Q1 Bins and Balls

4 Points

Suppose we have n bins and we throw m balls into the bins, uniformly randomly.

Q1.1 Simpler Coupon Collecting

1 Point

Let p be the probability that the first bin is empty after all m throws. Using the union bound, find the lower bound of the probability that none of the bins are empty.

- ○ 1
- ○ np
- ◉ $1 - np$
- ○ $1 - mp$
- ○ mp

Explanation

By union bound and symmetry, the probability of at least one of the bins being empty is at most np. Taking the complement of this event, the probability of none of the bins being empty is at least $1 - np$.

Q1.2 Simpler Hashing
1 Point

Let q be the probability that any two balls land in the same bin. Using the union bound, find the lower bound of the probability that every ball lands in a different bin.

- 1
- $\binom{n}{2}q$
- $1-\binom{n}{2}q$
- $1-\binom{m}{2}q$
- $\binom{m}{2}q$

Explanation
There are $\binom{m}{2}$ possible pairs of two balls. By union bound and symmetry, the probability that any two balls land in the same bin is at most $\binom{m}{2}q$. Taking the complement of this event, the probability of every ball landing in a different bin is at least $1-\binom{m}{2}q$.

Q1.3 Simpler Load Balancing
1 Point

Let r be the probability that any subset of k balls land in the first bin. Using the union bound, find the upper bound of the probability that the first bin has at least k balls.

- 1
- $\binom{n}{k}r$
- $1-\binom{n}{k}r$
- $1-\binom{m}{k}r$
- $\binom{m}{k}r$

Explanation
There are $\binom{m}{k}$ possible subsets of k balls. The event that the first bin has at least k balls is equal to the union of all the events that a subset of k balls lands in the first bin. By union bound and symmetry, this probability is at most $\binom{m}{k}r$.

Q1.4
1 Point

Suppose $1 < m < n$. Which of the following events cannot occur?

- ○ At least one bin has multiple balls
- ○ None of the bins have multiple balls
- ◉ None of the bins are empty

Explanation

There are less balls than bins. Thus, some bin must be empty.

For the other options: it is possible that 2 balls land in the same bin; that bin has multiple balls. It is also possible that every ball lands in a different bin, in which case no bin has multiple balls.