CS70 – Spring 2026

Lecture 3 : Jan. 27

Previous Lecture

- Proof types:
  - Direct Proof
  - Proof by Contraposition : instead of proving $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$
  - Proof by Contradiction : assuming $\neg P$ leads to a contradiction – so P is true!
  - Proof by Cases : consider an exhaustive set of cases and prove P holds in every case

- Some common pitfalls

- Today : Proof by Induction

Proof by Induction

Goal: Prove $(\forall n \in \mathbb{N})\ P(n)$

E.g. $P(n)$ is: $\sum_{i=0}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)$

Approach:

IND. HYPOTHESIS

BASE CASE — Prove $P(0)$

IND. STEP — For arbitrary $k \geq 0$, prove $P(k) \Rightarrow P(k+1)$

Hence $\forall n \in \mathbb{N}\ P(n)$ holds $\square$

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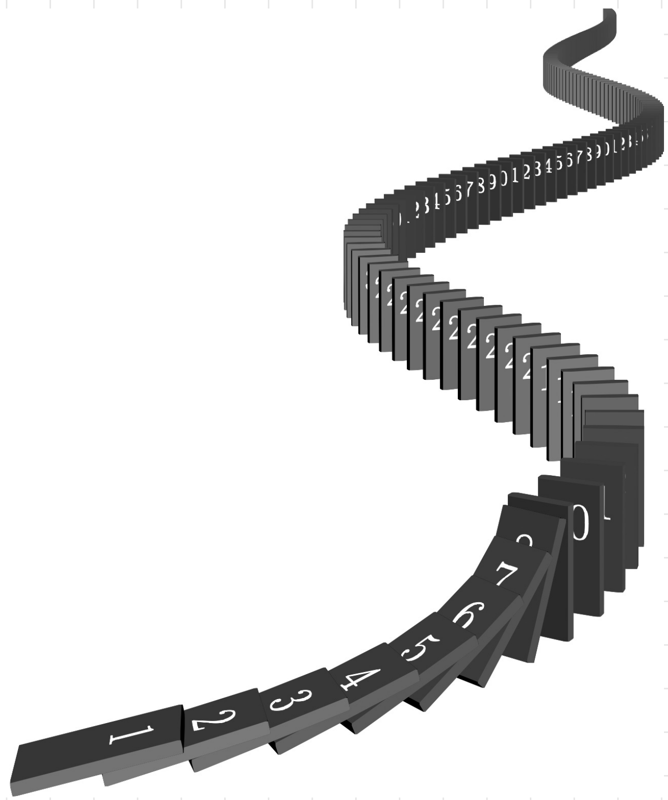


Theorem: $\forall n \in \mathbb{N}$ $\underbrace{\sum_{i=0}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1)}_{P(n)}$

Proof: By induction on n.

Base Case:

Inductive Step: For arbitrary $k \geq 0$, **assume** $P(k)$ and prove $P(k+1)$.

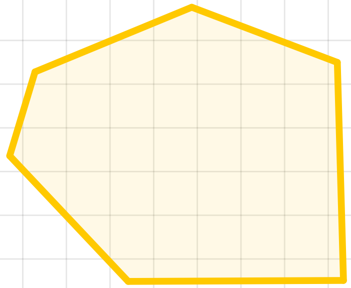
$P(k)$: $\sum_{i=0}^{k} i^2 = \frac{1}{6} k(k+1)(2k+1)$

$P(k+1)$: $\sum_{i=0}^{k+1} i^2 = \frac{1}{6} (k+1)(k+2)(2(k+1)+1)$

Theorem: For all $n \geq 3$, the sum of the interior angles of a convex polygon with n sides is $(n-2)\pi$

$P(n)$

E.g.:



$n = 6$

angle sum $= 4\pi$

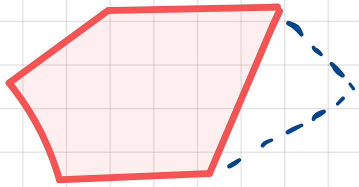
Proof: Induction on n.

Base case:

Inductive Step:

Notes

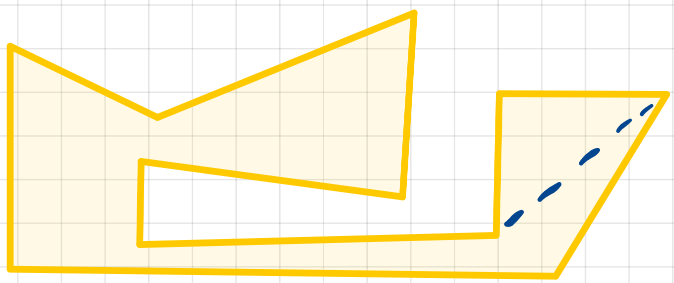
1. Invalid proof ("build-up" vs "break-down")



Start from a k-gon and add a triangle to get a (k+1)-gon

2. Theorem also holds for non-convex polygons

E.g.



... but harder to show you can always cut off a triangle

Strengthening the Induction Hypothesis

Theorem: $(\forall n \geq 1) \quad \sum_{i=1}^{n} \frac{1}{i^2} \leq 2$

Proof: Base case:

Inductive step:

Strengthening the Induction Hypothesis

Theorem: $(\forall n \geq 1) \quad \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$

Proof: **Base case**: $P(1)$: $\frac{1}{1^2} = 1 \leq 2 - \frac{1}{1} = 1$ ✓

Inductive step: $P(k)$: $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$

What about $P(k+1)$?

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

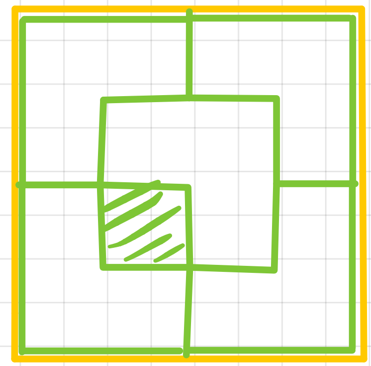
$$\leq 2 - \frac{1}{k+1} \quad ✓$$

algebra

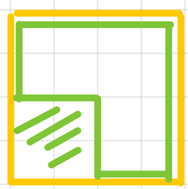
$\square$

Theorem: For any $n \geq 1$, any $2^n \times 2^n$ chessboard can be tiled with tiles, leaving one hole adj. to center

E.g. $n=2$:

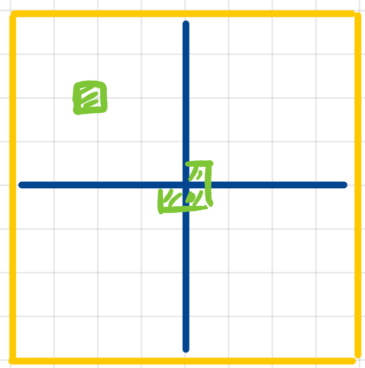


Proof: Base case $P(1)$: 2×2



✓

Inductive Step:



$2^{k+1} \times 2^{k+1}$ board:
divide into 4
$2^k \times 2^k$ boards

- Put the hole wherever you want
- Put three more holes adj. to center
- Tile each of the 4 $2^k \times 2^k$ boards with holes in these positions (assuming $P(k)$)
- Fill in the three holes at center with one tile $\square$

"Strong" Induction

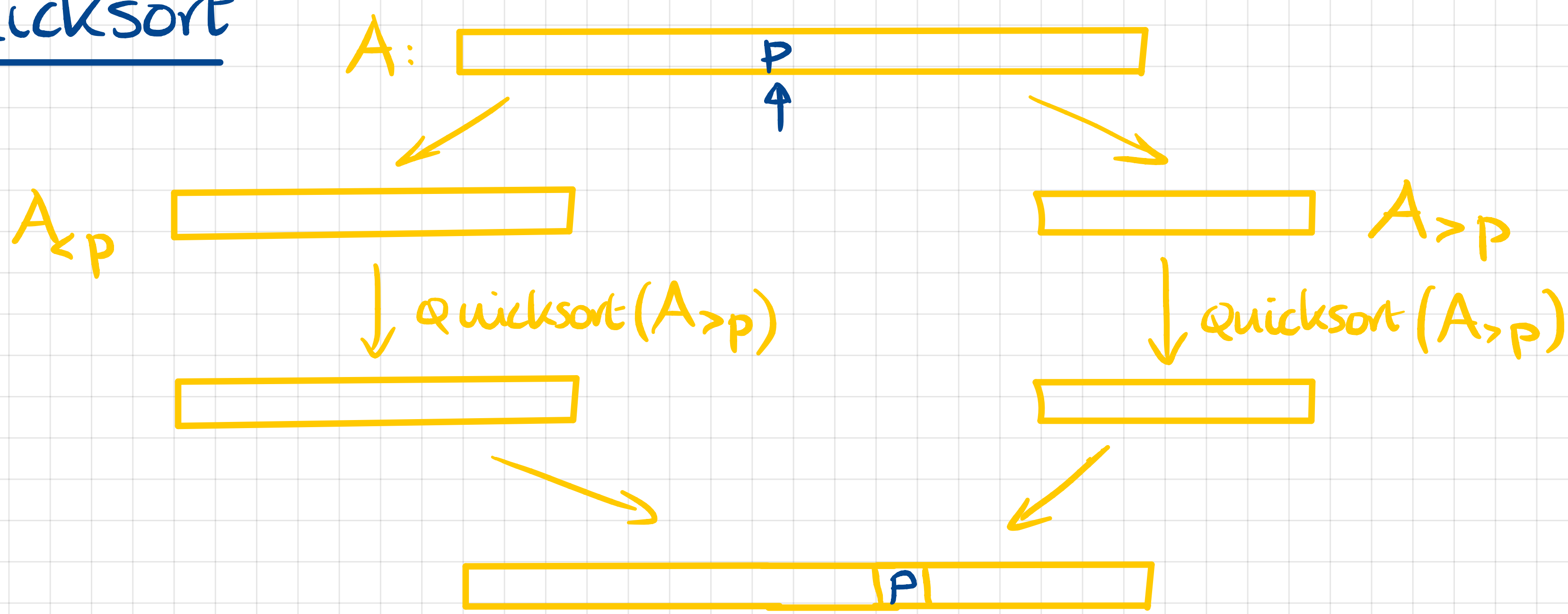
When proving $P(k+1)$, we can assume all of $P(0), P(1), \ldots, P(k)$ (not just $P(k)$)

Theorem [Euclid]: Every integer $n > 1$ can be written as the product of primes

Proof: Base case:

Inductive step:

Quicksort



Theorem: Quicksort (A) correctly sorts array A (of length n)

Proof: Base cases:

Inductive step:

Fibonacci Numbers

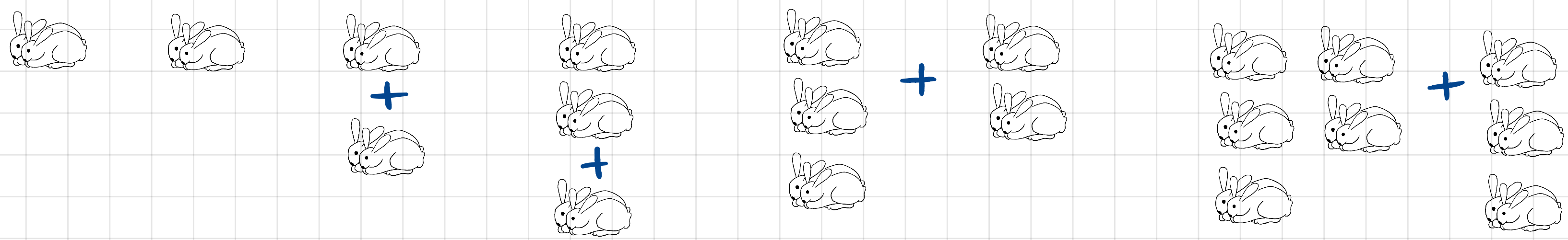
$F(0) = 0$; $F(1) = 1$

$F(n) = F(n-1) + F(n-2) \quad \forall n \geq 2$

Sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Fibonacci's Rabbits:

Time: 0 1 2 3 4 5 6



Theorem: $(\forall n \geq 0)\ F(n) = \dfrac{\varphi^n - \psi^n}{\sqrt{5}}$, where φ, ψ are the roots of $x^2 - x - 1$, i.e.,

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\psi = \frac{1-\sqrt{5}}{2} \approx -0.618$$

Note: $F(n) \approx \frac{1}{\sqrt{5}} \cdot 1.618^n$ for large n [E.g. $F(15) = 610$; $\frac{1.618^{15}}{\sqrt{5}} = 609.81$]

Theorem: $(\forall n \geq 0)\ F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$, where φ, ψ are the roots of $x^2 - x - 1$, i.e., $\varphi, \psi = \frac{1 \pm \sqrt{5}}{2}$

Proof: **Base cases**:

$$n=0: F(0) = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 \checkmark$$

$$n=1: F(1) = \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 \checkmark$$

Inductive step: Assume $F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$ & $F(n-1) = \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}}$

Then
$$F(n+1) = F(n) + F(n-1)$$
$$= \frac{\varphi^n - \psi^n}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}}$$

WHY? $\varphi^2 = \varphi + 1$
$\Rightarrow \varphi^{n+1} = \varphi^n + \varphi^{n-1}$
Same for ψ

$$= \frac{1}{\sqrt{5}} \left[(\varphi^n + \varphi^{n-1}) - (\psi^n + \psi^{n-1}) \right]$$
$$= \frac{1}{\sqrt{5}} \left[\varphi^{n+1} - \psi^{n+1} \right] \checkmark$$

Induction Proof Fails

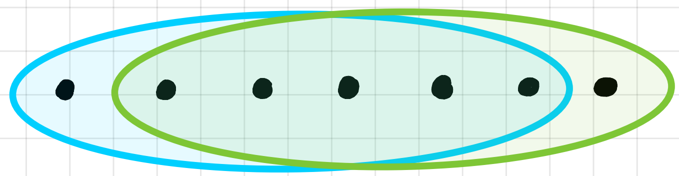
"Theorem": All iPhones are the same color

"Proof": Induction on the number, n, of iPhones

Base case: $n=1$ obvious ✓

Ind. step: Assume any set of k iPhones are same color

Consider a set of $k+1$ iPhones:



Blue set contains k iPhones $\Rightarrow$ all same color

Green " " " " " " " "

And both colors are the same since there's a phone in overlap

Hence $P(k) \Rightarrow P(k+1)$ $\square$

"Theorem": $\forall n \in \mathbb{N} \quad \sum_{i=0}^{n} i^2 = \frac{1}{6}(2n^3 + 3n^2 + n + 5)$

Note: Earlier we proved $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(2n^3+3n^2+n)$!

"Proof": Base case: easy ✓

Inductive Step:

$$\sum_{i=0}^{k+1} i^2 = \sum_{i=0}^{k} i^2 + (k+1)^2$$

$$= \frac{1}{6}(2k^3+3k^2+k+5) + (k+1)^2$$

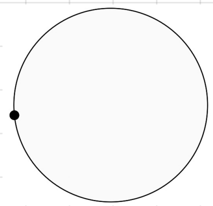
$$= \frac{1}{6}(2k^3+3k^2+k+5+6k^2+12k+6)$$

$$= \frac{1}{6}(2(k+1)^3 + 3(k+1)^2 + (k+1) + 5) \checkmark$$

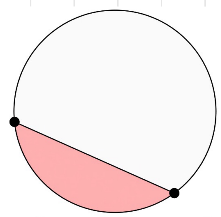
$\square$

Pitfalls of "Pattern Recognition"

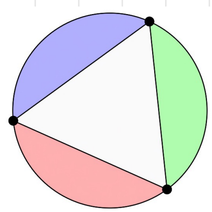
Place n points around a circle (general position) and draw a chord between every pair. How many regions?



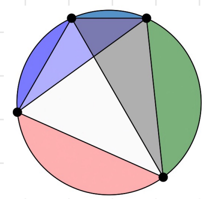
$n = 1$
$R(n) = 1$



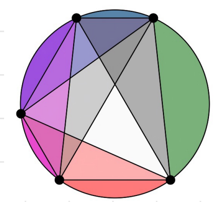
$n = 2$
$R(n) = 2$



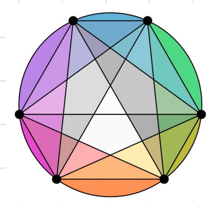
$n = 3$
$R(n) = 4$



$n = 4$
$R(n) = 8$



$n = 5$
$R(n) = 16$

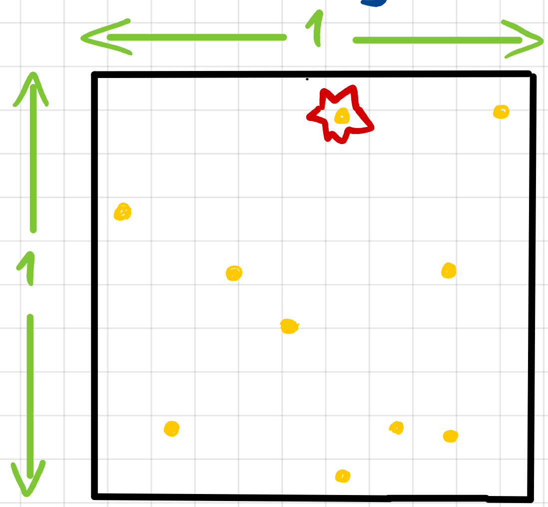


$n = 6$
$R(n) = ??$

"Guess": $R(n) = 2^{n-1}$

"Proof": Chords from each new point divide all previous regions

Fun Problem "Colliding Particles"

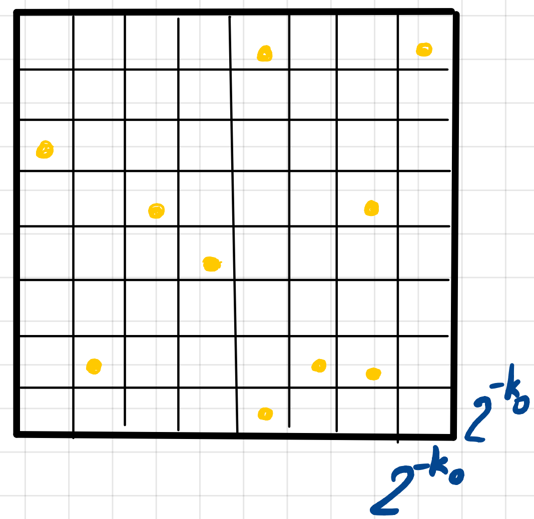


- particles in 1m × 1 m box
- when "awake", particles move at 1m/sec. in any direction we want — and can change direction
- awake particles wake up any particles they hit
- at time 0, there one awake particle

Claim: Can wake up all particles in ≤ 10 seconds

Claim: Can wake up <u>all</u> particles in $\leq$ 10 seconds

<u>Proof</u>: Repeatedly bisect box until ≤ 1 particle per cell:
say k_0 bisections $\rightarrow$ cells are $2^{-k_0} \times 2^{-k_0}$

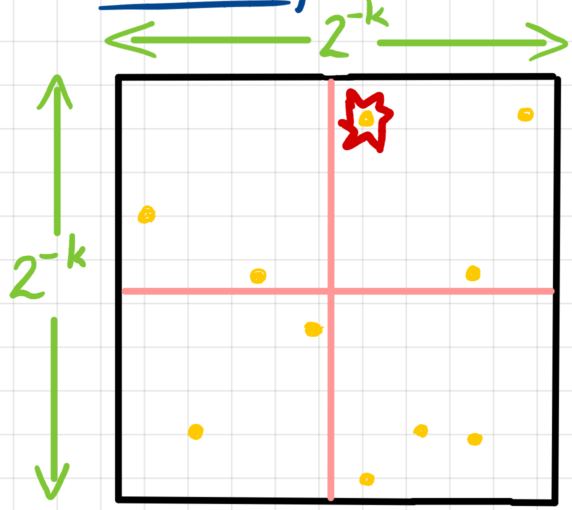


Induction on box dimension:

P(k): Can wake all particles in $2^{-k} \times 2^{-k}$ box in $\leq 10 \cdot 2^{-k}$ secs, starting with 1 awake particle

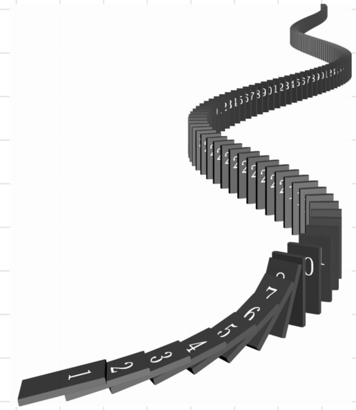
<u>Base</u>: $P(k_0)$ ✓ (takes 0 secs. for 1 cell)

<u>Ind. Step</u>:



- Bisect box into 4 $2^{-(k+1)} \times 2^{-(k+1)}$ boxes
- Awake particle visits one particle per box and wakes them up – takes time $\leq 2\sqrt{5} \times 2^{-k} < 5 \times 2^{-k}$ secs.
- Solve 4 subproblems on $2^{-(k+1)} \times 2^{-(k+1)}$ boxes <u>in parallel</u> $\longrightarrow$ time $\leq 10 \times 2^{-(k+1)} = 5 \times 2^{-k}$
- Total time $\leq (5 \times 2^{-k}) + (5 \times 2^{-k}) = 10 \times 2^{-k}$ ✓

Summary



- Principle of Induction
  - Base case $P(0)$ (or $P(1)$ etc.)
  - Inductive step : $\forall k \; P(k) \Rightarrow P(k+1)$
- Strengthening the hypothesis
- Strong induction : $P(0) \wedge P(1) \wedge \cdots \wedge P(k) \Rightarrow P(k+1)$
- Induction & recursion
- Some common errors

- Next lecture : Stable Matching