CS70 – Spring 2026

Lecture 10: Feb. 19

New Topic: Polynomials

- Basic properties of polynomials
- Polynomials mod p
- Application I: Secret Sharing
- Application II: Error-Correcting Codes

TODAY

NEXT LEC.

Polynomials (over the real numbers)

Defn: A polynomial in a single variable x is a function of the form

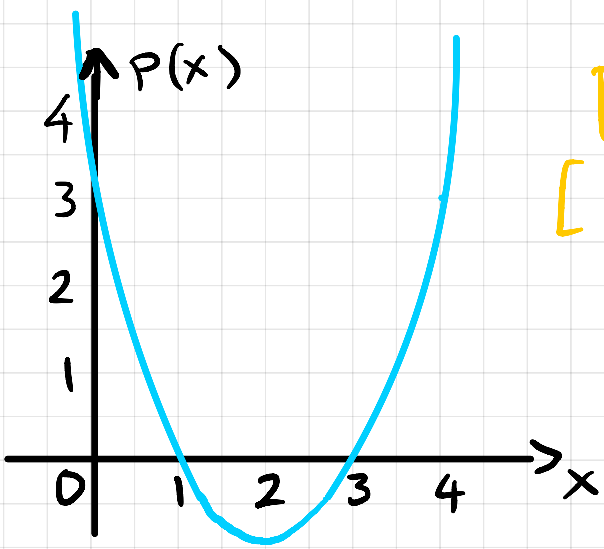
$$\boxed{p(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0}$$

degree d coefficients a_i (real)

Examples: $p(x) = 7x + 3$ (degree 1)

$p(x) = x^2 - 4x + 3$ (degree 2)

[$= (x-3)(x-1)$]

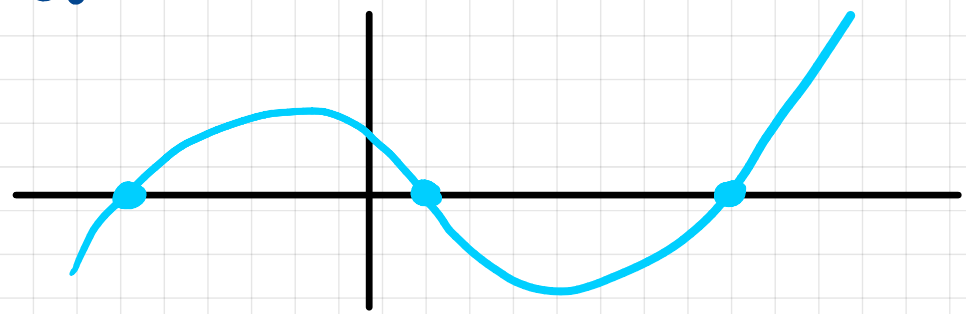


$\boxed{\text{Defn: } a \text{ is a root of } p(x) \text{ if } p(a) = 0}$

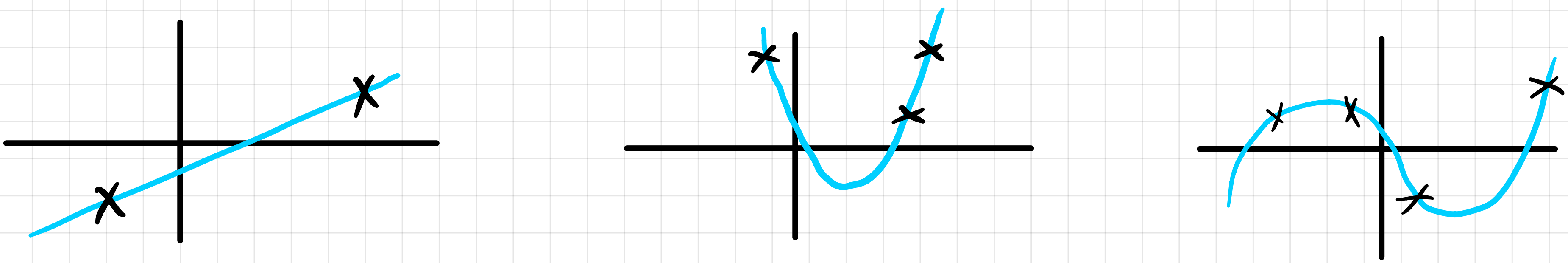
Polynomials: 2 Key Properties

Property 1: A non-zero polynomial of degree d has at most d roots

Eg. $d=3$ (cubic)



Property 2: Given $d+1$ pairs $(x_1,y_1),(x_2,y_2),\dots,(x_{d+1},y_{d+1})$, with all the x_i distinct, there exists a unique polynomial of degree $\leq d$ s.t.
$p(x_i)=y_i$ for $1 \leq i \leq d+1$



Property 1: A non-zero polynomial of degree d has at most d roots

Proof (sketch): Let $a_1, \ldots, a_d$ be distinct roots of p. We show p can be written as

$$p(x) = c(x-a_1)(x-a_2)\cdots(x-a_d) \qquad (*)$$

for some constant c.

Assuming $(*)$, if $a \neq a_1 \ldots a_d$ then

$$p(a) = c(a-a_1)(a-a_2)\cdots(a-a_d) \neq 0$$

So no other a is a root! $\square$

Proof of $(*)$: see Notes (induction on d)

Property 2 : Given $d+1$ pairs $(x_1, y_1), (x_2, y_2), \ldots, (x_{d+1}, y_{d+1})$, with all the x_i distinct, there exists a **unique** polynomial of degree $\leq d$ s.t. $p(x_i) = y_i$ for $1 \leq i \leq d+1$

Proof : We give an **algorithm**, called "Lagrange Interpolation", to construct $p(x)$

Method : Suppose we can construct "basis" polynomials $\Delta_i(x)$ for $1 \leq i \leq d+1$ s.t.

$$\Delta_i(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{if } x = x_j,\ j \neq i \end{cases}$$

$$\boxed{\text{Then we set } p(x) = \sum_{i=1}^{d+1} y_i \Delta_i(x)}$$

✓

Suppose we can construct "basis" polynomials $\Delta_i(x)$ for $1 \leq i \leq d+1$ s.t.

$$\Delta_i(x) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{if } x = x_j,\ j \neq i \end{cases}$$

Let $q_i(x) = (x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_{d+1})$

Then :
- degree of q_i is d
- $q_i(x_j) = 0 \quad \forall j \neq i$
- $q_i(x_i) = \prod_{j \neq i}(x_i - x_j) \neq 0$

So we can define $\boxed{\Delta_i(x) = \dfrac{q_i(x)}{q_i(x_i)}}$

$$= \frac{(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_{d+1})}{(x_i-x_1)\ \ (x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_{d+1})}$$

Example: Find degree-2 polynomial through $(0,-1)$, $(2,2)$, $(3,4)$

Proof that $p(x)$ is unique:

Suppose for ※ $\exists$ two different polynomials $p_1(x), p_2(x)$ of degree $\leq d$ that both go through the $d+1$ points $(x_1, y_1) \dots (x_{d+1}, y_{d+1})$

Then $q(x) := p_1(x) - p_2(x)$ is a non-zero poly. of degree $\leq d$.

But $q(x_i) = p_1(x_i) - p_2(x_i) = 0$ for $x_1, x_2, \dots, x_{d+1}$

So q is a poly. of degree $\leq d$ with $d+1$ roots! ※ Prop. 1

This concludes proof of Property 2. $\square$

Polynomials + Modular Arithmetic

Let p be prime

Integers mod p "behave like" real numbers, in that they support:

- 0 & 1 $\quad [x+0=x;\ x\cdot 0=0;\ x\cdot 1=x]$
- operations of addition, subtraction, multiplication, division (inverses)

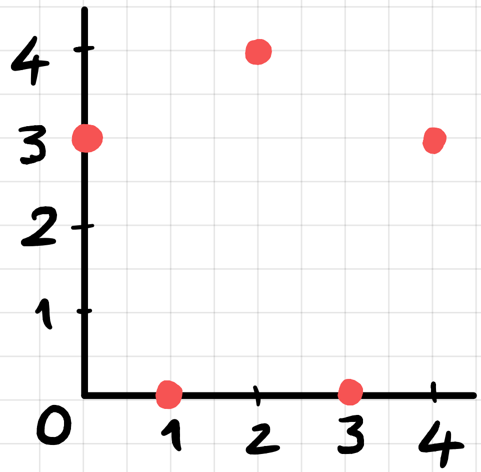
Technically, integers mod p are a field

Denote this field $\mathbb{Z}_p$ or $GF[p]$

Unlike $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$, $GF[p]$ is a finite field

Polynomials over GF[p]

Example: $p(x) = x^2 - 4x + 3 \pmod 5$



$p(0) = 3$
$p(1) = 0$
$p(2) = 4$
$p(3) = 0$
$p(4) = 3$

Key Fact: Polynomials over GF[p] "behave like" real polynomials, in that they satisfy Properties 1 & 2 !

We can use Property 2 to count polynomials over $GF[p]$

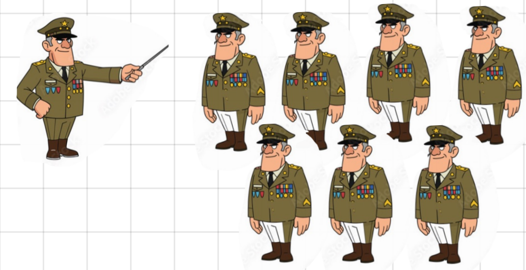
Q: How many different polys of degree $\leq d$ are there $\pmod p$?

A:

Q: Suppose I give you $k \leq d+1$ points. How many such polys. go through those points ?

A:

Application: Secret Sharing



Commander wants to share a secret code s among n generals s.t.

- any group of $\geq k$ generals can figure out s
- any group of $< k$ generals has no information about s

Method

- Commander constructs a random polynomial $p(x)$ of degree $k-1$ $(\text{mod } p)^*$ s.t. $p(0) = s$
- Commander gives each general i the value $p(i)$ $(i = 1, 2, \dots, n)$

* Need to choose $p > s$
Eg. $p = 1{,}000{,}003$
for 6-digit codes

- Commander constructs a random polynomial $p(x)$ of degree $k-1 \pmod{p}$ s.t. $p(0)=s$
- Commander gives each general i the value $p(i)$ $\quad (i=1,2,\ldots,n)$

Why does this work ?

(i) Sp. k generals get together

(ii) Sp. only $k-1$ (or fewer) generals get together

Example: $n = 7$ generals | $p = 11$
$k = 3$ | $s = 8$

Random degree-2 polynomial over GF[11]:

$$p(x) = x^2 + 6x + \underbrace{8}_{s}$$

$p(1) = 4 \rightarrow (1,4)$ to general 1
$p(2) = 2 \rightarrow (2,2)$ to gen. 2
$p(3) = 2 \rightarrow (3,2)$ to gen. 3
$\vdots$
$p(7) = 0 \rightarrow (7,0)$ to gen. 7

any 3 generals have 3 pts. on $p(x)$
$\Rightarrow$ can use Lagrange to recover $p(x)$

any 2 generals have 2 pts. on $p(x)$
$\Rightarrow$ consistent with $p=11$ values for $p(0)$

Example: $n = 7$ generals, $k = 3$ | $p = 11$, $s = 8$ | Polynomial $p(x) = x^2 + 6x + 8$

Suppose generals 1, 2, 7 get together
They have points $\boxed{(1,4),\ (2,2),\ (7,0)}$

Lagrange:

$$\Delta_1(x) = \frac{(x-2)(x-7)}{(1-2)(1-7)} = 6^{-1}(x-2)(x-7) \equiv \boxed{2(x-2)(x-7)} \pmod{11}$$

$[x_1 = 1]$

$$\Delta_2(x) = \frac{(x-1)(x-7)}{(2-1)(2-7)} = -5^{-1}(x-1)(x-7) \equiv \boxed{2(x-1)(x-7)} \pmod{11}$$

$[x_2 = 2]$

$$\Delta_3(x) = \langle \text{whatever} \rangle$$

$[x_3 = 7]$

Hence

$$p(x) = 4 \cdot \Delta_1(x) + 2 \cdot \Delta_2(x) + 0 \cdot \Delta_3(x)$$
$$= 8(x-2)(x-7) + 4(x-1)(x-7) = \ldots \equiv \boxed{x^2 + 6x + 8} \pmod{11}$$

Secret sharing works the same way over $\mathbb{R}$.
Why do we do it over $GF[p]$?

- Keeps all arithmetic exact and numbers moderate sized integers. (Note: even over $\mathbb{R}$, we would choose integer coeffs. for $p(x)$)
- Can precisely quantify how much info. any subset of generals have
- With real polynomials, smaller group of generals can use the fact that s is an integer to learn things about it !

Example: Sp. $\{p(x) = ax^2+bx+c$ over $\mathbb{R}$ and 2 generals have the points $(1,0)$ and $(2,6)$.

Then they know:

$$\left.\begin{aligned} p(1) &= a+b+c = 0 \\ p(2) &= 4a+2b+c = 6 \end{aligned}\right\}$$

solve for b, c in terms of a:

$$b = 5-3a$$
$$c = 2a-6$$

So $p(x) = ax^2+(5-3a)x+(2a-6)$

And a must be an integer

So the secret $s = p(0) = 2a-6$ is <u>even</u>!