

CS70 — SPRING 2026

LECTURE 3 : JAN. 27

Previous Lecture

- Proof types:

- Direct Proof

- Proof by Contraposition :

instead of proving $P \Rightarrow Q$,
prove $\neg Q \Rightarrow \neg P$

- Proof by Contradiction :

assuming $\neg P$ leads to a
contradiction — so P is true!

- Proof by Cases :

consider an exhaustive set of cases
and prove P holds in every case

- Some common pitfalls

- Today : Proof by Induction

Proof by Induction

Goal: Prove $(\forall n \in \mathbb{N}) P(n)$

E.g. $P(n)$ is: $\sum_{i=0}^n i^2 = \frac{1}{6} n(n+1)(2n+1)$

Approach:

BASE
CASE

Prove $P(0)$

IND.
STEP

For arbitrary $k \geq 0$, prove $P(k) \Rightarrow P(k+1)$

Hence $\forall n \in \mathbb{N} P(n)$ holds $\square$

IND. HYPOTHESIS



$$P(0) \Rightarrow P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \dots$$

Approach :

BASE
CASE

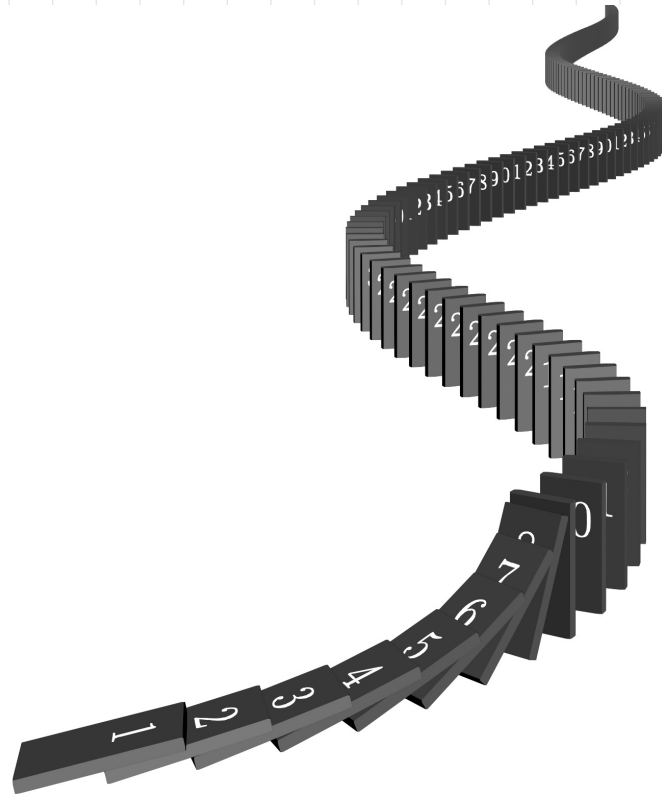
Prove $P(0)$

IND.
STEP

For arbitrary $k \geq 0$, prove $P(k) \Rightarrow P(k+1)$

Hence $\forall n \in \mathbb{N}$ $P(n)$ holds $\square$

IND. HYPOTHESIS



Theorem: $\forall n \in \mathbb{N} \quad \underbrace{\sum_{i=0}^n i^2 = \frac{1}{6} n(n+1)(2n+1)}_{P(n)}$

Proof: By induction on n .

Base Case: $P(0)$: $\sum_{i=0}^0 i^2 = \frac{1}{6} \cdot 0(0+1)(2 \cdot 0+1)$ i.e. $0=0$ ✓

Inductive Step: For arbitrary $k \geq 0$, assume $P(k)$ and prove $P(k+1)$.

IND.
HYP.

$$P(k): \sum_{i=0}^k i^2 = \frac{1}{6} k(k+1)(2k+1)$$



$$P(k+1): \sum_{i=0}^{k+1} i^2 = \frac{1}{6} (k+1)(k+2)(2(k+1)+1)$$

$$\sum_{i=0}^{k+1} i^2 = \sum_{i=0}^k i^2 + (k+1)^2 = \frac{1}{6} k(k+1)(2k+1) + (k+1)^2$$

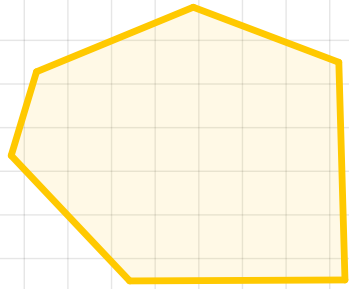
by Ind. Hypothesis $P(k)$

$$= \frac{1}{6} (k+1) [k(2k+1) + 6(k+1)]$$

$$= \frac{1}{6} (k+1)(k+2)(2k+3) \quad \checkmark P(k+1)$$

Theorem: For all $n \geq 3$, the sum of the interior angles of a convex polygon with n sides is $(n-2)\pi$

E.g.:



$$n = 6$$

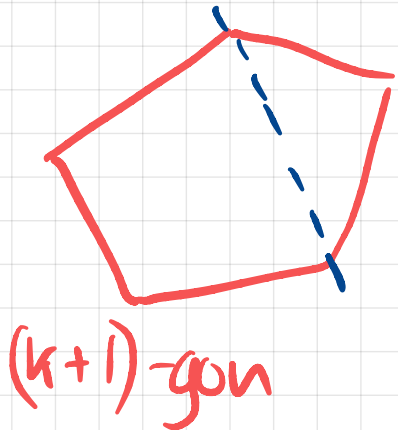
$$\text{angle sum} = 4\pi$$

$\nwarrow$
 $P(n)$

Proof: Induction on n .

Base case: $P(3)$:  triangle Angle sum of $\Delta = \pi$

Inductive Step: $P(k) \Rightarrow P(k+1)$



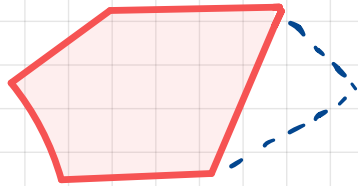
"Chop off" a Δ as shown — this gives a k -gon + a triangle

So angle-sum of $(k+1)$ -gon = angle-sum of k -gon + angle-sum of triangle

$$= (k-2)\pi \text{ by } P(k) + \pi = (k-1)\pi = (k+1-2)\pi$$

Notes

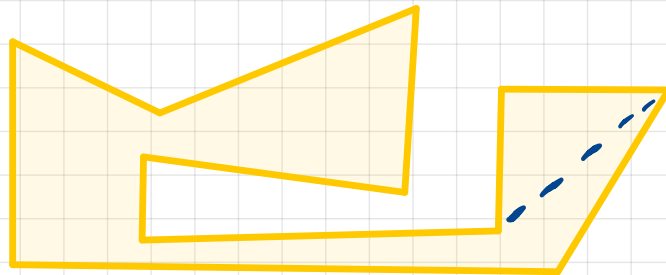
1. Invalid proof ("build-up" vs "break-down")



Start from a k -gon and add a triangle to get a $(k+1)$ -gon

2. Theorem also holds for non-convex polygons

E.g.



... but harder to show you can always cut off a triangle

Strengthening the Induction Hypothesis

Theorem: $(\forall n \geq 1) \sum_{i=1}^n \frac{1}{i^2} \leq 2$ $P(n)$

Proof: Base case: $P(1)$: $\sum_{i=1}^1 \frac{1}{i^2} \stackrel{?}{\leq} 2$
 $\rightarrow = 1 \leq 2$ ✓

Inductive step:

$$\forall k \geq 1, P(k) \Rightarrow P(k+1)$$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^k \frac{1}{i^2} + \frac{1}{(k+1)^2}$$

$$\leq 2 + \frac{1}{(k+1)^2}$$

by Ind. Hyp.
 $P(k)$

?

can't proceed
from here as r.h.s.
already exceeds 2!

Strengthening the Induction Hypothesis

Theorem: $(\forall n \geq 1) \sum_{i=1}^n \frac{1}{i^2} \leq 2 - \frac{1}{n}$ $P(n)$

Proof: Base case: $P(1)$: $\frac{1}{1^2} = 1 \leq 2 - \frac{1}{1} = 1$ ✓

Inductive step: $P(k)$: $\sum_{i=1}^k \frac{1}{i^2} \leq 2 - \frac{1}{k}$

What about $P(k+1)$?

$$\sum_{i=1}^{k+1} \frac{1}{i^2} = \sum_{i=1}^k \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\leq 2 - \frac{1}{k+1} \quad \checkmark$$

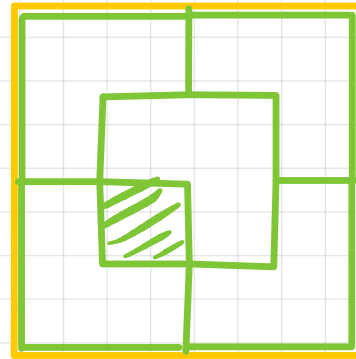
algebra

by ind.
hyp. $P(k)$



Theorem: For any $n \geq 1$, any $2^n \times 2^n$ chessboard can be tiled with $\square$ tiles, leaving one hole ^{in any position} ~~adj. to center~~

E.g. $n=2$:

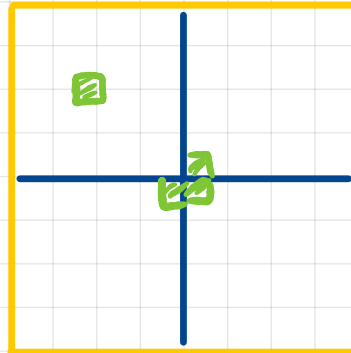


↑
strengthening
of hypothesis

Proof: Base case $P(1): 2 \times 2$



Inductive Step:



$2^{k+1} \times 2^{k+1}$ board:
divide into 4
 $2^k \times 2^k$ boards

assuming
 $P(k)$



- Put the hole wherever you want
- Put three more holes adj. to center
- Tile each of the 4 $2^k \times 2^k$ boards with holes in these positions
- Fill in the three holes at center with one $\square$ tile 

"Strong" Induction

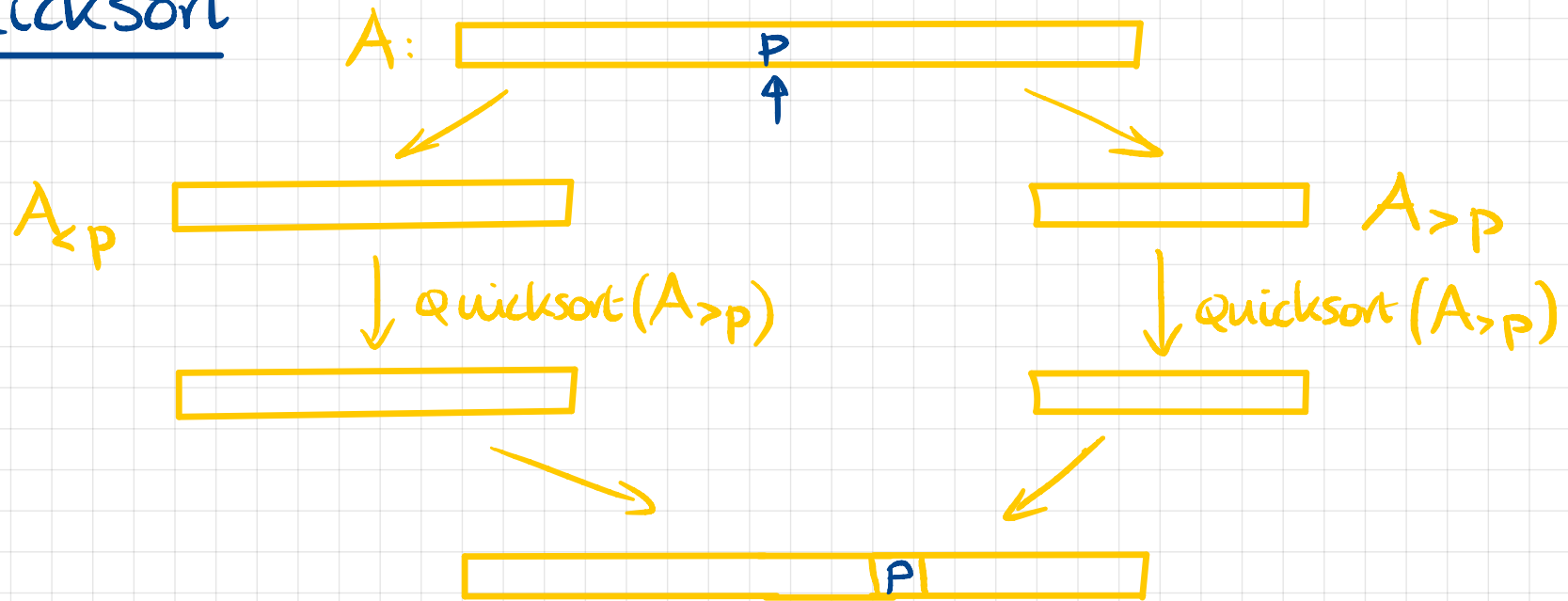
When proving $P(k+1)$, we can assume all of $P(0), P(1), \dots, P(k)$
(not just $P(k)$)

Theorem [Euclid]: Every integer $n > 1$ can be written as the product of primes

Proof: Base case: see Note 3

Inductive step: see Note 3

Quicksort



Theorem: Quicksort(A) correctly sorts array A (of length n)

Proof: Base cases: $n=0$ or 1 — do nothing ✓

Inductive step: $n = k+1$ for $k \geq 1$

Then $A_{<p}$ and $A_{>p}$ have lengths in $\{0, 1, \dots, k\}$

By strong induction hypothesis, recursive calls $\text{Quicksort}(A_{<p})$ & $\text{Quicksort}(A_{>p})$ return correctly sorted arrays

Hence final array A is correctly sorted $\square$

Fibonacci Numbers

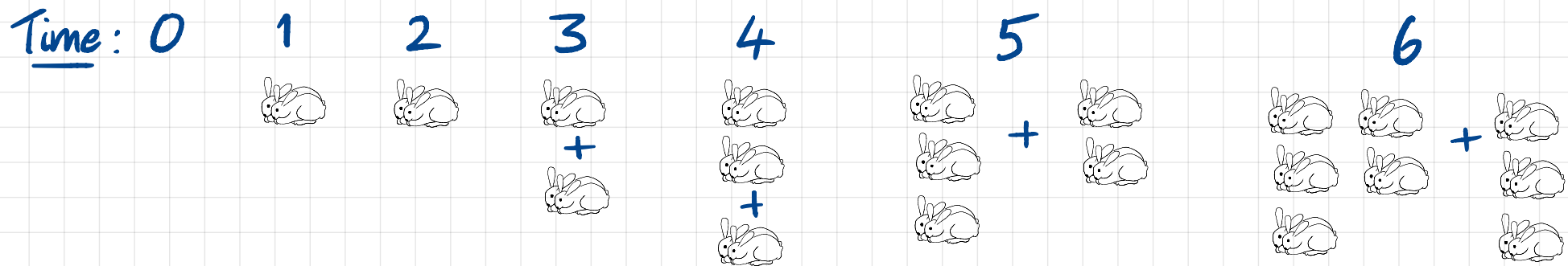
$$F(0) = 0; F(1) = 1$$

$$F(n) = F(n-1) + F(n-2) \quad \forall n \geq 2$$



Sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Fibonacci's Rabbits:



Theorem: ($\forall n \geq 0$) $F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$, where φ, ψ are the roots of $x^2 - x - 1$, i.e.,

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$$

$$\psi = \frac{1-\sqrt{5}}{2} \approx -0.618$$

Note: $F(n) \approx \frac{1}{\sqrt{5}} \cdot 1.618^n$ for large n [E.g. $F(15) = 610$; $\frac{1.618^{15}}{\sqrt{5}} = 609.81$]

Theorem : $(\forall n \geq 0) F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$, where φ, ψ are the roots of $x^2 - x - 1$, i.e., $\varphi, \psi = \frac{1 \pm \sqrt{5}}{2}$

Proof : Base cases : $n=0: F(0) = \frac{\varphi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 \checkmark$
 $n=1: F(1) = \frac{\varphi^1 - \psi^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 \checkmark$

Inductive step : Assume $F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}$ & $F(n-1) = \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}}$

$$\begin{aligned} \text{Then } F(n+1) &= F(n) + F(n-1) \\ &= \frac{\varphi^n - \psi^n}{\sqrt{5}} + \frac{\varphi^{n-1} - \psi^{n-1}}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}} [\varphi^n + \varphi^{n-1} - (\psi^n + \psi^{n-1})] \end{aligned}$$

WHY? $\varphi^2 = \varphi + 1$
 $\Rightarrow \varphi^{n+1} = \varphi^n + \varphi^{n-1}$
Same for ψ

$$= \frac{1}{\sqrt{5}} [\varphi^{n+1} - \psi^{n+1}] \checkmark$$

Note: Any function of the form

$$F(n) = A\varphi^n + B\gamma^n$$

satisfies the Inductive Step (because $\varphi^2 = \varphi + 1$
 $\gamma^2 = \gamma + 1$)

The coefficients A, B are determined by the base cases:

$$F(0) = A + B = 0 \quad \textcircled{1}$$

$$F(1) = A\varphi + B\gamma = 1 \quad \textcircled{2}$$

From ①, $B = -A$

$$\text{From ②, } A(\varphi - \gamma) = 1 \Rightarrow \sqrt{5}A = 1 \Rightarrow A = \frac{1}{\sqrt{5}}$$

$$\text{So } F(n) = \frac{1}{\sqrt{5}} \varphi^n - \frac{1}{\sqrt{5}} \gamma^n$$

Induction Proof Fails

"Theorem": All iPhones are the same color

"Proof": Induction on the number, n , of iPhones

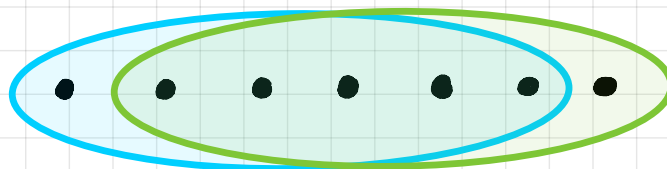
Base case: $n=1$ obvious ✓

Ind. step: Assume any set of k iPhones are same color

Consider a set of $k+1$ iPhones:



$k=1$
No overlap!



$k+1=7$
 $k=6$

Blue set contains k iPhones $\Rightarrow$ all same color
Green " " " " " " " " " " " "

And both colors are the same since there's
a phone in overlap

Hence $P(k) \Rightarrow P(k+1)$



"Theorem": $\forall n \in \mathbb{N} \quad \sum_{i=0}^n i^2 = \frac{1}{6}(2n^3 + 3n^2 + n + 5)$

Note: Earlier we proved $\frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}(2n^3 + 3n^2 + n)$!_o

"Proof": Base case: easy ✓

← ALERT! Actually base case fails!

Inductive Step:

$$\sum_{i=0}^{k+1} i^2 = \sum_{i=0}^k i^2 + (k+1)^2$$

Inductive Step
is correct!

$$= \frac{1}{6}(2k^3 + 3k^2 + k + 5) + (k+1)^2$$

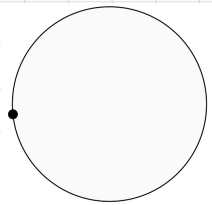
$$= \frac{1}{6}(2k^3 + 3k^2 + k + 5 + 6k^2 + 12k + 6)$$

$$= \frac{1}{6}(2(k+1)^3 + 3(k+1)^2 + (k+1) + 5) \quad \checkmark$$



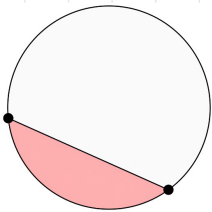
Pitfalls of "Pattern Recognition"

Place n points around a circle (general position) and draw a chord between every pair. How many regions?



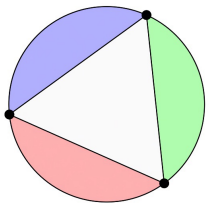
$$n = 1$$

$$R(n) = 1$$



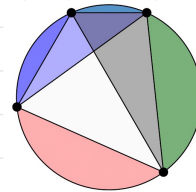
$$n = 2$$

$$R(n) = 2$$



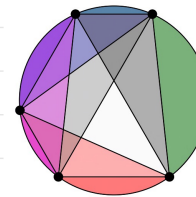
$$n = 3$$

$$R(n) = 4$$



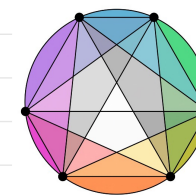
$$n = 4$$

$$R(n) = 8$$



$$n = 5$$

$$R(n) = 16$$



$$n = 6$$

$$R(n) = ??$$

$$32 ?$$

31

"Guess": $R(n) = 2^{n-1}$

$$R(6) = 31$$

$$R(7) = 57$$

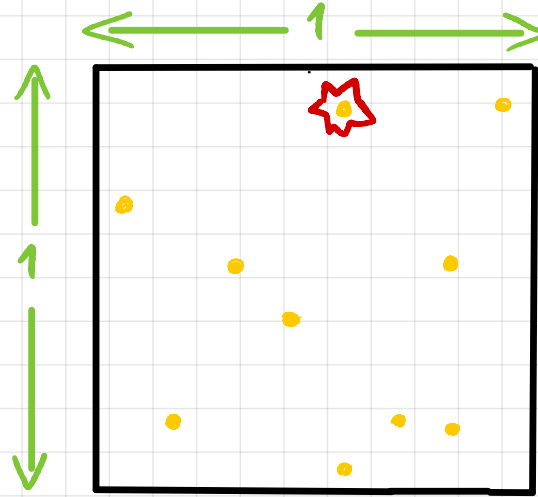
"Proof": Chords from each new point divide all previous regions

$$R(n) = \binom{n}{4} + \binom{n}{2} + 1$$

← happens to coincide with 2^{n-1} for $n \leq 5$

Fun Problem

"Colliding Particles"



- particles in $1\text{m} \times 1\text{m}$ box
- when "awake", particles move at 1m/sec. in any direction we want — and can change direction
- awake particles wake up any particles they hit
- at time 0, there one awake particle

Claim: Can wake up all particles in ≤ 10 seconds

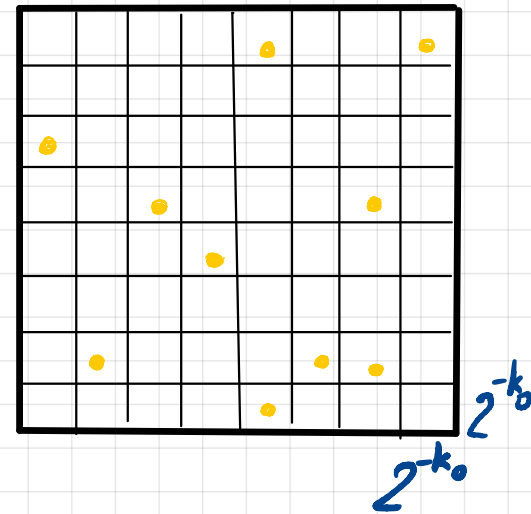
Claim: Can wake up all particles in ≤ 10 seconds

Proof: Repeatedly bisect box until ≤ 1 particle per cell:
say k_0 bisections $\rightarrow$ cells are $2^{-k_0} \times 2^{-k_0}$

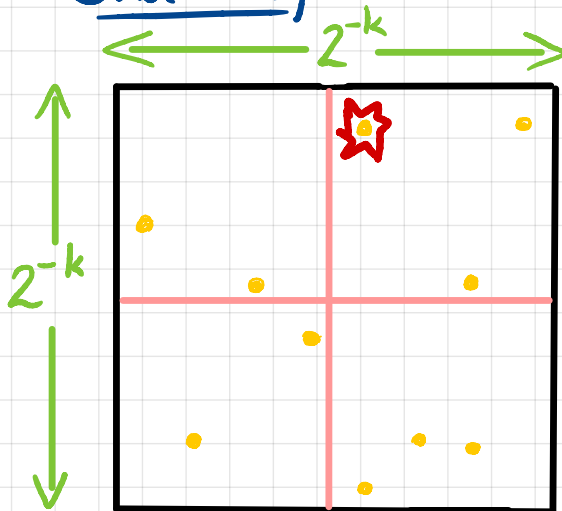
Induction on box dimension:

$P(k)$: Can wake all particles in $2^{-k} \times 2^{-k}$ box
in $\leq 10 \cdot 2^{-k}$ secs, starting with 1 awake particle

Base: $P(k_0)$ $\checkmark$ (takes 0 secs. for 1 cell)

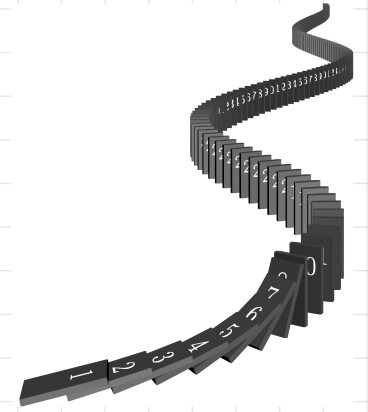


Ind. Step:



- Bisect box into 4 $2^{-(k+1)} \times 2^{-(k+1)}$ boxes
- Awake particle visits one particle per box and wakes them up — takes time $\leq 2\sqrt{5} \times 2^{-k} < 5 \times 2^{-k}$ secs.
- Solve 4 subproblems on $2^{-(k+1)} \times 2^{-(k+1)}$ boxes in parallel $\rightarrow$ time $\leq 10 \times 2^{-(k+1)} = 5 \times 2^{-k}$
- Total time $\leq (5 \times 2^{-k}) + (5 \times 2^{-k}) = 10 \times 2^{-k} \checkmark$

Summary



- Principle of Induction
 - Base case $P(0)$ (or $P(1)$ etc.)
 - Inductive step : $\forall k \ P(k) \Rightarrow P(k+1)$
- Strengthening the hypothesis
- Strong induction : $P(0) \wedge P(1) \wedge \dots \wedge P(k) \Rightarrow P(k+1)$
- Induction & recursion
- Some common errors
- Next lecture : Stable Matching