: $(v_1, v_2), (v_2, v_3), \ldots (v_{k-1}, v_k)$

Same basic idea, but can't go "head-to-tail" on edge

- $\Rightarrow$ Not a path: $(1, 10), (10, 5), (4, 5), (4, 11)$
- $\Rightarrow$ Path: $(1, 10), (10, 8), (8, 4), (4, 11)$

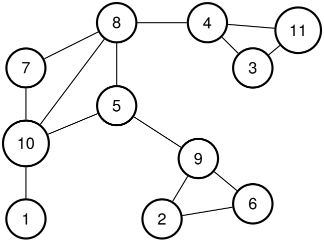
Paths, walks, cycles, tours... are analogous to undirected

A graph with no cycles is *acyclic* – directed acyclic graph is "dag"

	no rep vertices	no rep edges	start = end
Walk			
Path	✓	✓	
Tour			✓
Cycle	✓*	✓	✓

(* except start=end)

Connectivity: Undirected Graph



u and *v* are connected if there is a path between *u* and *v*

⇒ Walk or path – does it matter? No! (Cut out between repeated vertices)

A graph is connected if all pairs of vertices are connected

If a vertex *x* is connected to every other vertex, is graph connected? Yes!

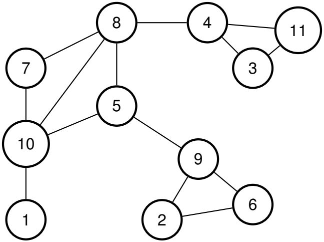
Proof idea: For any pair u, v, use path from *u* to *x* and then from *x* to *v*

⇒ Remember: undirected!

⇒ Gives walk between *u* and *v* □

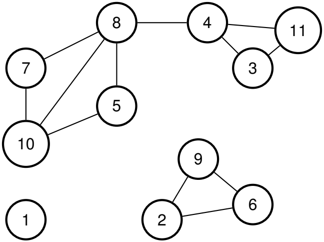
May not be a simple path! But we already said walk or path doesn't matter.

Connectivity and Connected Components



Is the graph above connected? Yes!

Connectivity and Connected Components



Is the graph above connected? Yes!

How about now? No! No path from vertex 1 to vertex 10.

A connected component is a maximal set of connected vertices.

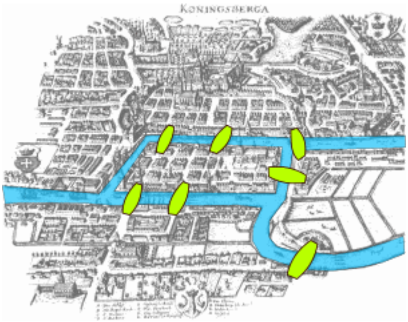
$\Rightarrow$ Connected Components? $\{1\}, \{10, 7, 5, 8, 4, 3, 11\}, \{2, 9, 6\}$

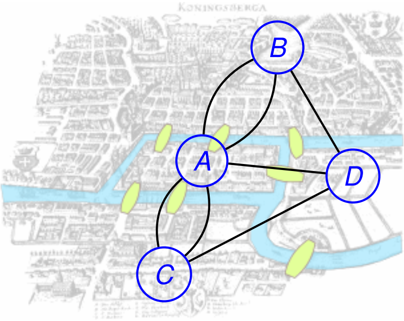
Quick Check: Is $\{10, 7, 5\}$ a connected component? No! Not *maximal*.

Seven Bridges of Königsberg (1736)

Can you make a tour visiting each bridge exactly once?

“Konigsberg bridges” by Bogdan Giuşcă - License





Idea: Model with a graph – each region a node (recall map coloring!)
Add an edge for each bridge connection   Need a multi-graph!

Now: Is there a tour in the multi-graph that visits each edge exactly once?
$\Rightarrow$ *Note importance of abstraction to “get at the heart of the matter”*

Eulerian Tour

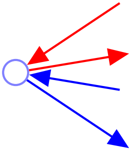
An Eulerian Tour is a tour that covers the graph using each edge exactly once.

Theorem: Any undirected multi-graph has an Eulerian tour if and only if it is connected and all vertices have even degree.

Proof of only if: Eulerian $\implies$ connected and all vertices have even degree

Given an Eulerian Tour: it is connected, so the graph is connected.

Non-start/stop vertices: Tour enters and leaves on each visit.



Start/stop vertex: Initially leaves, then enters at end.

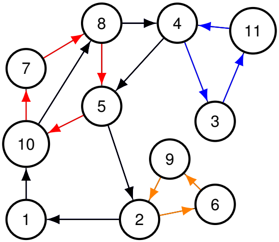
For every vertex: Uses two incident edges per visit. Tour uses every edge exactly once $\implies$ every vertex has even degree. $\square$

When you enter, you can leave. Not The Hotel California (Timestamp: 4:10)

Finding a Tour

Proof of if: Even degrees + connected $\implies$ Eulerian tour

We will give an algorithm – with illustration!



1. Take a walk starting from v (1) on “unused” edges ... until you get back to v
2. Remove tour, C (halt if no edges left)
3. Let $G_1, \ldots, G_k$ be connected components
   Each is touched by C
      Why? G was connected
   Let v_i be (first) node in G_i touched by C
   Example: $v_1 = 1$, $v_2 = 10$, $v_3 = 4$, $v_4 = 2$.
4. Recurse on $G_1, \ldots, G_k$ starting from v_i
5. Splice together
   1,10,7,8,5,10 ,8,4,3,11,4 5,2,6,9,2 and to 1!

Recursive/Inductive Algorithm – Important Facts

1. Take a walk from arbitrary node v, until you get back to v

Claim: We do get back to v!
Proof: Even degree. If we enter, we can leave except (possibly) for v. □

2. Remove tour, C, from G

Resulting graph may be disconnected (removed edges)
Let components be $G_1, \ldots, G_k$, and let v_i be first vertex of C that is in G_i
Always possible? Does tour C touch every G_i?
G_1 (component with $v \in G_1$): $v_1 = v$
G_i with $v \notin G_i$: No path v to G_i after C removed, so edge in C connected it

Claim: Each vertex in each G_i has even degree and is connected.
Proof: Tour C has even incidences to any vertex v (even - even = even). □

3. Find Eulerian tour T_i of G_i from at v_i. Strong induction (G_i is smaller)

4. Splice T_i into C where v_i first appears in C.

Visits every edge exactly once:
Visits each edge in C exactly once.
Remaining edges: each in a G_i, visited exactly once (by induction). □

Eulerian Graphs

A graph is Eulerian if it has an Eulerian tour.

Question: One of the following statements is false. Which one?

(A) Removing a tour from an Eulerian graph leaves a graph with all even-degree vertices.

(B) After removing a set of edges E' in a connected graph, every connected component is incident to an edge in E'.

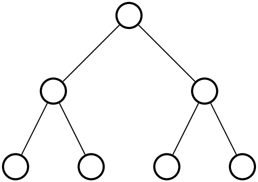
(C) Removing a tour leaves a connected graph.

(D) If one walks on new edges in an Eulerian graph, starting at v, one gets back to v.

Answer: (C) is false

Trees

A common picture of a tree (in computer science):



This is a binary tree, which has certain properties:

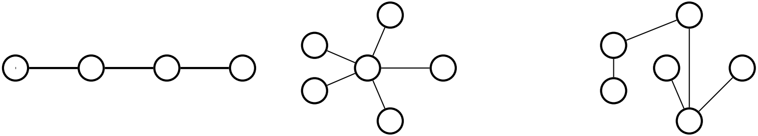
- It is rooted (has a root node)
- Every edge represents a parent/child relationship
  - Every parent has at most 2 children
  - A child is a “left” or “right” child

None of these properties are necessary for a tree!

Trees In General

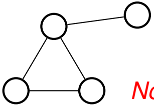
Definitions: A tree is...

... a connected graph without a cycle.
... a connected graph with $|V|-1$ edges.
... a connected graph where any edge removal disconnects it.
... a connected graph where any edge addition creates a cycle.

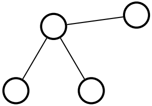


No cycle and connected? *Yes.*
$|V|-1$ edges and connected? *Yes.*
Removing any edge disconnects it? *Harder to check, but yes.*
Adding any edge creates cycle. *Harder to check, but yes.*

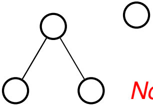
To tree or not to tree, that is the question:



Not a tree



Tree



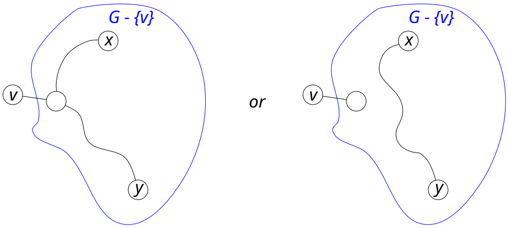
Not a tree

Equivalence of (First Two) Definitions

Theorem:
"G connected with $|V|-1$ edges" $\iff$ "G is connected and has no cycles."

Lemma: If v is degree 1 in connected graph G, $G-\{v\}$ is connected.
Proof:

For $x, y \in G-\{v\}$:
Since G is connected, there is path between x and y in G
Path cannot use v (degree 1) or it would repeat a vertex
$\implies$ every pair in $G-\{v\}$ is connected in $G-\{v\}$
$\implies$ $G-\{v\}$ is connected. $\square$



Proof of "Only If"

Theorem:
"G connected with $|V|-1$ edges" $\implies$ "G is connected and has no cycles."

Proof: By induction on $|V|$.

Base Case ($|V|=1$)*:* There are $|V|-1=0$ edges, so no cycles.

Induction Hypothesis: Any G with $|V|=k$ and $|E|=k-1$ is conn w/ no cycles

Induction Step: Prove that in G with $|V|=k+1$ and $|E|=k$, conn w/ no cycles

Claim: G has a degree 1 node.

Proof: First, connected $\implies$ every vertex degree ≥ 1.
Sum of degrees is $2|E|=2(|V|-1)=2|V|-2$
Average degree $(2|V|-2)/|V|=2-2/|V|$, so must be a degree 1 vertex.
Cuz not everyone is bigger than average! □

By degree 1 removal lemma, $G-\{v\}$ is connected.

$G-\{v\}$ has k vertices and $k-1$ edges so by induction hypothesis
$\implies$ no cycle in $G-\{v\}$.

Add v back to get G: no cycle since degree 1 cannot participate in cycle. □

Proof of "If"

Theorem:
"G is connected and has no cycles" $\implies$ "G connected with $|V|-1$ edges"

Proof: By induction on $|V|$.

Base Case ($|V|=1$)*:* Cannot have any edges, and $|V|-1=0$ edges.

Induction Hypothesis: Any connected G with $|V|=k$ no cycles has $|E|=k-1$

Induction Step: Any connected G with $|V|=k+1$ no cycles has $|E|=k$

Pick an arbitrary vertex $v \in V$ and walk using untraversed edges.
Finitely many edges, so must stop ("get stuck") at some vertex w.

Claim: w has degree 1.
Proof: Can't visit any vertex more than once since no cycle.
Entered w. Didn't leave. Only one incident edge. □

Remove w and single edge connecting it: can't create cycle.
Removal does not disconnect graph (by degree 1 lemma).

So $G-\{w\}$ is conn w/ no cycles and k vertices $\implies$ has $k-1$ edges (by I.H.)
G has one more edge, or k edges. □

Concept Check: Trees

Let G be a connected graph with $|V| - 1$ edges.

Question: Which of the following are true?

(A) Removing a degree 1 vertex can disconnect the graph.

(B) One can use induction on smaller objects.

(C) The average degree is $2 - 2/|V|$.

(D) There is a Hotel California: a degree 1 vertex.

(E) Everyone can be smarter than average.

Answer: (B), (C), (D) are true

Lecture Summary

Graphs:
- Definitions, basic properties (degree, path, cycle, tour, ...)
- Degree-sum formula (sum of degrees is $2|E|$)
- Connected: Path between every pair of nodes
- Connected Component: Maximal set of connected vertices

Euler tour and condition for existence (even degree vertices)
- Necessary: Existence of tour $\implies$ connected, even degree
- Sufficient: Recursive algorithm for finding an Eulerian tour

Trees:
- Definitions – *four* of them – all equivalent
- Equivalence of definitions
  - $\Rightarrow$ Two proved - others “left as an exercise for the reader”