CS70 – SPRING 2026

LECTURE 13: MAR. 3

Last Lecture

- Sets A, B have same cardinality $\Leftrightarrow \exists$ bijection $f: A \to B$
- Set A is countable $\Leftrightarrow \exists$ bijection between A and (some subset of) $\mathbb{N}$
- Some countable sets (proved by enumeration):
  - all finite sets
  - $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Z} \times \mathbb{Z}, \{0,1\}^*, \mathbb{N}[x], \ldots$
- Some uncountable sets (proved by diagonalization):
  - $\mathbb{R}$, $[0,1]$, $\mathcal{P}(\mathbb{N})$, Cantor set, ...

Today

Q: Is there any (mathematical/technical) problem that computers cannot solve ?

A: Yes! [Turing: Halting Problem]

Q': Is there any true mathematical theorem that cannot be proved ?

A': Yes! [Gödel: Incompleteness Theorem]

Why ?

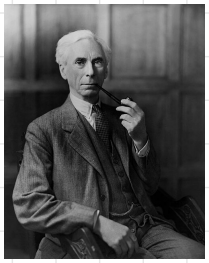
- Great application of Diagonalization
- Core philosophy of Computer Science

$\mathbb{R}$ uncountable — Cantor OR Dedekind ?
— Simons Foundation "Quanta"

Timeline

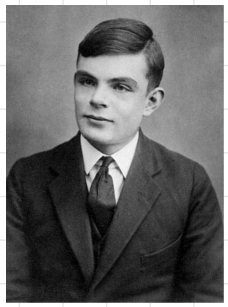
~1900: Hilbert's Program: formalize all of Mathematics via axioms & inference rules

early 1900s: Russell & others: logical paradoxes

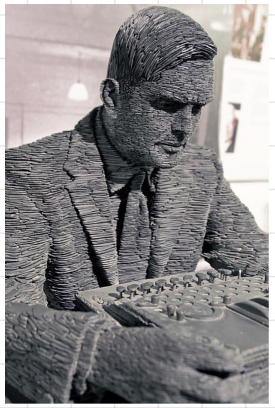


1931: Gödel: Not possible for arithmetic — must exist true statements that aren't provable

1936: Turing: There are mathematical problems that no computer can solve



Turing's Theorem Ingredients:

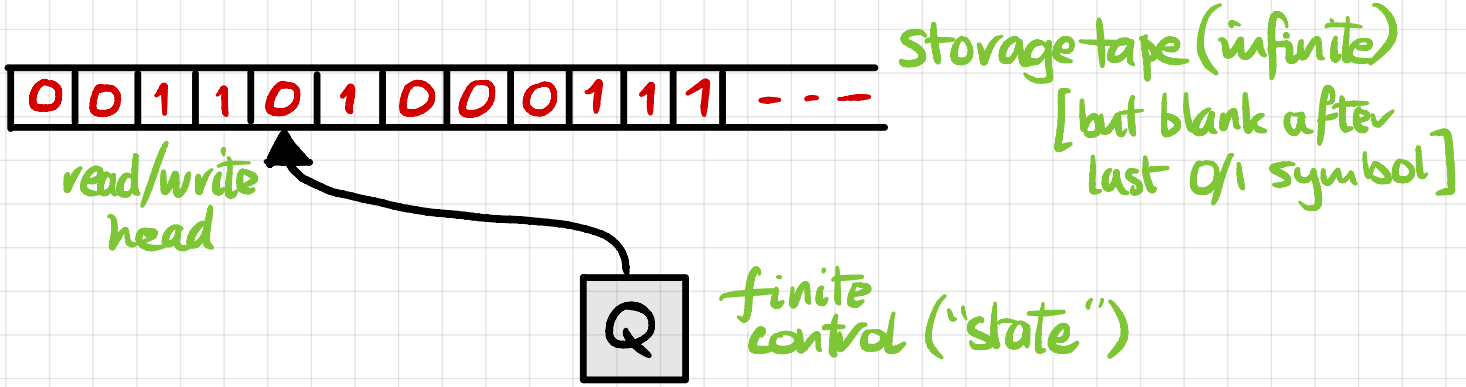


1. A model of computation
   (to define what's possible/impossible)

2. Programs = Data (= binary strings)

3. Self-reference (≈ diagonalization)

1. Model of Computation: Turing Machine



transition function

$$\delta: \underbrace{Q}_{\text{state}} \times \underbrace{\{0,1,\textcircled{b}\}}_{\text{tape symbol}} \longrightarrow Q \times \{0,1\} \times \underbrace{\{L,R,S\}}_{\text{move}}$$

Primitive, but equivalent to (i.e., can simulate) any other model, e.g.:

- Random Access Machine
- High Level Programming Language
- Quantum Computer

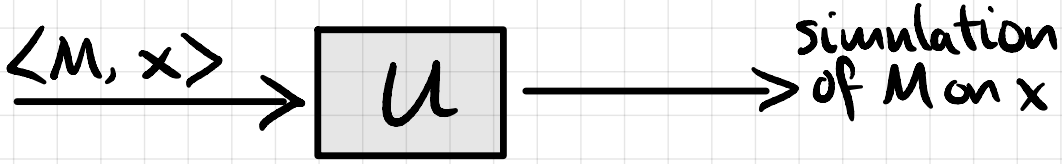
⋮ ⋮

2. Programs = Data

Turing machine/Computer program is just a finite string of bits

⇒ it can be fed to another program as input !

Universal TM [Turing 1936]



Note: Turing invented programmable computers long before they were designed or built !

[Manchester Baby 1948, Mark 1 1949
Grace Hopper Compilers, 1952]

3. Self-Reference

Logical paradoxes [B. Russell etc.]

- "This sentence is false"
- The barber shaves those – and only those – men who do not shave themselves (where everyone is clean shaven)
- S = the set of all sets that do not contain themselves
- "'yields a falsehood when appended to its own quotation' yields a falsehood when appended to its own quotation"

Quines

Computer programs that take no input and output their own source code

Example in Ruby:

```
eval s = "print 'eval s = '; p s"
```

Halting & Non-Halting Programs

```
program count_by_threes (x)
    count = 0
    while count ≠ x
        count = count + 3
    output "Done"
```

- if x is a non-negative multiple of 3, then count_by_threes(x) halts (& outputs "Done")
- otherwise, count_by_threes doesn't halt/loops forever

Halting Problem: Distinguish these two behaviors

The Halting Problem [Turing 1936]

Input: program (TM) P & input x for P

Question: does P halt on input x ?

> **Turing's Theorem**: There is no computer program (TM) that solves the Halting Problem on all inputs

Note: We say that the Halting Problem is uncomputable or undecidable

Proof of Turing's Theorem: Use diagonalization

Assume for ※ that ∃ program that solves the HP.

Since programs are just finite strings in $\{0,1\}^*$, and so are their inputs, can write a table:

programs \ inputs	P_0	P_1	P_2	P_3	...
P_0	(H)	L	H	H	...
P_1	L	(L)	L	H	...
P_2	H	H	(H)	L	...
P_3	L	L	L	(L)	...

Construct following program:

Turing (P)

if P halts on input P

then LOOP

else HALT

uses HP

Program Turing is NOT in the table ※

└ because it behaves differently from each P_i on input P_i

Hence our assumption that ∃ program for HP must be false □

The "Easy" Halting Problem

Input: Program P

Question: Does P halt on input 0 ?

Theorem: "Easy" HP is undecidable

Proof: Show that if EasyHP decidable then so is HP

```
program TestHalt(P, x)
    construct a program P' that, on input 0,
        simulates P on x
    return TestEasyHalt(P')
```

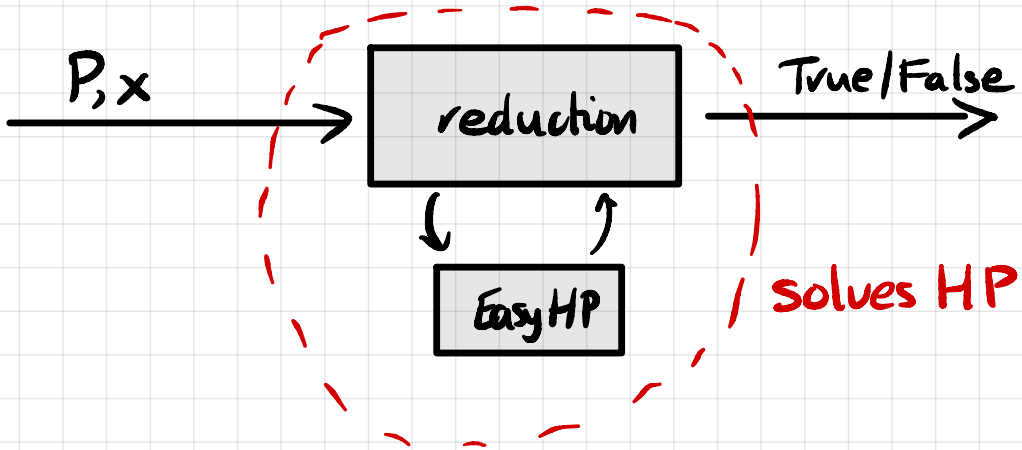
Claim: TestHalt(P, x) = TRUE

$\Leftrightarrow$

P halts on x

$\blacksquare$

We just used a reduction from HP to EasyHP:



The reduction shows that if EasyHP is solvable then so is HP ⨳

So EasyHP is not solvable

Program Equivalence

Input: Two programs P_1, P_2

Question: Do P_1, P_2 behave the same (halt/not halt) on all inputs?

Theorem: Program Equiv. is undecidable

Proof: Ex.

Other Undecidable Problems

1. Most questions about software behavior, e.g.,:

   Input: Program P, Specification S

   Question: Does P satisfy S?

2. Wang Tiling Problem

   Input: Finite set of Wang tiles, e.g., 

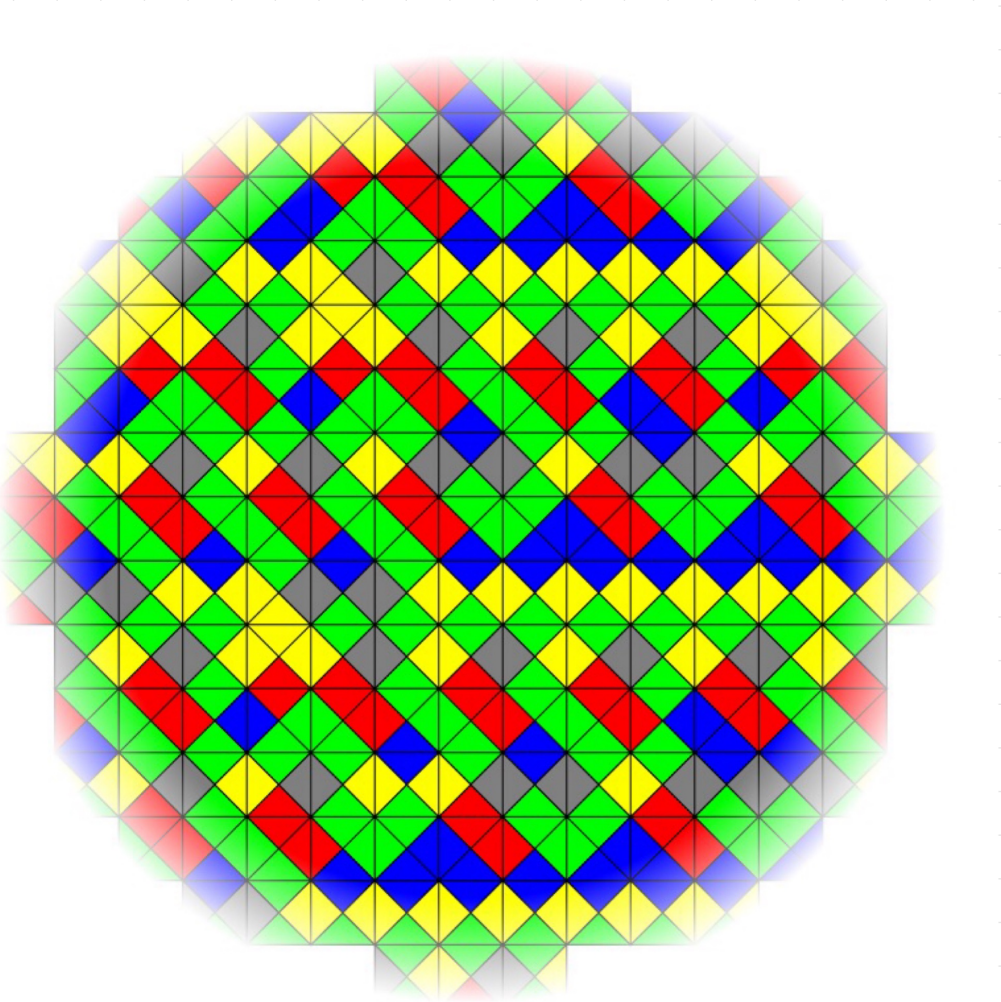
   Question: Can we tile the infinite plane with these tiles?

3. Matrix Mortality Problem

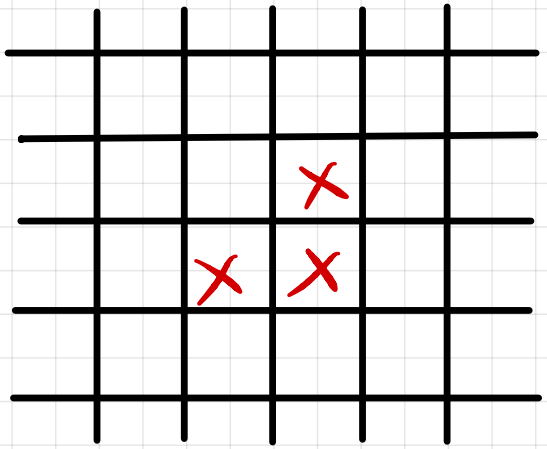
   Input: Finite set of $n \times n$ integer matrices

   Question: Can we multiply them, in any order (repetitions allowed) to get the zero matrix?

[Note: Undecidable for 6 3×3 matrices, or for 2 15×15 matrices]



4. Conway's Game of Life



< 2 occupied nbrs. $\longrightarrow$ die

> 3 " " $\longrightarrow$ die

$= 3$ " " $\longrightarrow$ born

Input: Initial config. I; final config. F

Question: Is F reachable from I in a finite no. of generations ?

5. Diophantine Equations [Hilbert's 10th Problem]

Input: Polynomial equation with integer coefficients

e.g. $x^2 + y^2 + 1 = 0$ NO

$4x^2 + xy^2 + 5 = 0$ $x = -1, y = 3$

Question: Does the equation have an integer solution ?

Gödel's Incompleteness Theorem

There does not exist a proof system for arithmetic that is both consistent and complete

Can't prove both P and $\neg P$

Can prove all true statements, i.e., $\forall P$, can prove either P or $\neg P$

Gödel's Theorem: Sketch of Proof

Write a statement in arithmetic of the form

S: "This statement is not provable"

Case (i): S is provable.

Then by consistency S is true, so S is not provable ※

Case (ii): S is not provable.

Then by completeness S is false, so S is provable ※

Writing S is tricky – relies on clever arithmetic coding of statements ("Gödel numbering")

General form of S: $\neg \exists z$ (z is a proof of this statement)

self-reference

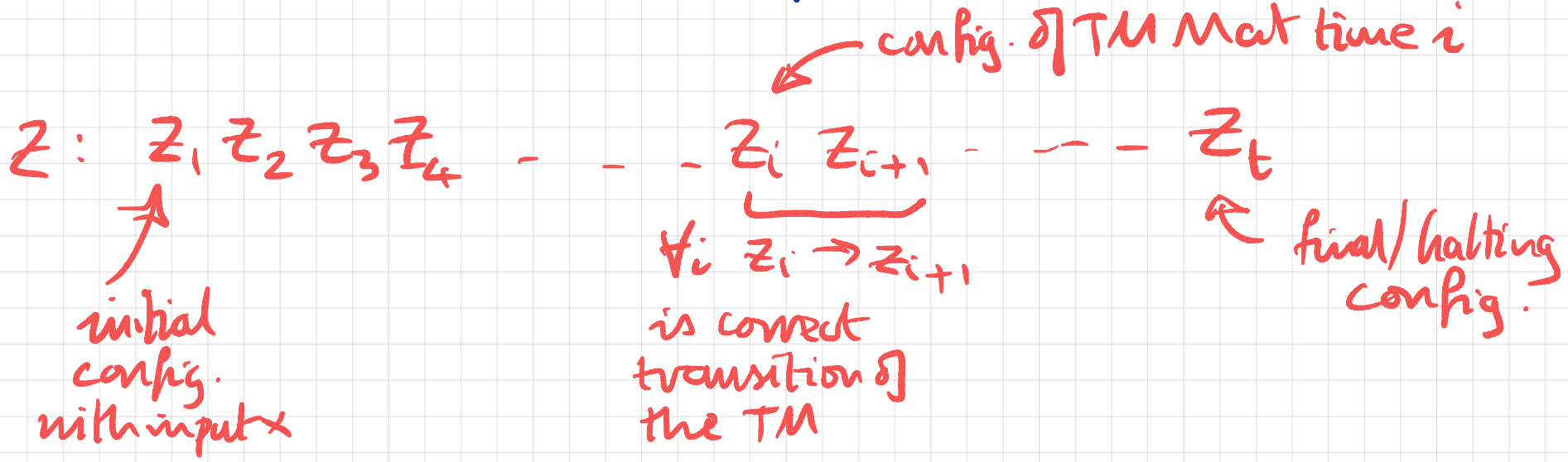
Proof of Gödel's Theorem via Halting Problem

1. Given $\langle M, x \rangle$, write an arithmetic statement of the form

$$\varphi_{M,x} : M \text{ halts on input } x$$

Form of $\varphi_{M,x}$:

$$\exists z \, (z \text{ encodes a halting execution sequence of } M \text{ on input } x)$$



Proof of Gödel's Theorem via Halting Problem (cont.)

2. Solve Halting Problem by searching for a proof of either $\varphi_{M,x}$ or $\neg\varphi_{M,x}$

```
Program Test Halt (M, x)
    construct φ_{M,x}
    for each possible proof q
        if q is a proof of φ_{M,x} output "True"
        if q is a proof of ¬φ_{M,x} output "False"
```

If proof system is complete & consistent, this program always decides the H.P.

Hence can't be complete & consistent □

Note: Uses fact that proofs can be enumerated & checked!