CS70 – Spring 2026

Lecture 1: Jan. 20

Topic 1 : Logic & Proofs

Goals :

1. Learn mathematical language & notation
2. Learn to write convincing arguments (e.g., to justify why your programs work as intended)

Propositional Logic

Proposition: A statement that is either true or false

Examples:

- $\sqrt{3}$ is irrational ✓ T
- $6 - 2 = 3$ ✓ F
- 1 billion is a big number ✗
- Julius Caesar was 5'8" tall ✓ ?
- $3x + 17 = 42$ ✗
- $42/23$ ✗
- Julius Caesar was short ✗

Combining Propositions

$$\left.\begin{array}{ll} P \wedge Q & \text{"AND"} \\ P \vee Q & \text{"OR"} \\ \neg P & \text{"NOT"} \end{array}\right\} \text{"connectives"}$$

Examples:

F P : "3 is even" $\qquad$ Q : "$2+2=4$" T

$P \wedge Q$: F

$P \vee Q$: T

$\neg P$: T

Truth Tables

define connectives

P	Q	$P \wedge Q$	$P \vee Q$	
T	T	T	T	
T	F	F	T	
F	T	F	T	
F	F	F	F	

Truth Tables

define connectives

P	Q	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Another connective : $P \Rightarrow Q$ "IMPLIES"

Example : "If you pass the exam, you'll get into College"

Q: How can this be false ?

A: If you pass exam but don't get into college

If pigs can whistle then horses can fly T

Logical Equivalences

Fact: $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$

We write $(P \Rightarrow Q) \equiv \neg P \vee Q$

Why? Check the truth tables!

P	Q	$P \Rightarrow Q$	$\neg P$	$\neg P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example: "If you pass the exam you'll get into College"

$$\equiv$$

"Either you fail the exam or you'll get into College"

The <u>contrapositive</u> of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$

The <u>converse</u> of $P \Rightarrow Q$ is $Q \Rightarrow P$

<u>Exercise</u> : Use truth tables to check that :

- $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$

  [If you don't get into College then you didn't pass the exam]

- $(P \Rightarrow Q) \not\equiv (Q \Rightarrow P)$

  [If you get into College then you passed the exam]

One more connective : $P \Leftrightarrow Q$ "IF & ONLY IF"

This is defined by : $(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Predicates & Quantifiers : First Order Logic

Propositions : Aristotle is a philosopher
Plato is a philosopher

↓

Predicate : Phil(Aristotle)
Phil(Plato)

where Phil(x) denotes "x is a philosopher"

Quantifiers (over some universe $\mathcal{U}$) :

$(\forall x \in \mathcal{U})\ P(x)$ — universal "for all"

$(\exists x \in \mathcal{U})\ P(x)$ — existential "exists"

Example :

P(x) : x is divisible by 2
Q(x) : x 3
R(x) : x 6

Q: How do we write : "A nat. number x is div. by 6 if & only if it's div. by both 2 and 3" ?

$$(\forall x \in \mathbb{N})\ \big(R(x) \iff (P(x) \wedge Q(x))\big)$$

More examples:

"$x \mid y$" means x divides y

"209 has a divisor larger than 17"

$$(\exists x \in \mathbb{N})(x > 17 \wedge x \mid 209)$$

"$f(x) = x^2 - 4x + 3$ has exactly two distinct real roots"

$$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(f(x) = 0 \wedge f(y) = 0 \wedge x \neq y)$$
$$\wedge\ \forall x \forall y \forall z\ \Big(\big(f(x) = 0 \wedge f(y) = 0 \wedge f(z) = 0\big) \Rightarrow \big((x = y) \vee (y = z) \vee (x = z)\big)\Big)$$

"There is no largest integer"

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(y > x)$$

$$(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(y > x) \quad \#$$

Negation: De Morgan's Laws

$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

Ex: Check using truth tables!

With quantifiers:

$\neg(\forall x\, P(x)) \equiv \exists x\, (\neg P(x))$

$\neg(\exists x\, P(x)) \equiv \forall x\, (\neg P(x))$

Example

$\neg(\exists x \forall y \exists z\; P(x,y,z)) \equiv \forall x \exists y \forall z\, (\neg P(x,y,z))$

Fun Example

Bob is on trial for murder.

Bob's attorney never lies.

$P \Rightarrow Q$

Judge: "If Bob commited this murder, he didn't act alone"

Attorney: "That's not true!"

Q: Did the attorney help Bob?

A: NO! Attorney is saying that Bob commited the murder and acted alone

Fun Example 2

[R. Smulyan]

"A watched kettle never boils unless it is watched"

Q: true/false/undetermined ?

A: "P unless Q" $\equiv \neg P \Rightarrow Q \equiv P \vee Q$

Here P is $W(x) \Rightarrow \neg B(x)$

Q is $W(x)$

So "P unless Q" $\equiv P \vee Q$

$\equiv \forall x \left(\left(W(x) \Rightarrow \neg B(x) \right) \vee W(x) \right)$

$\equiv$ TRUE (if $W(x)$ is false then $W(x) \Rightarrow \neg B(x)$ is true)

1. No one who is going to a party fails to brush his/her hair

$\forall x\,(P(x) \Rightarrow B(x))$

2. No one looks fascinating if he/she is untidy

$\forall x\,(U(x) \Rightarrow \neg F(x))$ $\quad F(x) \Rightarrow \neg U(x)$

3. Opium-eaters have no self-command

$\forall x\,(O(x) \Rightarrow \neg S(x))$

4. Everyone who has brushed his/her hair looks fascinating

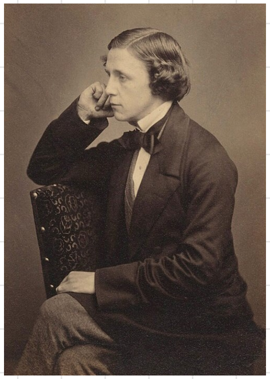
$\forall x\,(B(x) \Rightarrow F(x))$

5. No one wears kid gloves unless he/she is going to a party

$\forall x\,(G(x) \Rightarrow P(x))$

6. A person is untidy if she/he has no self-command

$\forall x\,(\neg S(x) \Rightarrow U(x))$



Lewis Carroll
Symbolic Logic
1897

Q: What can we say about someone who is wearing kid gloves?

$$\forall x\,(G(x) \Rightarrow P(x) \Rightarrow B(x) \Rightarrow F(x) \Rightarrow \neg U(x) \Rightarrow S(x) \Rightarrow \neg O(x))$$

Summary

- Propositions
- Connectives $\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftrightarrow$
- Truth tables ; logical equivalence $\equiv$
- Implications

$$P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P \quad \text{(contrapositive)}$$

$$\not\equiv \quad Q \Rightarrow P \quad \text{(converse)}$$

- Predicates & Quantifiers :

$$\forall x\, P(x) \qquad \exists x\, P(x)$$

- De Morgan's Laws :

$$\neg \forall x\, P(x) \equiv \exists x\, (\neg P(x)) \qquad \neg \exists x\, P(x) \equiv \forall x\, (\neg P(x))$$