

Q1 Bins and Balls

4 Points

Suppose we have n bins and we throw m balls into the bins, uniformly randomly.

Q1.1 Simpler Coupon Collecting

1 Point

Let p be the probability that the first bin is empty after all m throws. Using the union bound, find the lower bound of the probability that none of the bins are empty.

- ☐ 1
- ☐ np
- ☒ $1 - np$
- ☐ $1 - mp$
- ☐ mp

Explanation

By union bound and symmetry, the probability of at least one of the bins being empty is at most np . Taking the complement of this event, the probability of none of the bins being empty is at least $1 - np$.

Q1.2 Simpler Hashing

1 Point

Let q be the probability that any two balls land in the same bin. Using the union bound, find the lower bound of the probability that every ball lands in a different bin.

- ☐ 1
- ☐ $\binom{n}{2}q$
- ☐ $1 - \binom{n}{2}q$
- ☒ $1 - \binom{m}{2}q$
- ☐ $\binom{m}{2}q$

Explanation

There are $\binom{m}{2}$ possible pairs of two balls. By union bound and symmetry, the probability that any two balls land in the same bin is at most $\binom{m}{2}q$. Taking the complement of this event, the probability of every ball landing in a different bin is at least $1 - \binom{m}{2}q$.

Q1.3 Simpler Load Balancing

1 Point

Let r be the probability that any subset of k balls land in the first bin. Using the union bound, find the upper bound of the probability that the first bin has at least k balls.

- ☐ 1
- ☐ $\binom{n}{k}r$
- ☐ $1 - \binom{n}{k}r$
- ☐ $1 - \binom{m}{k}r$
- ☒ $\binom{m}{k}r$

Explanation

There are $\binom{m}{k}$ possible subsets of k balls. The event that the first bin has at least k balls is equal to the union of all the events that a subset of k balls lands in the first bin. By union bound and symmetry, this probability is at most $\binom{m}{k}r$.

Q1.4

1 Point

Suppose $1 < m < n$. Which of the following events cannot occur?

- ☐ At least one bin has multiple balls
- ☐ None of the bins have multiple balls
- ☒ None of the bins are empty

Explanation

There are less balls than bins. Thus, some bin must be empty.

For the other options: it is possible that 2 balls land in the same bin; that bin has multiple balls. It is also possible that every ball lands in a different bin, in which case no bin has multiple balls.