CS70 – Spring 2026

Lecture 4: Jan. 29

Summary

- Proof techniques
  - Direct Proof
  - Proof by Contraposition
  - Proof by Contradiction
  - Proof by Cases
  - Proof by Induction

- This lecture: Application to Stable Matching

Stable Matching Problem

Input: n jobs, n candidates

for each job, a ranked list of all candidates
- - - - candidate, - - - - - - - - - jobs

} "preference lists"

Goal: match up jobs & candidates in a "good" way

Motivation for CS70

"Killer app" for proofs!

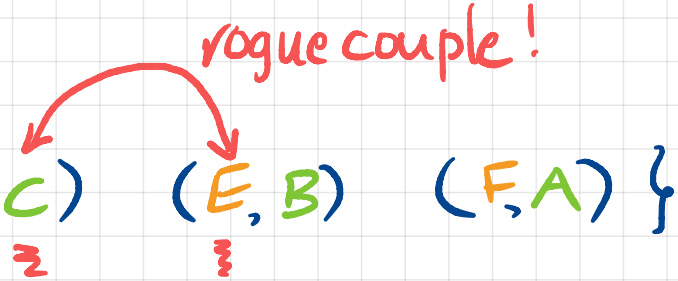
Example

Databricks :	A	C	B
Ericsson :	C	A	B
FedEx :	A	C	B

Annie :	E	F	D
Bertha :	F	E	D
Caro :	F	E	D

$\{(D,B), (E,C), (F,A)\}$

stable

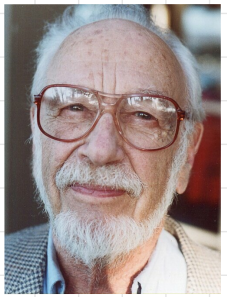


Defn : A rogue couple in a matching is a job & candidate who prefer each other to their current partners

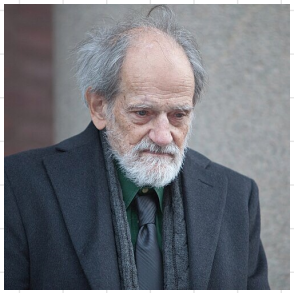
Defn : A matching is stable if it has no rogue couples

National Residency Matching Program (NRMP)

- matches residency applicants with hospitals
- chaotic system until 1950
- early 1950s: switch to centralized algorithm, then to stable matching
- formalized by Gale & Shapley in 1962
- crowning achievement in combinatorial algorithms
- CS70: "killer app" for proofs



David Gale



Lloyd Shapley

Do stable matchings always exist?

Roommates Problem:

Input: 2n people, each with a pref. list for all 2n-1 others

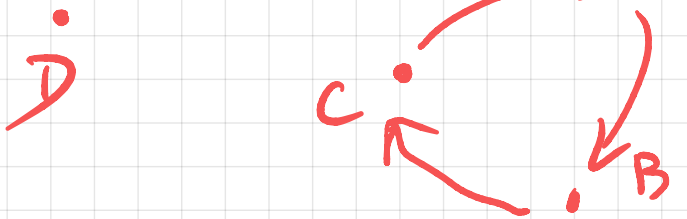
Goal: find a stable matching

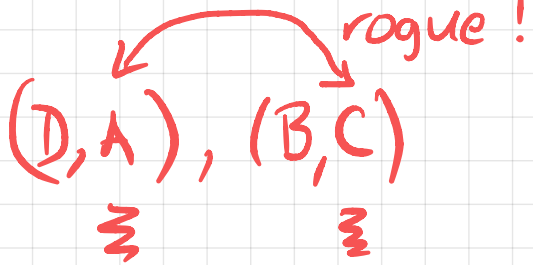
Example:

Annie : B C D
Bertha : C A D
Caro : A B D
Daphne : <whatever>

Claim: There is no stable matching for this input!

Proof:





Possible algorithm :

while a rogue couple (J, C) exists
switch the partners of J & C

Problem : may create new rogue couples*

Note : If above algorithm works for stable matching then it should also work for roommates !

So : Look for an algorithm that exploits asymmetry

Note : Roth/Vande Vate : ∃ carefully chosen seq. of switches that works for stable matching (but complicated & hard to describe)

The Propose-and-Reject Algorithm

REPEAT

1 "DAY":

1. Each job makes an offer to the highest ranked candidate on its list who hasn't rejected it
2. Each candidate says "maybe" to the job she likes best among her offers, and rejects the others (puts offer "on hold")
3. If a job was rejected, it crosses that candidate off its list

UNTIL no rejections occurred

OUTPUT resulting matching

Q1: Does this algorithm terminate?

Q2: Does it always output a matching?

Q3: Is that matching always stable?

Example

Databricks : ~~A~~ ~~C~~ B
Ericsson : C A B
FedEx : A C B

Annie : E F D
Bertha : F E D
Caro : F E D

	Day 1	Day 2	Day 3
Annie	~~D~~, F	F	F
Bertha			D
Caro	E	~~D~~, E	E

output :
{(A, F), (B, D), (C, E)}

Another Example

Jobs

1 : A B C D
2 : B D A C
3 : B A D C
4 : A B C D

Candidates

A : 2 1 3 4
B : 1 4 2 3
C : 2 3 1 4
D : 4 1 2 3

Day 1 Day 2 Day 3 Day 4

A
B
C
D

Exercise: Run the algorithm on this input & check it outputs the pairing $\{(A,1)\ (B,4)\ (C,3)\ (D,2)\}$

Check also that this pairing is stable

Analysis of P-&-R Algorithm

Theorem: The algorithm always terminates

Proof: On every day when alg. doesn't terminate, at least one candidate is crossed off a job's list.

Total length of all lists $= n^2$

$\Rightarrow$ alg. terminates after $\leq n^2$ iterations/days

□

Improvement Lemma

Suppose job J offers to candidate C on day k.
Then on every day $i \geq k$, C has on hold an offer from a job she likes at least as much as J.

Proof: Induction on i

Base case: $i = k$ – true by defn. of algorithm

Ind. step: Assume for some $i \geq k$ & prove for $i+1$

On day i, C has on hold an offer from job $J' \geq J$

On day $i+1$, J' again offers to C (along, possibly, with others)

So after day $i+1$ C has offer from $J'' \geq J' \geq J$

□

Theorem: The algorithm always outputs a matching

Proof: By contradiction

Suppose for ※ that some job J is not matched at end of algorithm

Then J must have made an offer to every candidate — and got rejected

By Improvement Lemma, every candidate therefore has an offer at termination

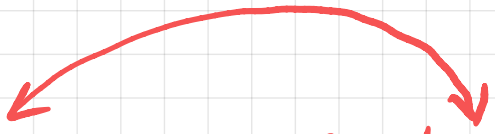
So n candidates have offers from n-1 jobs

※

Theorem: The output matching is always stable

Proof: Direct proof.

Sp. output matching is:



$$M = \{ \ldots (J, C), \ldots, (J', C'), \ldots \}$$

and J likes C' more than C.

By defn. of algorithm, J proposed to C' before proposing to C — and got rejected

By Improvement Lemma, C' must like J' more than J

Hence J, C' is NOT a rogue couple

Since J & C' were arbitrary, $\nexists$ any rogue couple

□

Example

Databricks	:	A	C	B
Ericsson	:	C	A	B
FedEx	:	A	C	B
Annie	:	E	F	D
Bertha	:	F	E	D
Caro	:	F	E	D

Matching output by alg:

$\{(A,F), (B,D), (C,E)\}$

Only other stable matching:

$\{(A,E), (B,D), (C,F)\}$

Q: What's special about the output matching?

A: Both E and F get their top choice candidate, so couldn't do any better. D gets last choice — but in fact must be paired with B in <u>every</u> stable matching!
Hence <u>all</u> jobs do as well as they possibly can!

Definition: For a job J, the optimal candidate for J is the best candidate J has is any stable matching

Definition: The job-optimal matching is the one in which each job is matched with its optimal candidate

ALERT: Is this even a valid matching?

(How do we know that two jobs don't have the same optimal candidate?)

NOTE: In previous example, output matching is job optimal.

Theorem: The matching output by the algorithm is job-optimal

Proof: Enough to prove $\forall k \geq 0$ $P(k)$, where

> $P(k)$: On day k, no job gets rejected by its opt. candidate

Proof by induction on k

Base case: $k = 0$ ✓ (nothing to prove)

Inductive step: Sp no such rejections through day k — $P(k)$

$P(k) \Rightarrow P(k+1)$ Day $k+1$: Sp. for ✱ that job J gets rejected by its opt. candidate C^* in favor of some job J^*

Since C^* is opt. for J, $\exists$ stable matching

$$M = \{\dots (J, C^*) \dots (J^*, C) \dots\}$$

Claim: J^*, C^* are a rogue couple

Inductive step: Sp no such rejections through day k

Day k+1: Sp. for ✱ that job J gets rejected by its opt. candidate C^* in favor of some job J^*

Since C^* is opt. for J, ∃ stable matching

$$M = \{\ldots (J, C^*) \ldots (J^*, C) \ldots\}$$

Claim: J^*, C^* are a rogue couple

Proof of Claim: Certainly C^* prefers J^* to J ✓

What about J^*?

By ind. hypothesis, J^* has not yet been rejected by its optimal candidate

Hence, since J^* is offering to C^* now, J^* must like C^* at least as much as its opt. candidate, hence at least as much as C (partner in stable M) ✓

Thus J^*, C^* are a rogue couple □

Theorem: The job-optimal matching is candidate-pessimal

Proof: By contradiction

Let $\mathcal{M} = \{\ldots (J, C) \ldots\}$ be job-optimal

Sp. for ※ that $\mathcal{M}$ is not candidate-pessimal

Then ∃ stable matching

$$\mathcal{M}' = \{\ldots (J, C') \ldots (J^*, C) \ldots\}$$

s.t. C prefers J to J^*

Claim: J, C is a rogue couple in $\mathcal{M}'$

Proof of Claim: C prefers J to J^* by assn.

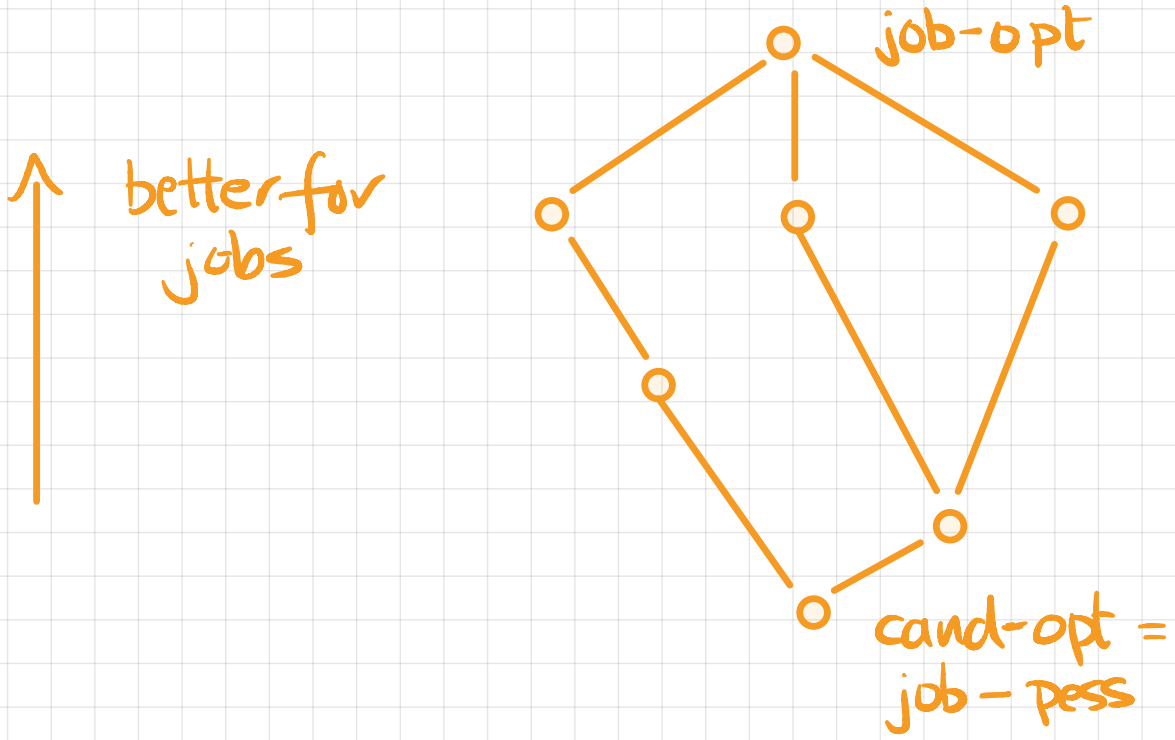
Also, J prefers C to C' since $\mathcal{M}$ is job-optimal

Hence $\mathcal{M}'$ is not stable ※ □

Q: How many stable matchings can there be?

A: Between 1 and exponentially many (in n)

Fact: Stable matchings form a lattice (partial order)



$M \preceq M'$ iff every job does at least as well in M' as in M

Exercise: Come up with a simple instance of Stable Matching that has only one stable matching!

Another Example

Jobs

1 : A C D B
2 : B A C D
3 : C D B A
4 : D B A C

Candidates

A : 3 4 2 1
B : 1 3 4 2
C : 4 2 1 3
D : 2 1 3 4

Job-optimal: $\{(1,A),(2,B),(3,C),(4,D)\}$

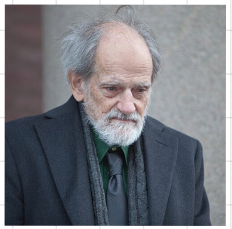
Cand.-optimal: $\{(1,B),(2,D),(3,A),(4,C)\}$

Also stable : $\{(1,C),(2,A),(3,D),(4,B)\}$

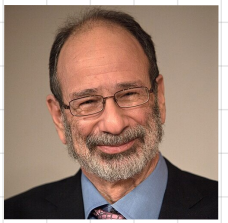
Ex: Check this is stable
Find another stable matching
How do these four matchings fit into the lattice?

N.R.M.P. Revisited

- Originally (1950s) hospital-optimal
- Switched to candidate-optimal in 1990s
- various bells-and-whistles, e.g.
  - allows incomplete pref. lists
  - allows couples to apply jointly
- used in many other contexts, e.g.
  - organ donor matching
  - school districts
  - internet content delivery
- Nobel Prize in Economics 2012 (Shapley-Roth)



Lloyd Shapley



Alvin Roth

Summary

- Stable matching: an important real-world problem
- Not obvious that a stable matching always exists!
- We proved that a stable matching exists by designing & analyzing a clever algorithm
- Also: notions of optimality among stable matchings

- Next week: Graphs!