CS70 – Spring 2026

Lecture 2: Jan. 22

Previous Lecture

- Propositions
- Connectives $\wedge$ $\vee$ $\neg$ $\Rightarrow$ $\Leftrightarrow$
- Truth tables ; logical equivalence $\equiv$
- Implications

$$P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P \quad \text{(contrapositive)}$$

$$\not\equiv Q \Rightarrow P \quad \text{(converse)}$$

- Predicates & Quantifiers :

$$\forall x\, P(x) \qquad \exists x\, P(x)$$

- De Morgan's Laws :

$$\neg \forall x\, P(x) \equiv \exists x\, (\neg P(x)) \qquad \neg \exists x\, P(x) \equiv \forall x\, (\neg P(x))$$

Today : Proofs

Goals:

- Clearly specify our claims (e.g., about behavior of programs or systems)
- Convince ourselves & others that these claims are valid

Q: What is a proof?

A: A sequence of statements (propositions), each of which follows from the preceding ones by a valid law of reasoning

A proof may also use basic facts we assume without proof (axioms) and other facts we've already proved (lemmas)

Keep in mind:

- Proofs are a "social process" — a contract between the prover and the reader
- Writing good proofs is an art (like writing good code)

Proof Techniques

1. Direct proof
2. Proof by contraposition
3. Proof by contradiction
4. Proof by cases
5. Proof by induction ← next lecture

1. Direct Proof

Goal : Prove $P \Rightarrow Q$

Approach : Assume P

$\vdots$

logical steps / axioms / lemmas

Therefore Q $\square$

Theorem: For any integers a, b, c with $a \neq 0$, if $a \mid b$ and $a \mid c$ then $a \mid (b+c)$

Proof:

Example: Divisibility by 11

E.g.: $23738 \longrightarrow 2373 - 8 = 2365$

$\longrightarrow 236 - 5 = 231$

$\longrightarrow 23 - 1 = 22$

$\longrightarrow 2 - 2 = 0$ ✓

"Delete last digit & subtract it from remaining number"

Denote by reduce(n) the number obtained from n by this rule

Claim: n is divisible by 11 $\iff$ reduce(n) is divisible by 11

Theorem: For any integer $n \geq 10$, $11 \mid n \iff 11 \mid \text{reduce}(n)$

Proof: Need to prove two things:

(i) $\forall n \geq 10, \quad 11 \mid n \Rightarrow 11 \mid \text{reduce}(n)$

(ii) $\forall n \geq 10, \quad 11 \mid \text{reduce}(n) \Rightarrow 11 \mid n$

Theorem: For any integer $n \geq 10$, $11 \mid n \iff 11 \mid \text{reduce}(n)$

Proof: Need to prove two things:

(i) $\forall n \geq 10, \quad 11 \mid n \Rightarrow 11 \mid \text{reduce}(n)$

(ii) $\forall n \geq 10, \quad 11 \mid \text{reduce}(n) \Rightarrow 11 \mid n$

Theorem: For any integer n, $n^3 - n$ is divisible by 6

Proof:

Proof by Contraposition

Goal: Prove $P \Rightarrow Q$

Recall: $P \Rightarrow Q \equiv \neg Q \Rightarrow \neg P$

Approach:

Assume $\neg Q$

$\vdots$

logical steps/ axioms/ lemmas

$\vdots$

Therefore $\neg P$

Hence $P \Rightarrow Q$ $\square$

Theorem: Let n be an integer. If $n^2 - 4n + 7$ is even then n is odd.

Proof:

Theorem: For a real number x, if $x^3 - x > 0$ then $x > -1$.

Proof:

Proof by Contradiction "Reductio ad absurdum"

Goal: Prove P

Approach:

Assume $\neg P$

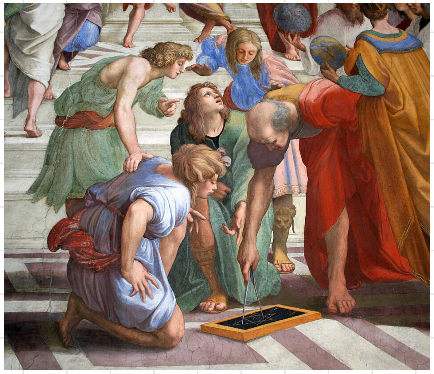
$\vdots \qquad\qquad \vdots$

Therefore R $\qquad$ Therefore $\neg R$

Since $\neg P \Rightarrow (R \wedge \neg R) \equiv$ False,
P must be True. $\square$

Theorem [Euclid]: There are infinitely many primes.

Proof:



Theorem: $\sqrt{2}$ is irrational

Proof:

Proof by Cases

Goal : Prove P

Approach :

Case 1

$\vdots$

P

Case 2

$\vdots$

P

$\vdots$

Case K

$\vdots$

P

P holds in all cases, so P is True $\square$

Theorem: For all real x, $|x+3| - x \geq 3$

Proof:

Theorem: There exist irrational numbers x, y s.t. x^y is rational.

Proof:

Proof Fails

Example 1

"Theorem": $2 = 0$

"Proof": Let $x = y = 1$

Since $x = y$, we have $x^2 - y^2 = 0$

Factorizing: $(x+y)(x-y) = 0$

Dividing both sides by $(x-y)$: $x+y = 0$

But $x = y = 1$, so we have $2 = 0$ $\square$

Example 2

"Theorem": $2 = 1$

"Proof": Clearly $4 - 6 = 1 - 3$

Add $9/4$ to both sides:

$$4 - 6 + 9/4 = 1 - 3 + 9/4$$

Both sides are perfect squares:

$$(2 - 3/2)^2 = (1 - 3/2)^2$$

Taking square roots:

$$2 - 3/2 = 1 - 3/2$$

Adding $3/2$ to both sides:

$$2 = 1 \qquad \square$$

Example 3

"Theorem": $9 < 4$

"Proof": Clearly $-3 < 2$

Squaring both sides:

$$9 < 4 \qquad \square$$

Example 4

"Theorem": For any positive real x, $x + \frac{1}{x} \geq 4$

"Proof": Assuming $x + \frac{1}{x} \geq 4$, since $x > 0$ we can multiply both sides by x to get

$$x^2 + 1 \geq 4x$$

Hence $(x-2)^2 \geq 0$

This is true for any real x.

Hence $x + \frac{1}{x} \geq 4$ for all pos. real x $\square$

Summary

- Proof types:
  - Direct Proof
  - Proof by Contraposition
  - Proof by Contradiction
  - Proof by Cases

- Some common pitfalls

- Next lecture: Proof by Induction