Graphs – Part 2

CS70: Discrete Mathematics and Probability Theory

UC Berkeley – Summer 2025

Lecture 6

Ref: Still Note 5

Lecture 6

Planar Graphs

- Euler's Formula
- Planar Six Color Theorem
- Planar Five Color Theorem!

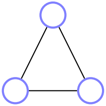
Some other important types of graphs:

- Complete Graphs
- Hypercubes

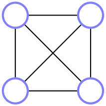
Planar Graphs

A planar graph can be drawn in the plane without edge crossings

A planar embedding is a planar drawing of a graph with no edge crossings



Planar?

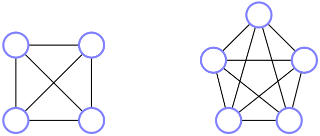


Planar?

Complete Graphs and Planarity

The complete graph with n vertices has n vertices with all connections

Notation: K_n is the complete graph with n vertices ("K" is for Kuratowski)



K_4 K_5

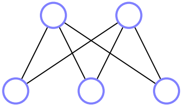
Last slide: K_4 is planar

Is K_5 planar?

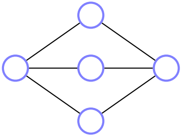
Bipartite Graphs

A bipartite graph $G = (V, E)$ is one where vertices can be partitioned into two sets A and B such that edges are only between these two sets: $E \subseteq A \times B$.

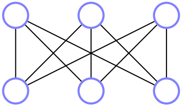
Consider:



$K_{2,3}$



Also $K_{2,3}$



$K_{3,3}$

In fact has *all possible* edges in $A \times B$: complete bipartite graph

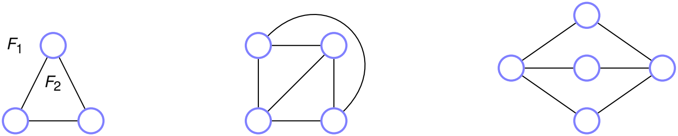
$\Rightarrow$ Complete bipartite graph with $|A| = n$ and $|B| = m$ is denoted $K_{n,m}$

$\Rightarrow$ So the graph above is $K_{2,3}$

Question: Is $K_{2,3}$ planar?

Question: Is $K_{3,3}$ planar?

Euler's Formula



Faces: connected regions of a planar embedding – including "outside" region!

How many faces for

- K_3 (triangle)?
- K_4 (complete on four vertices)?
- $K_{2,3}$ (complete two/three bipartite)?

Variables: v is number of vertices, e is number of edges, f is number of faces

Euler's Formula: Connected planar graph has $v + f = e + 2$ (*any embedding!*)

K_3 $(v = 3, f = 2, e = 3)$: $3 + 2 = 3 + 2$ Good!
K_4: $4 + 4 = 6 + 2$! Good!
$K_{2,3}$: $5 + 3 = 6 + 2$! Good!

3 examples! Proven? Nope!!!!

Euler's Formula

Theorem: If $G = (V, E)$ is a connected planar graph, then $v + f = e + 2$.

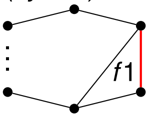
Proof: We proceed by induction on e.

Base case $(e = 0)$*:* Conn? $v = 1$ No edges? $f = 1$ so: $v + f = 2$ and $e + 2 = 2$

Induction Step: Prove when $e = k + 1$, $v + f = k + 3$

Case 1 (no cycles): A tree! $e = v - 1 \implies v = k + 2$, $f = 1$, $v + f = k + 3$. ✓

Case 2 (cycles): Find a cycle – remove a bounding edge:



Outer face.

Without edge: k edges, $f - 1$ faces, v vertices
Still connected, so I.H. says $v + (f - 1) = k + 2 \implies v + f = k + 3$
Induction step done! □

Core idea: Removing a cycle edge (RHS) reduces faces by one (LHS)

For a tree: Removing an edge disconnects (recall equivalent definitions!)

Concept Check: Euler's Formula and Proof

Euler's formula: $v + f = e + 2$

Proof idea: Remove an edge $\Longrightarrow$ decrease faces by one so sides stay equal

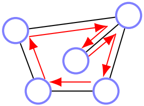
Question: Does removing an edge from a planar embedding always decrease the number of faces?

Answer:

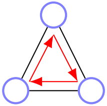
Too Many Edges?

Difficult to embed when “too many” edges: What is “too many”?

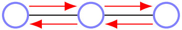
For planar graphs with $|E| \geq 2$, define face “sides”:



Sides: walk around face boundary



Smallest interior 3 sides



Smallest exterior 4 sides

First graph: 6 sides
Smallest interior face: 3 sides
Smallest exterior face: 4 sides

Count sides by faces: at least 3 sides each, so total is $\geq 3f$

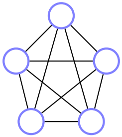
Count sides by edges: each edge has two sides, so total is $2e \quad \Longrightarrow \; 2e \geq 3f$

Euler's formula: $f = e + 2 - v$

So: $2e \geq 3(e+2-v) \;\Longrightarrow\; 2e \geq 3e+6-3v \;\Longrightarrow\; e \leq 3v-6$

Very important! No planar graph can have more than $3v-6$ *edges!*

Non-planarity of K_5 and $K_{3,3}$

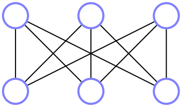


How many vertices in K_5?

$\Rightarrow$ So $3v-6 = 3\cdot 5 - 6 = 15-6 = 9$

How many edges in K_5?

Is $e \le 3v-6$? No! So K_5 is not planar.



How many vertices in $K_{3,3}$?

$\Rightarrow$ So $3v-6 = 3\cdot 6 - 6 = 15-6 = 12$

How many edges in $K_{3,3}$?

Is $e \le 3v-6$? Yes... we need to work a little harder...

Important: In a bipartite graph, cycles must have an even number of edges

So for *bipartite graphs*, number of sides is $\ge 4f$ (not just $3f$)

What happens if we modify previous bound for a bipartite-specific bound?

You work it out! Conclusion is too many edges: $K_{3,3}$ is not planar

More Coolness with K_5, $K_{3,3}$, and Planarity

We saw K_5 and $K_{3,3}$ are not planar.

$\Rightarrow$ No graph which *contains* K_5 or $K_{3,3}$ can be planar.

$\Rightarrow$ "*contains*" means more than just exact appearance of graphs

So: "Graph contains K_5 or $K_{3,3}$" $\Longrightarrow$ "Graph is not planar"

Amazingly, the converse is true:

"Graph does not contain K_5 or $K_{3,3}$" $\Longrightarrow$ "Graph is planar"

$\Rightarrow$ Proof is beyond the scope of this class

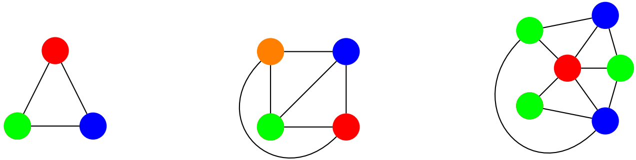
$\Rightarrow$ So K_5 and $K_{3,3}$, and expansions of them, are the only non-planar graphs

$\Rightarrow$ Leads to an efficient algorithm for testing planarity!

Proved by Kuratowski – that's why his name is immortalized on K_5 and $K_{3,3}$!

Graph Coloring

Given $G = (V, E)$, a coloring of G assigns colors to vertices V where endpoints of each edge have different colors.



First one (K_3): 3 colors is necessary and sufficient

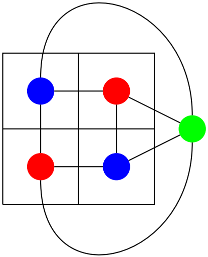
Second one (K_4): 4 colors are necessary and sufficient

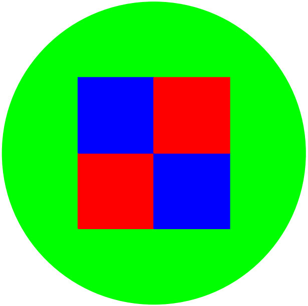
Third one: 3 colors are necessary and sufficient
  Determined by number of vertices? No!
  Determined by maximum vertex degree? No!

Something more interesting to explore here...

Planar graph coloring $\equiv$ map coloring

Vertices represent regions, edge means "shares a border"





Four color theorem is about planar graphs!

Six Color Theorem

An easy warm-up....

Theorem: Every planar graph can be colored with at most six colors.

Proof Sketch: We prove this by induction on v.

Base Case $(v = 1)$*:* Only one color needed!

Induction hypothesis: Any graph with $v = k$ can be colored with 6 colors.

Inductive step: We prove a graph with $v = k + 1$ can be colored with 6 colors.

Recall (from Euler's formula): $e \leq 3v - 6$ for any planar graph where $v > 2$.

Sum of vertex degrees is $2e \implies$ *average* degree $= \frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v}$.

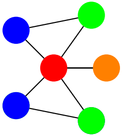
So there exists a vertex with degree < 6: remove it!

After removal: k vertices, planar $\implies$ I.H. says can color with 6 colors.

So color and add v back in. Has ≤ 5 neighbors, so a "spare color" for v.

Graph colored with 6 colors – completes inductive step – completes proof. $\square$

Five Color Theorem: Preliminary Observation



Pick two colors and look at just vertices with those colors – try blue and green

Ignoring other vertices can disconnect graph – look at connected components

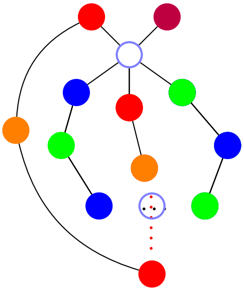
In any connected component (with two colors), can flip colors

Even with switched colors, still a valid coloring for full graph

Five Color Theorem

Theorem: Every planar graph can be colored with five colors.

Proof: As before, there's a degree 5 vertex – consider neighbors.



Uses $<$ 5 colors? Recurse, use 5th color here... Done!

Look at blue-green components – neighbors connected?
  No? $\Rightarrow$ Swap colors in green component.
  Now green neighbor is blue – only 4 colors – Done!

Look at red-orange components– neighbors connected?
  No? $\Rightarrow$ Swap colors in orange component.
  Now orange neighbor is red – only 4 colors – Done!

Now: blue-green connected and red-orange connected

Planar, so paths must cross at a vertex

Color of intersection vertex?
  On blue-green path, so blue or green
  On red-orange path, so red or orange
  Impossible!

All *possible* cases led to 5-coloring. Done! □

Five Color Theorem – Flow

Steps/ideas in 5-color theorem:

(A) There is a degree 5 vertex cuz Euler – remove, recursively color

(B) Option 1: Only 4 colors used for neighbors – done

(C) Option 2: Subgraph of 1st and 3rd colors disconnects 1st and 3rd neighbors – flip one – now only 4 colors on neighbors

(D) Option 3: Subgraph of 2nd and 4th colors disconnects 2nd and 4th neighbors – flip one – now only 4 colors on neighbors

(E) With 5 colors on neighbors, options 2 and 3 can't both fail cuz planarity

(F) In all possible options, end with 4 colors on neighbors, so can complete coloring

(G) Done!

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof: Not Today!

Number of Edges?

Question: How many edges in K_n?

Vertex 1 connected to vertices $2, \ldots, n$ ($n-1$ vertices)
Vertex 2 connected to vertices $3, \ldots, n$ ($n-2$ vertices)
Vertex 3 connected to vertices $4, \ldots, n$ ($n-3$ vertices)
Vertex 4 connected to vertices $5, \ldots, n$ ($n-4$ vertices)
...

Total number of edges: $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$ (remember 7 year old Gauss?)

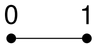
Hypercubes

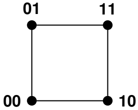
Complete graphs: hard to disconnect, but need lots of edges
$\frac{|V|(|V|-1)}{2}$ edges

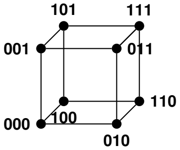
Trees: fragile (removing any edge disconnects), but very few edges
$|V|-1$ edges

Hypercubes: Really connected, with $\frac{|V|\log|V|}{2}$ edges!
Vertices map to binary strings

$G=(V,E)$
$V=\{0,1\}^n$ (len n binary strings – n is the "dimension" of the hypercube)
$E=\{\{x,y\} \mid x$ and y differ in one bit position$\}$







2^n vertices: number of n-bit strings!

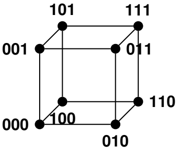
2^n vertices each of degree $n \implies$ sum of degrees is $n2^n$
$\implies$ number of edges is $\frac{n2^n}{2}=n2^{n-1}$

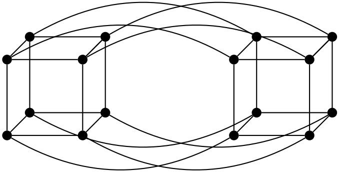
Recursive Definition

A 0-dimensional hypercube is a node labeled with the empty string of bits.

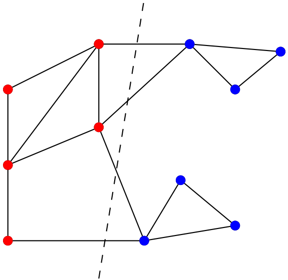
An *n*-dimensional hypercube consists of:

- An $(n-1)$ dimensional hypercube called the 0-subcube (add “0” to the front of each label)
- An $(n-1)$ dimensional hypercube called the 1-subcube (add “1” to the front of each label)
- Edges added between corresponding nodes in the 0-subcube and the 1-subcube





Cuts in Graphs



A cut in a graph partitions it into two pieces.

$\Rightarrow$ For $S \subseteq V$, have cut $(S, V - S)$

$\Rightarrow$ In picture: S is red, $V - S$ is blue

Cut edges have one endpoint in S and one in $V - S$

$\Rightarrow$ Can visualize by cutting apart sides (OK viz for small graphs...)

The size of a cut is the number of cut edges.

Question: What is the size of the cut above?

Cut size is a measure of how connected a graph is.

Cuts in Hypercubes

Trees can have large vertex sets with size 1 cuts
$\Rightarrow$ Easy to disconnect!

Complete graphs have very large cuts: $|S| \cdot (|V| - |S|)$
$\Rightarrow$ Very hard to disconnect!

What about *hypercubes*?
$\Rightarrow$ Far fewer edges than a complete graph
$\Rightarrow$ Still good connectivity (robustness), as we'll prove next

Theorem: For any cut $(S, V - S)$ in a hypercube, with $|S| \leq |V|/2$, the cut size is $\geq |S|$.

Restatement: For any cut in the hypercube, the number of cut edges is at least the size of the smaller side.

For example: *Any* cut that splits the graph in half has at least $|V|/2$ edges.

Proof of Hypercube Cut Size

Theorem: For any cut $(S, V - S)$ in a hypercube, with $|S| \leq |V|/2$, the cut size is $\geq |S|$.

Proof: By induction on n (the dimension of the hypercube)

Base Case $(n = 1)$*:* $V = \{0, 1\}$, so $|V| = 2$ and $|V|/2 = 1$. All S with $|S| \leq 1$:

If $|S| = 0$: no edges crossing the cut, which is $\geq |S|$ ✓

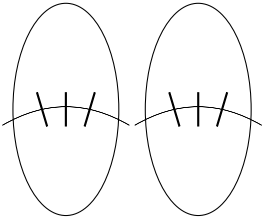
If $|S| = 1$: one edge crosses the cut, which is $\geq |S|$ ✓

Induction Step Idea

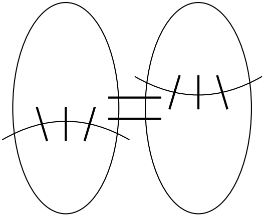
Recall: An n-dimensional hypercube is made up of two $(n-1)$-dimensional hypercubes (a 0-subcube and 1-subcube), joined together.

So lower bound edges cut in hypercube by adding:

1. Lower bound edges of cut inside the 0-subcube
2. Lower bound edges of cut inside the 1-subcube
3. Lower bound edges of cut between the 0-subcube and 1-subcube.



Sometimes (1) and (2) are enough



Some cases need all 3

Induction Step: First Part

Induction Hypothesis: For k-dimensional hypercube, any cut $(S, V - S)$ with $|S| \leq \frac{1}{2}2^k$, the cut size is $\geq |S|$.

Induction Step: For $(k+1)$-dimensional hypercube, any cut $(S, V - S)$ with $|S| \leq \frac{1}{2}2^{k+1}$, the cut size is $\geq |S|$.

Some notation:

0-subcube $H_0 = (V_0, E_0)$; 1-cube $H_1 = (V_1, E_1)$; connecting edges E_x
Full $(k+1)$-dimensional hypercube: $H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
$S_0 = S \cap V_0$ and $S_1 = S \cap V_1$

Case 1: $|S_0| \leq \frac{1}{2}2^k$ and $|S_1| \leq \frac{1}{2}2^k$

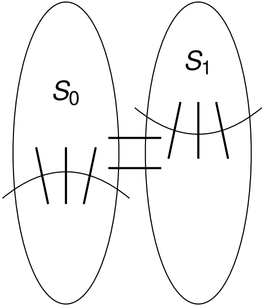
Both S_0 and S_1 are "small sides" in their subcube. By induction hypothesis:

Edges cut in H_0 are $\geq |S_0|$
Edges cut in H_1 are $\geq |S_1|$

Total cut edges $\geq |S_0| + |S_1| = |S|$ (case 1 complete...)

Induction Step: Case 2

Case 2: $|S_0| > \frac{1}{2}2^k$



$|V_0| = 2^k$ so $|V_0 - S_0| \le \frac{1}{2}2^k$

Ind hypothesis $\Longrightarrow$ edges cut in H_0 is $\ge |V_0| - |S_0|$

$|S| = |S_0| + |S_1|$ and $|S| \le \frac{1}{2}2^{k+1}$, so $|S_1| \le \frac{1}{2}2^k$

(i.e., S_0 is big, so S_1 must be small)

Ind hypothesis $\Longrightarrow$ edges cut in H_1 is $\ge |S_1|$

Edges in E_x connect corresponding nodes:

At most $|S_1|$ vertices in S_0 linked to S_1 in S

Remaining $|S_0| - |S_1|$ in S_0 must cross the cut

$\Longrightarrow$ edges in cut from E_X is $\ge |S_0| - |S_1|$

Total edges cut:

$\ge |S_1| + (|V_0| - |S_0|) + (|S_0| - |S_1|) = |V_0| = 2^k$

$|S| \le \frac{1}{2}2^{k+1} = 2^k$, so edges cut is $\ge |S|$

Case 3: $|S_1| > \frac{1}{2}2^k$ (*same as case 2*) □

Hypercubes and Decision Problems

A decision problem is a function with a yes/no answer

Examples

- Is the input number even?
- Is the input number prime?
- Does the input graph have an Eulerian tour?

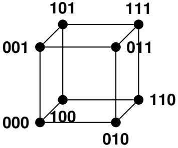
Decision problems are central to computer science!

View *n*-bit inputs as a hypercube

Define cut: $S =$ inputs with a “no” answer (or “yes”)

- Edges in cut: inputs where flipping one bit changes answer “no-to-yes”
- The cut is the “frontier” where no’s change to yes’s

Hypercubes and Communication Networks



Vertices are processors

Edges are communication links

$2^n = N$ communicating nodes

Communicate *a* to *b*:

- ⇒ Which bits flip to turn *a* into *b*?
- ⇒ Change bits one at a time: each gives a communication link to use
- ⇒ At most *n* bits change – so at most $n = \log_2 N$ "hops"

Cool things:

- Short distance (logarithmic) between any two processors
- Easy routing algorithm (which bits need to flip?)
- Not too many communication links needed (they're expensive!)
- Robust network (highly connected – hard to disconnect)

Summary

Planar graphs and planar embeddings

Euler's formula: $v+f=e+2$.

Proof: removing an edge from a cycle removes a face (and keeps connected)

Euler's formula consequence: $e \leq 3v-6$

Use to show that K_5 is not planar

Modify slightly to show that $K_{3,3}$ is not planar

Coloring Planar Graphs

Can color with 6 colors! Easy proof – just needs existence of deg ≤ 5 vertex

Can color with 5 colors! Argue about intersection of paths in the plane

Can color with 4 colors! Proof.. well, it's possible

Graph connectivity

Trees: few edges, but fragile (easily disconnected)

Complete: very robust, but many, many edges

Hypercube: very connected with modest edges

Beautiful structure – bits, bits, bits!