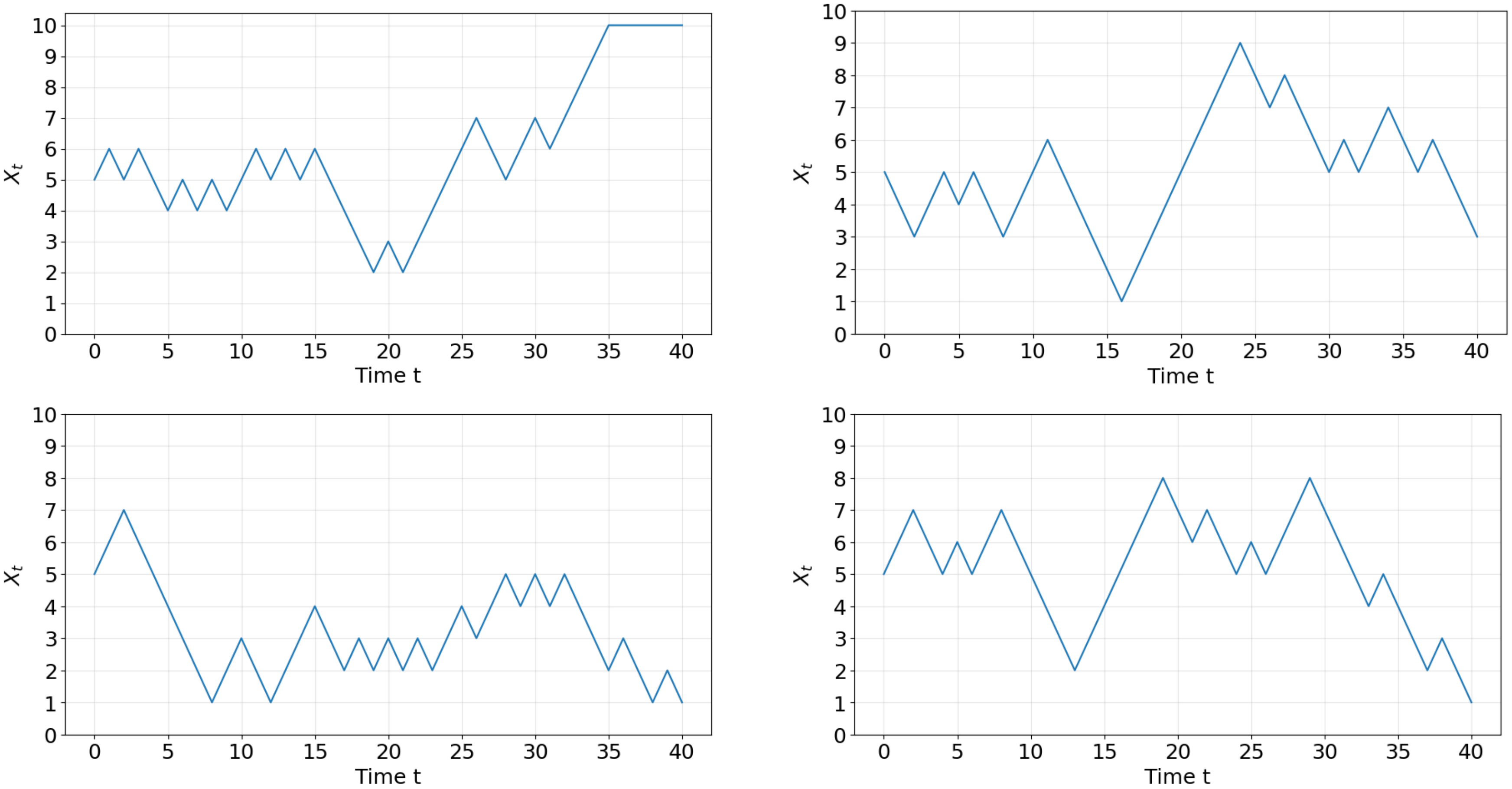
# CS70 @ UC Berkeley, Spring 2026

## Lecture 26

## Markov Chains I

April 28, 2026

# Sample Trajectories from a Stochastic Process



# Stochastic Process

**Definition (Random process or stochastic process):** A random process is a family $\{X_t, t \in T\}$ of random variables indexed by some set $T$.
E.g., $T = \mathbb{N}$ (discrete-time) or $T = [0, \infty)$ (continuous-time)

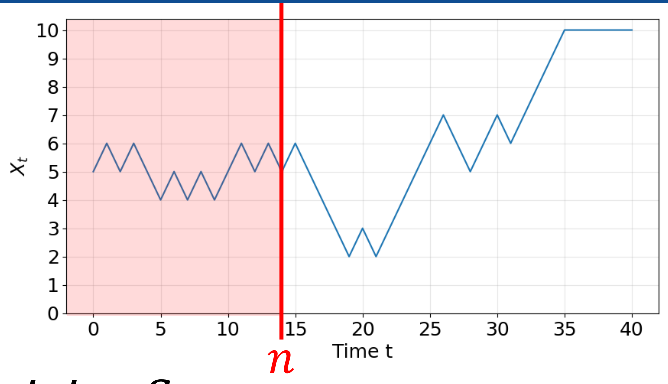
**Interpretation:** as time passes, $X_t$ "evolves" in a random but prescribed way.
**Examples of stochastic processes:**

- Random walk
- Speech recognition
- Protein sequence evolution
- There are many other applications in physics, biology, chemistry, economics, finance, computer science, and engineering.

**Stochastic processes $\{X_t, t \in T\}$ are distinguished by**

- State space $S$ (i.e., $X_t: \Omega \to S$, the possible values of $X_t$)
- Time index set $T$.
- Dependence relation among $X_t$ over time.

# Discrete-time, Discrete-space Markov Chains




In CS70, we will consider:

- Finite state space $S$.
- Discrete-time $T = \mathbb{N}$.

- **Markov property:** For all $n \in T$ and $a_0, \dots, a_{n-1}, i, j \in S$,

$$\mathbb{P}[X_{n+1} = j \mid X_0 = a_0, X_1 = a_1, \dots, X_{n-1} = a_{n-1}, X_n = i] = \mathbb{P}[X_{n+1} = j \mid X_n = i]$$

- Intuitively, this condition says that the future depends on the current state, but not on how you got there.
- $\mathbb{P}[X_{n+1} = j \mid X_n = i]$ is called the transition probability.
- If the transition probability $\mathbb{P}[X_{n+1} = j \mid X_n = i]$ does not depend on the time $n$, then the Markov chain is said to be homogeneous, and we denote

$$P_{ij} := \mathbb{P}[X_{n+1} = j \mid X_n = i].$$

# Transition Probability and Initial Distribution

$$P_{ij} := \mathbb{P}[\, X_{n+1} = j \mid X_n = i \,].$$

**Remarks:**

- $0 \leq P_{ij} \leq 1$ for all $i, j \in S$
- $\sum_{j \in S} P_{ij} = 1$ for all $i \in S$.
- The distribution of $X_0$ and the transition probability matrix $\boldsymbol{P} = \left(P_{ij}\right)$ completely specify a Markov Chain.

The **initial distribution** of $X_0$

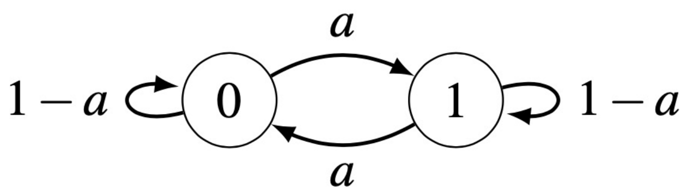
$$\mathbb{P}[X_0 = i] = \pi_0(i), \quad i \in S$$

satisfies $\pi_0(i) \geq 0$ for all $i \in S$ and $\sum_{i \in S} \pi_0(i) = 1$.

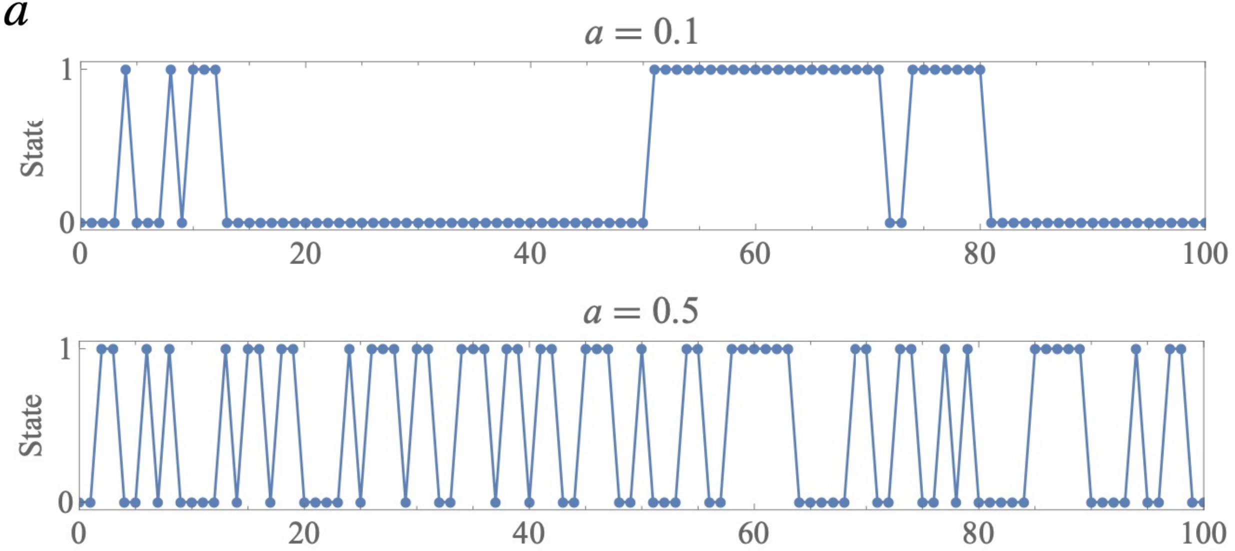
# A Symmetric Two-State Markov Chain

A symmetric two-state Markov Chain with $S = \{0,1\}$, $X_0 \sim \text{Uniform}(S)$,

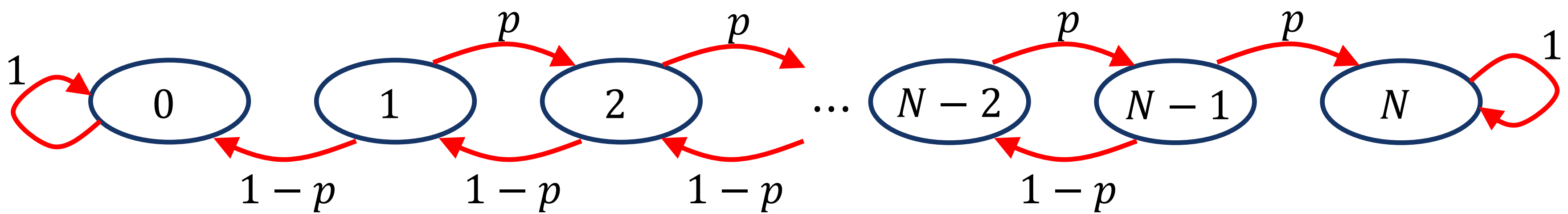
$$P = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 1-a & a \\ a & 1-a \end{bmatrix}.$$




**Simulations**



# 1D Random Walk with Two Absorbing Boundaries

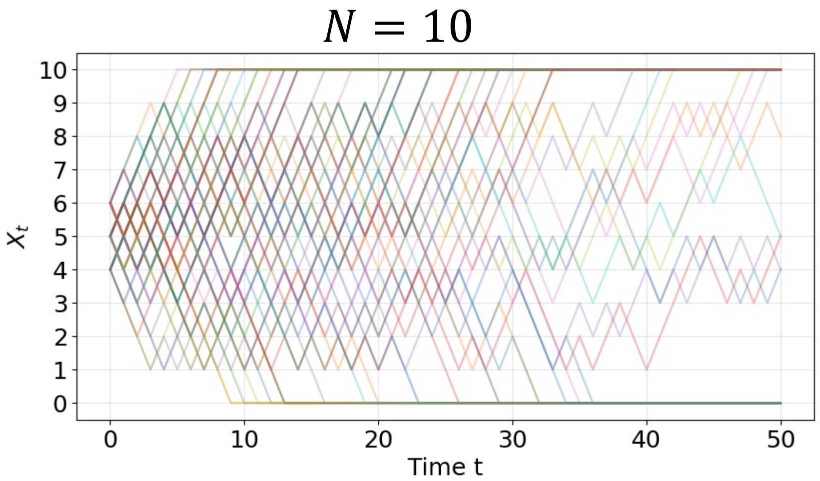


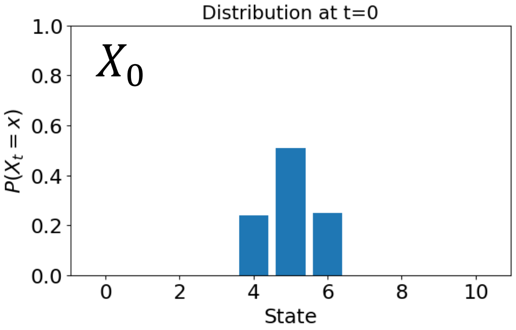

Transition Matrix for $N = 6$:

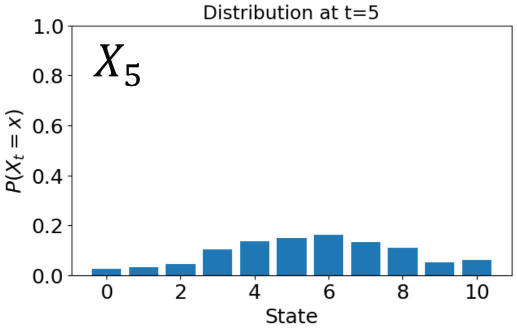
$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 1-p & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

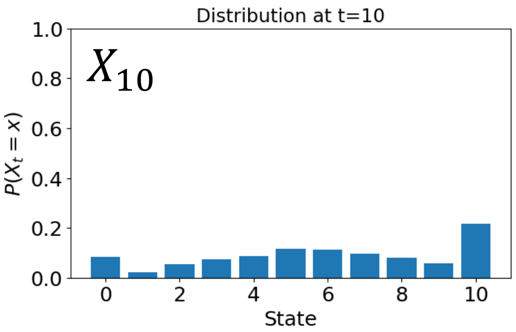
Tri-diagonal matrix

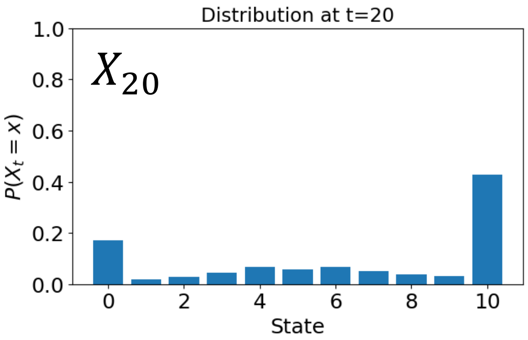
# Simulations of 1D RW with Two Absorbing Boundaries

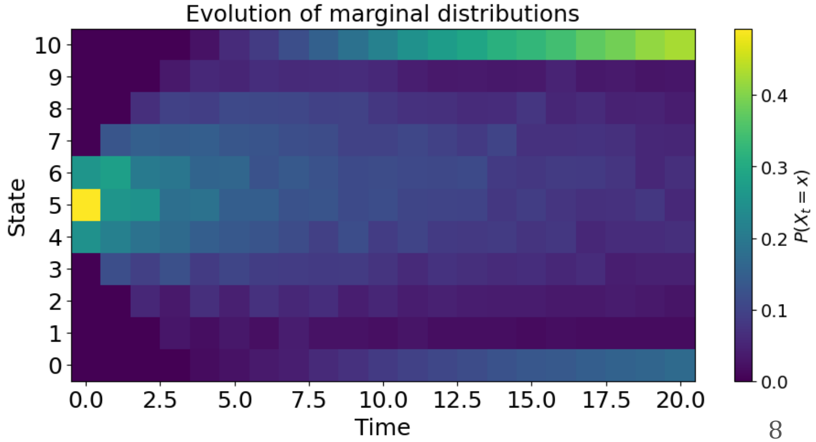


# What is the Distribution of $X_n$ at Time $n$?

First, note that the joint probability of $X_0, \dots, X_n$ is given by

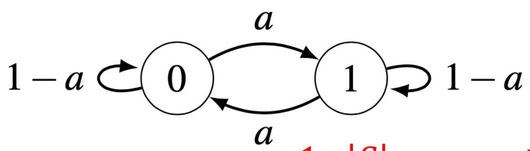
$$\mathbb{P}[X_0 = i_0, X_1 = i_1, \dots, X_n = i_n]$$
$$= \mathbb{P}[X_0 = i_0]\, \mathbb{P}[X_1 = i_1 | X_0 = i_0] \cdots \mathbb{P}[X_n = i_n | X_0 = i_0, \dots, X_{n-1} = i_{n-1}]$$
$$= \pi_0(i_0) P_{i_0,i_1} \cdots P_{i_{n-1},i_n}$$

Consequently, the marginal probability of $X_n$ can be computed as

$$\mathbb{P}[X_n = i_n] = \sum_{i_0,\dots,i_{n-1} \in S} \mathbb{P}[X_0 = i_0, \dots, X_{n-1} = i_{n-1}, X_n = i_n]$$

$$= \sum_{i_0,\dots,i_{n-1} \in S} \pi_0(i_0) P_{i_0,i_1} \cdots P_{i_{n-1},i_n} = [\boldsymbol{\pi}_0 \boldsymbol{P}^n]_{i_n}$$

$1 \times |S|$    $|S| \times |S|$




For the two-state example with $S = \{0,1\}$,

$1 \times |S|$ row vector → $[\mathbb{P}[X_n = 0] \quad \mathbb{P}[X_n = 1]] = \boldsymbol{\pi}_0 \boldsymbol{P}^n$

# $n$-step Transition Probability

First, note that the joint probability of $X_0, \dots, X_n$ is given by

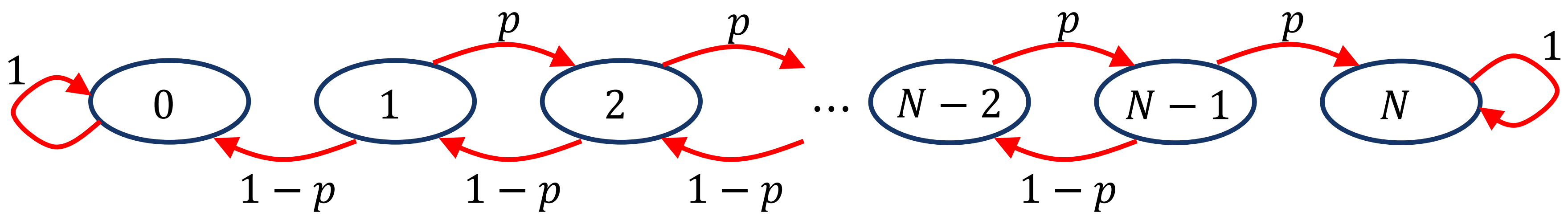
$$\mathbb{P}[X_0 = i_0, X_1 = i_1, \dots, X_n = i_n]$$
$$= \mathbb{P}[X_0 = i_0]\, \mathbb{P}[X_1 = i_1 | X_0 = i_0] \cdots \mathbb{P}[X_n = i_n | X_0 = i_0, \dots, X_{n-1} = i_{n-1}]$$
$$= \pi_0(i_0) P_{i_0,i_1} \cdots P_{i_{n-1},i_n}$$

The conditional probability of $X_n$ given $X_0 = i_0$ can be computed as follows:

$$\mathbb{P}[X_n = i_n, X_0 = i_0] = \sum_{i_1,\dots,i_{n-1} \in S} \mathbb{P}[X_0 = i_0, \dots, X_{n-1} = i_{n-1}, X_n = i_n]$$
$$= \sum_{i_1,\dots,i_{n-1} \in S} \pi_0(i_0) P_{i_0,i_1} \cdots P_{i_{n-1},i_n} = \pi_0(i_0) [\boldsymbol{P}^n]_{i_0,i_n}$$

$$\mathbb{P}[X_n = i_n | X_0 = i_0] = \frac{\mathbb{P}[X_n = i_n, X_0 = i_0]}{\mathbb{P}[X_0 = i_0]} = \frac{\pi_0(i_0) [\boldsymbol{P}^n]_{i_0,i_n}}{\pi_0(i_0)} = [\boldsymbol{P}^n]_{i_0,i_n}$$

# 1-d Random Walk with two absorbing boundaries




Transition Matrix for $N = 6$:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 1-p & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

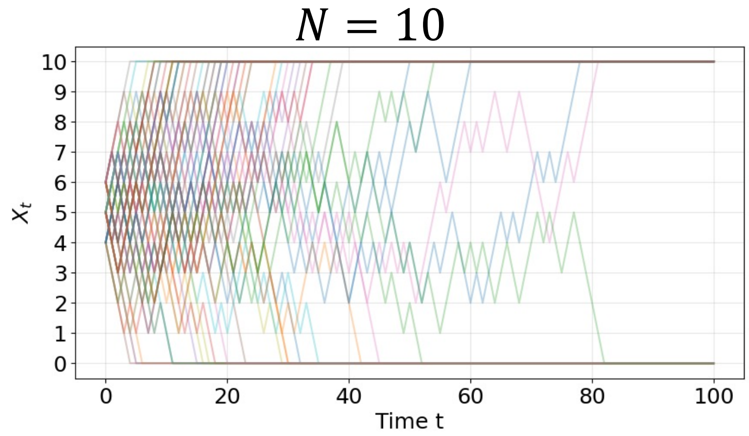
Tri-diagonal matrix

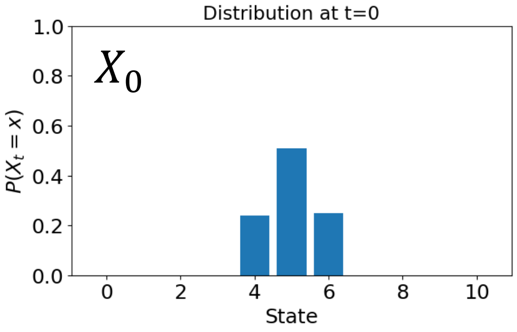
**We will now show how to compute these two quantities:**

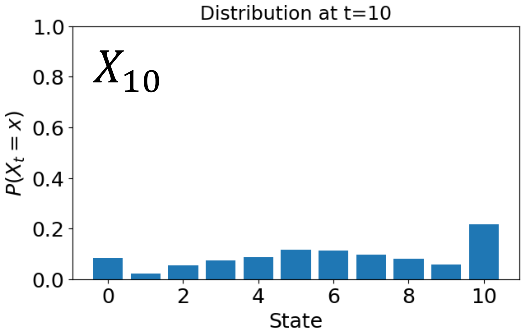
$\alpha(i) = \mathbb{P}(\text{ Hit 0 before } N \mid X_0 = i\ )$

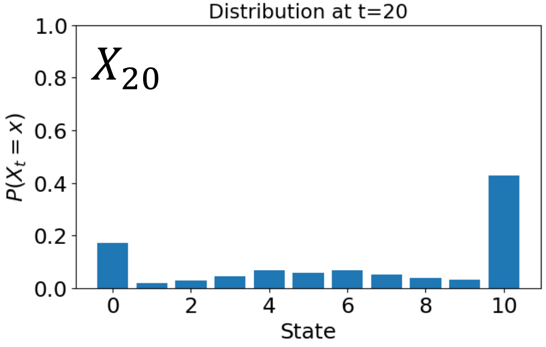
$\beta(i) =$ Expected time until hitting either 0 or $N$, given that $X_0 = i$.

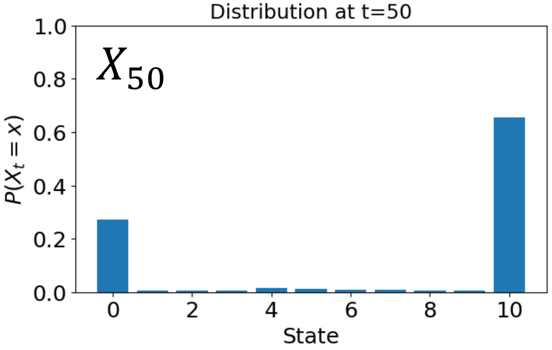
# Simulations with $p = 0.55$

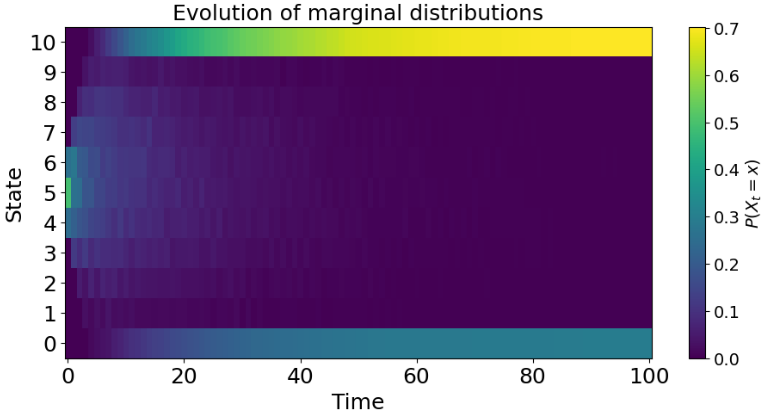


# Hitting Probability

**First-step analysis:** Let $A$ be any event.
$\{X_1 = j, \text{for } j \in S\}$ partitions the sample space $\Omega$.
Hence, by the Law of Total Probability

$$\mathbb{P}[\,A \mid X_0 = i\,] = \sum_{j \in S} \mathbb{P}[\,A \mid X_0 = i, X_1 = j\,]\,\mathbb{P}[\,X_1 = j \mid X_0 = i\,]$$

Let $A$ = "event of hitting 0 before $N$". Then, $\alpha(i) = \mathbb{P}(\,A \mid X_0 = i\,)$.

$\alpha(0) = 1$ and $\alpha(N) = 0$ (boundary conditions)

For $i \neq 0, N$,

By the Markov property — By time-homogeneity

$$\mathbb{P}[\,A \mid X_0 = i, X_1 = j\,] = \mathbb{P}[\,A \mid X_1 = j\,] = \mathbb{P}[\,A \mid X_0 = j\,] = \alpha(j)$$

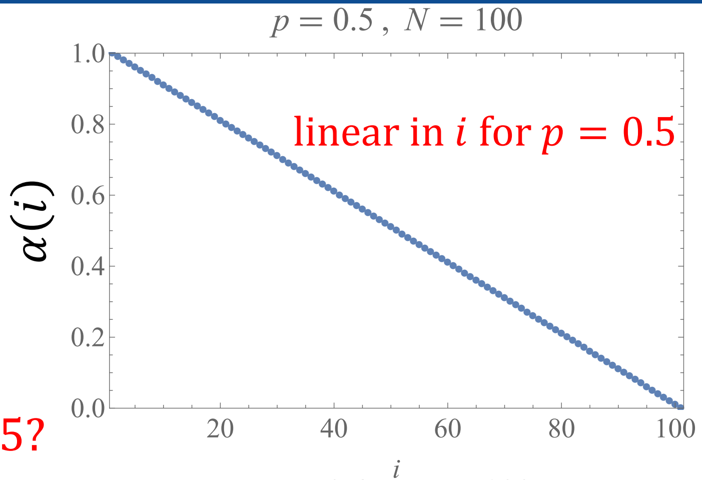
In the 1D RW case, we obtain the following recursion:

$$\alpha(i) = \sum_{j \in S} P_{ij}\alpha(j) = (1-p)\alpha(i-1) + p\alpha(i+1)$$

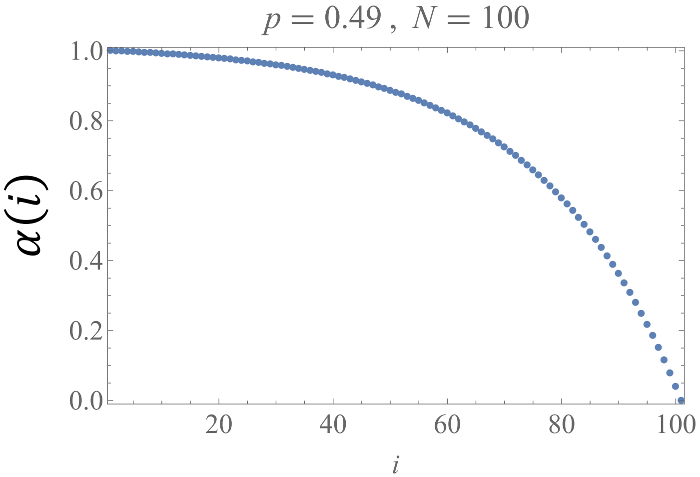
# Hitting Probability

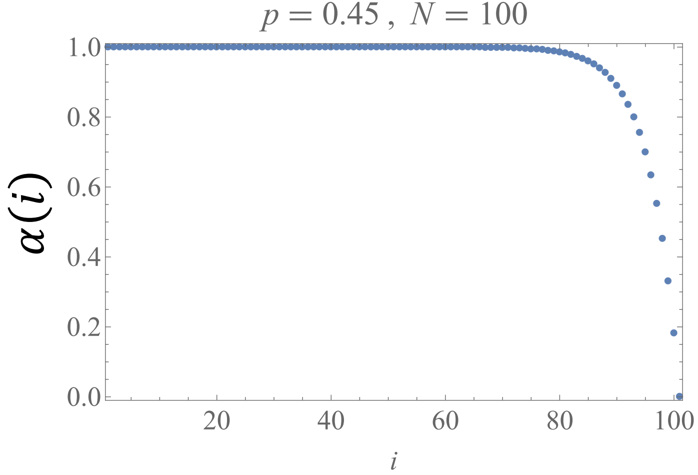
$$\alpha(i) = \begin{cases} \dfrac{N-i}{N}, & \text{if } p = \dfrac{1}{2} \\ \dfrac{r^i - r^N}{1 - r^N}, & \text{if } p \neq \dfrac{1}{2} \end{cases}$$

where $r = \frac{1-p}{p}$




What would happen to $\alpha(i)$ as $p$ deviates from 0.5?

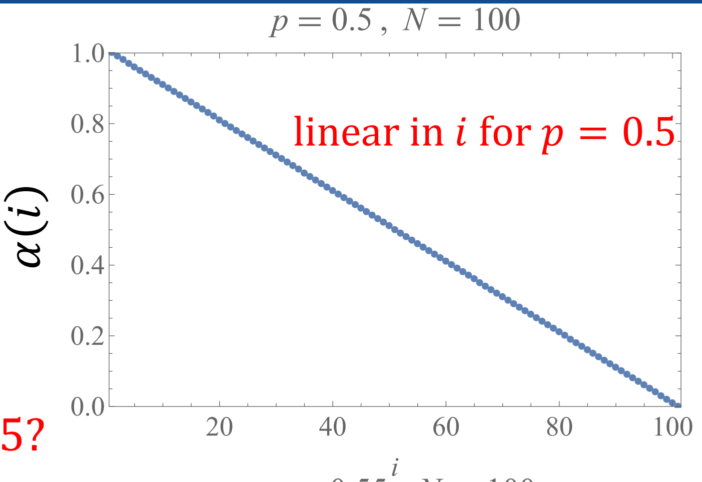


# Hitting Probability

$$\alpha(i) = \begin{cases} \dfrac{N-i}{N}, & \text{if } p = \dfrac{1}{2} \\[2ex] \dfrac{r^i - r^N}{1 - r^N}, & \text{if } p \neq \dfrac{1}{2} \end{cases}$$

where $r = \frac{1-p}{p}$

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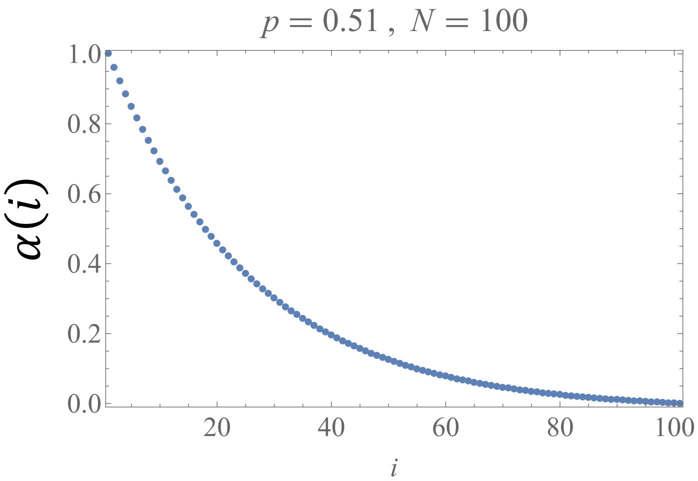


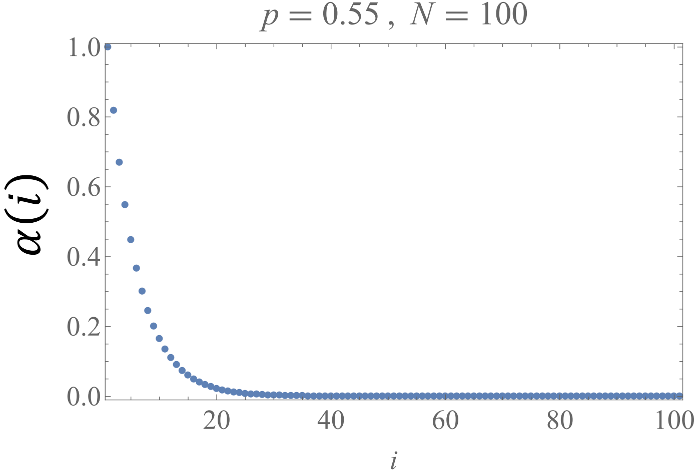

# Hitting Probability

$$\alpha(i) = \begin{cases} \dfrac{N-i}{N}, & \text{if } p = \dfrac{1}{2} \\ \dfrac{r^i - r^N}{1 - r^N}, & \text{if } p \neq \dfrac{1}{2} \end{cases}$$

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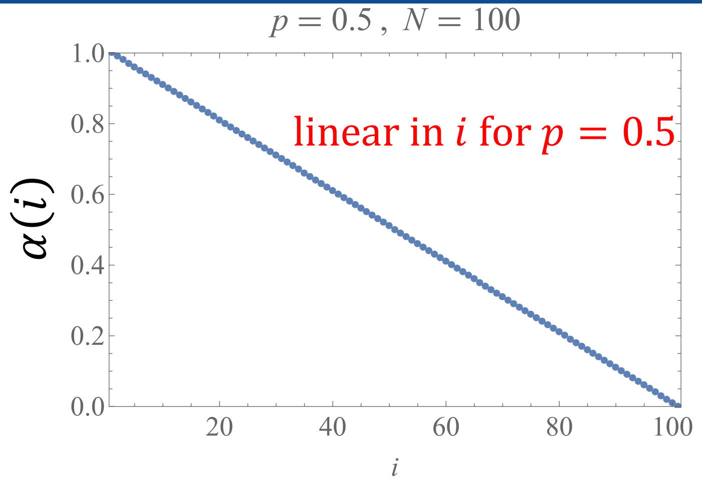
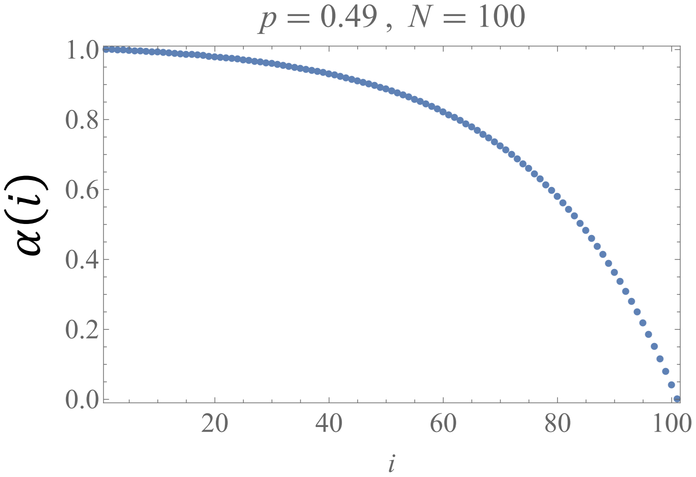
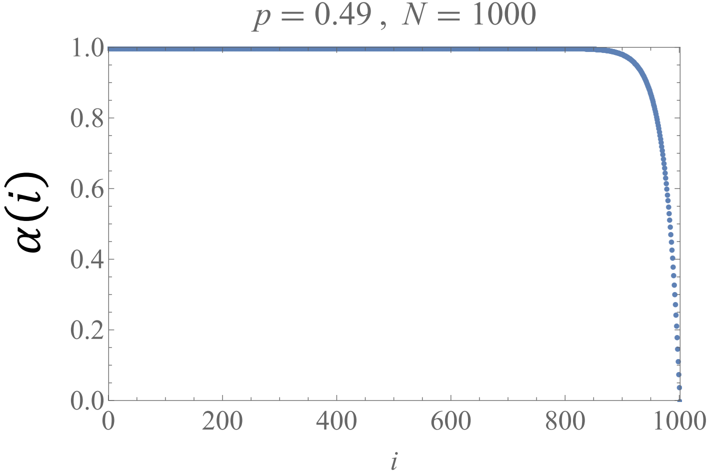



What would happen to $\alpha(i)$ as $N$ increases?



# Expected Hitting Time

**Definition (Conditional Expectation):** For any $b \in \mathcal{B}$ with $\mathbb{P}[Y = b] > 0$, the conditional expectation of $X$ given $Y = b$ is

$$\mathbb{E}[X \mid Y = b] = \sum_{a \in \mathcal{A}} a \ \mathbb{P}(X = a \mid Y = b).$$

(from Lecture 21)

Let $W$ = waiting time until either 0 or $N$ is hit. $\beta(i) = \mathbb{E}[W \mid X_0 = i]$

$\beta(0) = \beta(N) = 0$ (boundary conditions). For $i \neq 0, N$

$$\beta(i) = \sum_{t=1}^{\infty} t\,\mathbb{P}[W = t \mid X_0 = i] = \sum_{t=1}^{\infty} t \sum_{j \in S} \mathbb{P}[W = t \mid X_0 = i, X_1 = j]\,\mathbb{P}[X_1 = j \mid X_0 = i]$$

For 1D RW:

$$= \sum_{t=1}^{\infty} t\left[(1-p)\mathbb{P}[W = t \mid X_0 = i, X_1 = i-1] + p\mathbb{P}[W = t \mid X_0 = i, X_1 = \mathrm{i}+1]\right]$$

$$= (1-p)\mathbb{E}[W \mid X_0 = i, X_1 = i-1] + p\mathbb{E}[W \mid X_0 = i, X_1 = i+1]$$

$1 + \mathbb{E}[W \mid X_0 = i-1]$ $\quad$ $1 + \mathbb{E}[W \mid X_0 = i+1]$

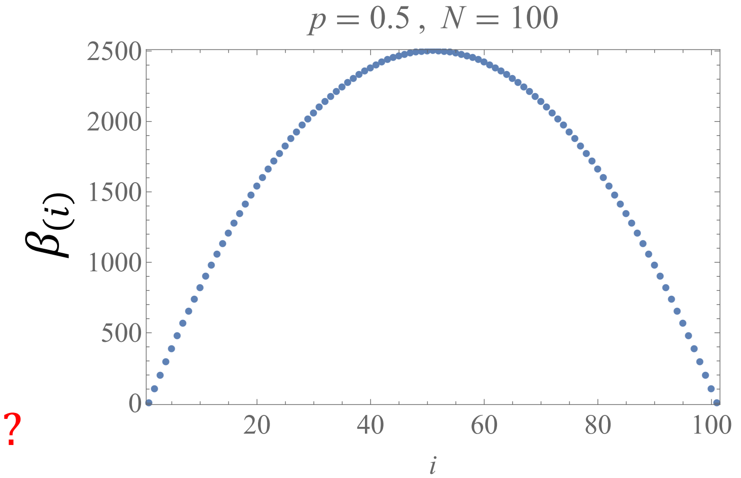
(again by the Markov property and time-homogeneity)

$$= 1 + (1-p)\beta(i-1) + p\beta(i+1)$$

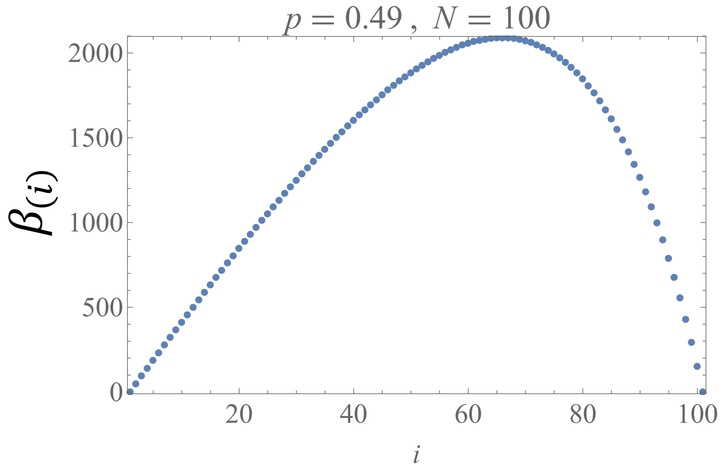
# Expected Hitting Time

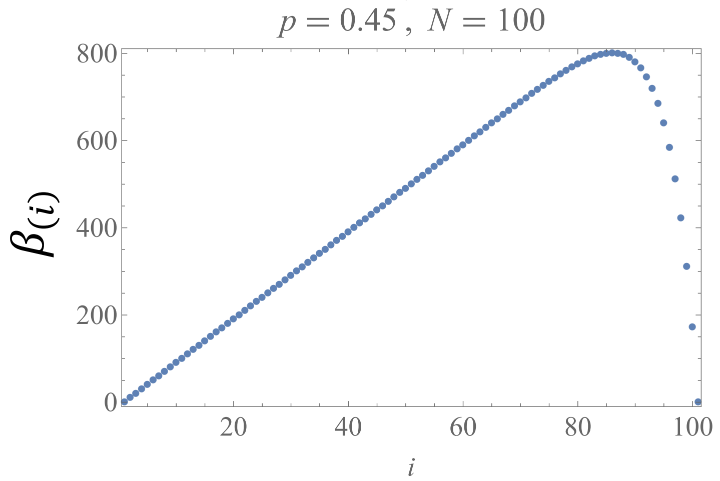
$$\beta(i) = \begin{cases} i(N-i), & \text{if } p = \frac{1}{2} \\ \frac{1}{1-2p}\left[i - N\frac{1-r^i}{1-r^N}\right], & \text{if } p \neq \frac{1}{2} \end{cases}$$

where $r = \frac{1-p}{p}$




What would happen to $\beta(i)$ as $p$ deviates from 0.5?



# Expected Hitting Time

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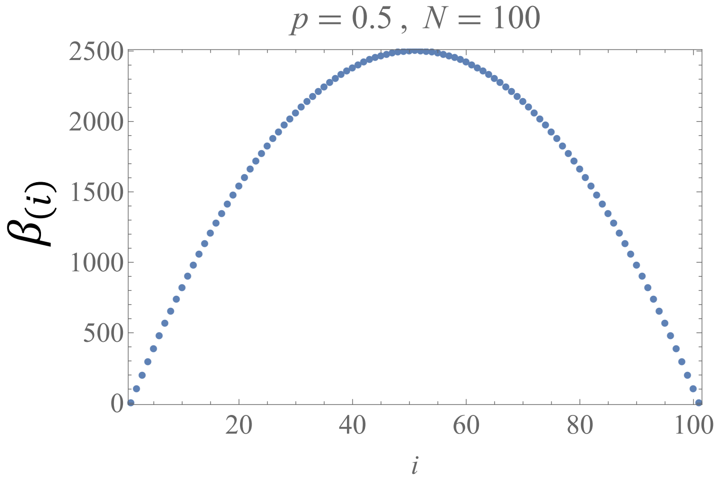
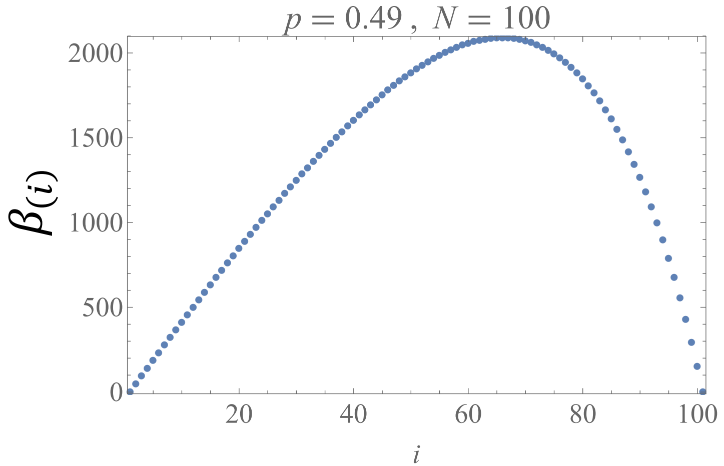
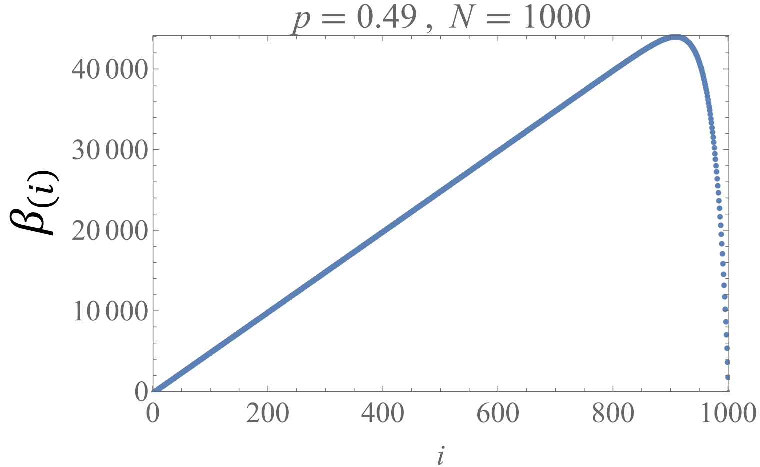



What would happen to $\beta(i)$ as $N$ increases?



# Solving the Hitting Probability Recursion

$\alpha(0) = 1$ and $\alpha(N) = 0$

For $1 \leq i \leq N-1$, $\quad (1-p)\alpha(i-1) + p\alpha(i+1) = \alpha(i) = p\alpha(i) + (1-p)\alpha(i)$

Let $\Delta_k = \alpha(k) - \alpha(k-1)$.

Then, the above equation becomes $p\Delta_{i+1} = (1-p)\Delta_i$

$\Delta_2 = r\Delta_1 \qquad$ where $r = \frac{1-p}{p}$

$\Delta_3 = r\Delta_2 = r^2\Delta_1$

$\vdots$

$\Delta_N = r^{N-1}\Delta_1$

$\Delta_1 + \Delta_2 + \cdots + \Delta_i = \alpha(i) - \alpha(0) = \Delta_1\left(1 + r + r^2 + \cdots + r^{i-1}\right)$

$\Delta_1 + \Delta_2 + \cdots + \Delta_N = \alpha(N) - \alpha(0) = -1$

$\Rightarrow \Delta_1 = -1/(1 + r + r^2 + \cdots + r^{N-1})$

$$\Rightarrow \alpha(i) = 1 - \frac{\left(1 + r + r^2 + \cdots + r^{i-1}\right)}{\left(1 + r + r^2 + \cdots + r^{N-1}\right)} = 1 - \frac{1 - r^i}{1 - r^N} = \frac{r^i - r^N}{1 - r^N}$$

since $(1-x)\left(1 + x + \cdots + x^{k-1}\right) = 1 - x^k$