CS70 – SPRING 2026

LECTURE 5 : FEB. 3

Today: Graphs

Goals

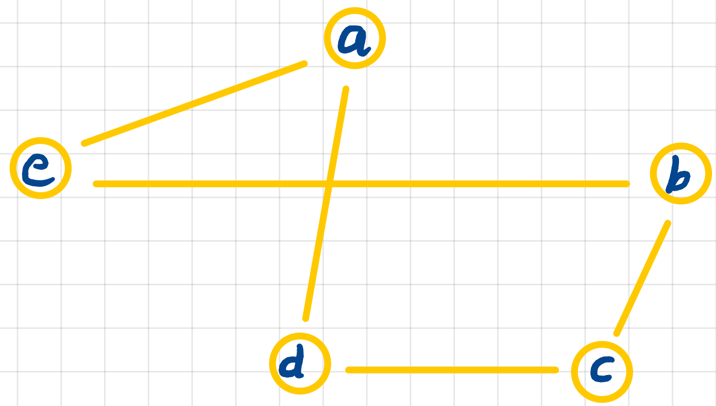
- Understand graphs as an important model for many phenomena in CS & real world
- Explore some basic properties of graphs:
  - paths in graphs
  - trees & complete graphs
  - planar graphs
  - connectivity & hypercubes

next lecture

Graphs

$G = (V, E)$

vertices   edges

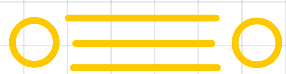


$V = \{a, b, c, d, e\}$

$E = \{\{a,e\}, \{a,d\}, \{e,b\}, \{b,c\}, \{c,d\}\}$

Undirected graph

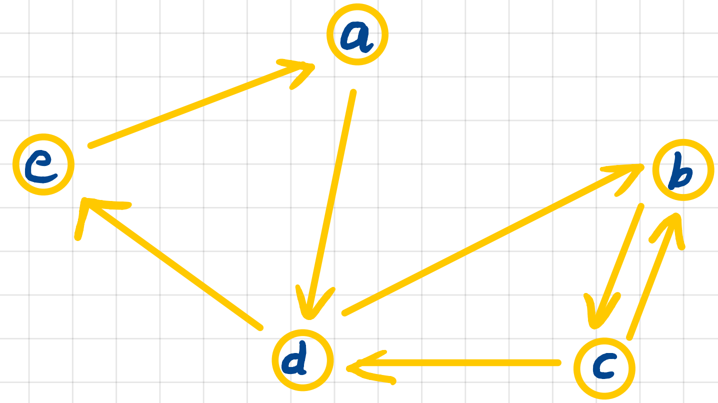
An edge is an unordered pair of vertices

Unless otherwise stated, we don't allow loops  or multiple edges 

I.e., our graphs are simple graphs, not multigraphs

Directed Graphs

$G = (V, E)$

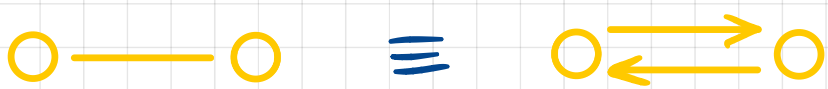


$V = \{a, b, c, d, e\}$

$E = \{(e,a), (d,e), (a,d), (d,b), (b,c), (c,b), (c,d)\}$

Same, except edges are now <u>ordered</u> pairs of vertices

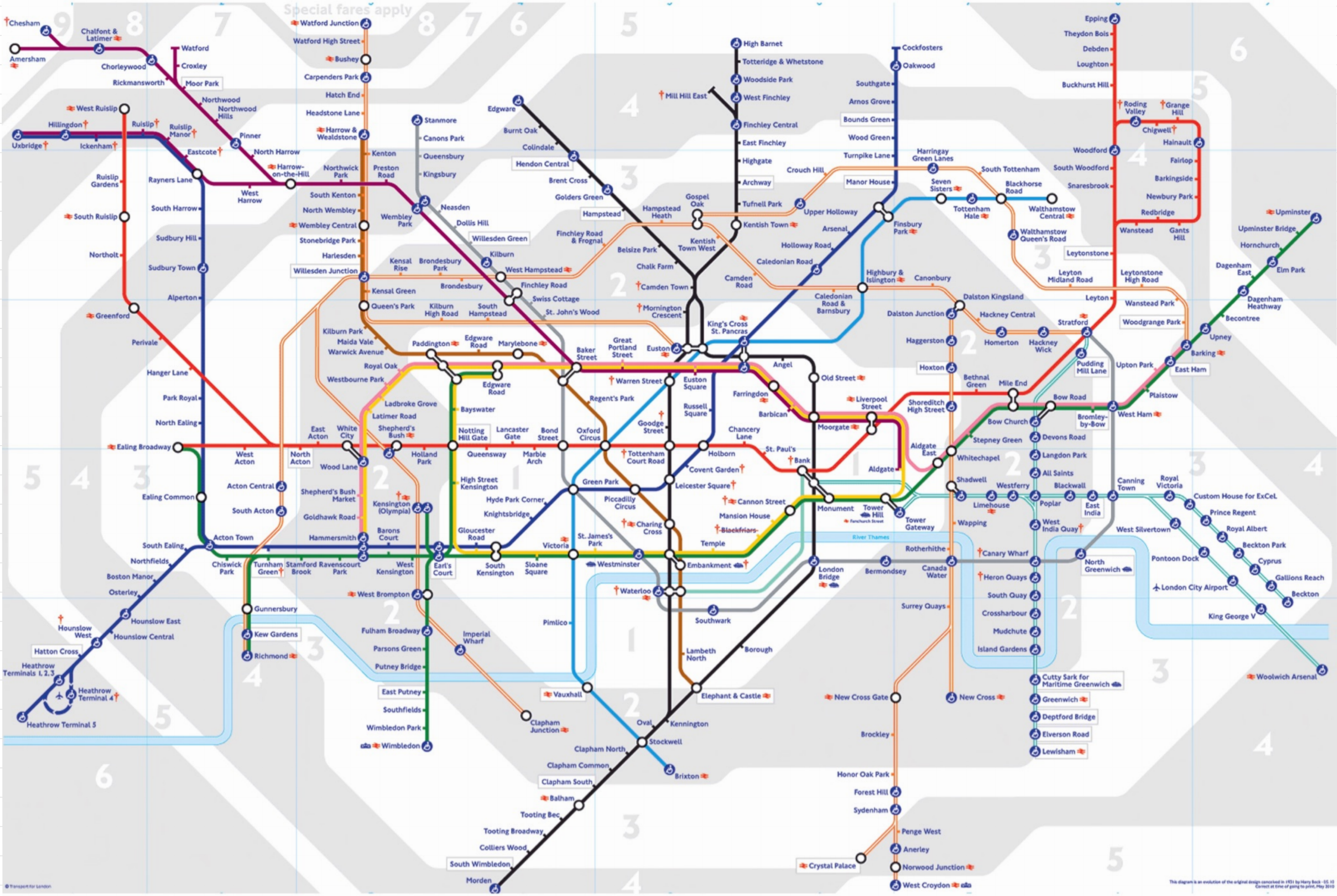
<u>Note</u>: Undirected graphs are a special case of directed graphs:



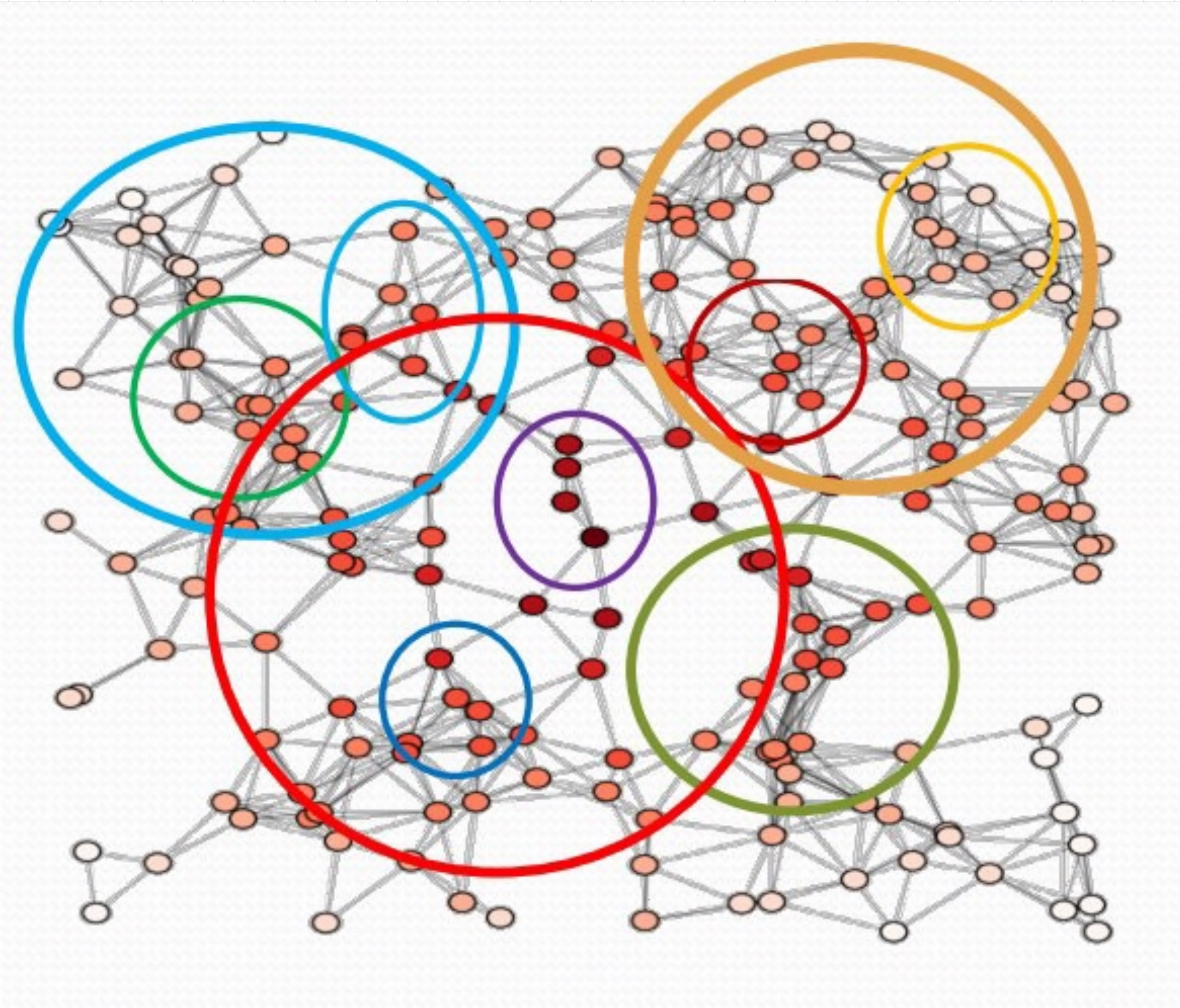
Examples of Undirected Graphs

- road networks (no one-way streets)
- power grids
- social networks ("knows")

⋮



Special fares apply
Chesham
Amersham
Chalfont & Latimer
Chorleywood
Rickmansworth
Watford
Croxley
Moor Park
Northwood
Northwood Hills
Pinner
North Harrow
Harrow-on-the-Hill
West Harrow
West Ruislip
Ruislip
Ruislip Manor
Hillingdon
Uxbridge
Ickenham
Eastcote
Ruislip Gardens
Rayners Lane
South Ruislip
South Harrow
Sudbury Hill
Sudbury Town
Northolt
Alperton
Greenford
Perivale
Hanger Lane
Park Royal
North Ealing
Ealing Broadway
Ealing Common
West Acton
North Acton
East Acton
Acton Central
South Acton
Acton Town
South Ealing
Northfields
Boston Manor
Osterley
Hounslow East
Hounslow Central
Hounslow West
Hatton Cross
Heathrow Terminals 1, 2, 3
Heathrow Terminal 4
Heathrow Terminal 5
Chiswick Park
Turnham Green
Stamford Brook
Ravenscourt Park
Gunnersbury
Kew Gardens
Richmond
Watford Junction
Watford High Street
Bushey
Carpenders Park
Hatch End
Headstone Lane
Harrow & Wealdstone
Kenton
Northwick Park
Preston Road
South Kenton
North Wembley
Wembley Central
Wembley Park
Stonebridge Park
Harlesden
Willesden Junction
Kensal Green
Queen's Park
Kensal Rise
Brondesbury Park
Brondesbury
Kilburn High Road
South Hampstead
Stanmore
Canons Park
Queensbury
Kingsbury
Neasden
Dollis Hill
Willesden Green
Kilburn
West Hampstead
Finchley Road
Swiss Cottage
St. John's Wood
Edgware
Burnt Oak
Colindale
Hendon Central
Brent Cross
Golders Green
Hampstead
Hampstead Heath
Finchley Road & Frognal
Belsize Park
Chalk Farm
Camden Town
Mornington Crescent
Gospel Oak
Kentish Town West
Kentish Town
Camden Road
High Barnet
Totteridge & Whetstone
Woodside Park
West Finchley
Mill Hill East
Finchley Central
East Finchley
Highgate
Archway
Tufnell Park
Crouch Hill
Upper Holloway
Arsenal
Holloway Road
Caledonian Road
Caledonian Road & Barnsbury
Highbury & Islington
Canonbury
Cockfosters
Oakwood
Southgate
Arnos Grove
Bounds Green
Wood Green
Turnpike Lane
Manor House
Harringay Green Lanes
Finsbury Park
Seven Sisters
South Tottenham
Blackhorse Road
Tottenham Hale
Walthamstow Central
Walthamstow Queen's Road
Leytonstone
Leyton Midland Road
Leytonstone High Road
Leyton
Wanstead Park
Woodgrange Park
Stratford
Hackney Wick
Homerton
Hackney Central
Dalston Kingsland
Dalston Junction
Haggerston
Hoxton
Epping
Theydon Bois
Debden
Loughton
Buckhurst Hill
Roding Valley
Grange Hill
Chigwell
Hainault
Fairlop
Barkingside
Newbury Park
Woodford
South Woodford
Snaresbrook
Redbridge
Wanstead
Gants Hill
Upminster
Upminster Bridge
Hornchurch
Elm Park
Dagenham East
Dagenham Heathway
Becontree
Upney
Barking
East Ham
Upton Park
Plaistow
West Ham
Bromley-by-Bow
Bow Road
Bow Church
Mile End
Bethnal Green
Pudding Mill Lane
Devons Road
Langdon Park
All Saints
Kilburn Park
Maida Vale
Warwick Avenue
Paddington
Edgware Road
Marylebone
Baker Street
Great Portland Street
Euston
Royal Oak
Westbourne Park
Ladbroke Grove
Latimer Road
Bayswater
Warren Street
Euston Square
King's Cross St. Pancras
Angel
Old Street
Regent's Park
Russell Square
Farringdon
Barbican
Goodge Street
Chancery Lane
Liverpool Street
Shoreditch High Street
Moorgate
White City
Shepherd's Bush
Wood Lane
Holland Park
Notting Hill Gate
Queensway
Lancaster Gate
Marble Arch
Bond Street
Oxford Circus
Tottenham Court Road
Holborn
St. Paul's
Bank
Aldgate
Aldgate East
Whitechapel
Stepney Green
Shadwell
Limehouse
Westferry
Poplar
Blackwall
East India
Canning Town
Royal Victoria
Custom House for ExCeL
Prince Regent
Royal Albert
Beckton Park
Cyprus
Gallions Reach
Beckton
West Silvertown
Pontoon Dock
London City Airport
King George V
Woolwich Arsenal
Shepherd's Bush Market
Goldhawk Road
Kensington (Olympia)
Hammersmith
Barons Court
High Street Kensington
Hyde Park Corner
Knightsbridge
Green Park
Piccadilly Circus
Covent Garden
Leicester Square
Cannon Street
Mansion House
Blackfriars
Temple
Monument
Tower Hill
Fenchurch Street
Tower Gateway
Wapping
Rotherhithe
Canada Water
Canary Wharf
West India Quay
Heron Quays
South Quay
Crossharbour
Mudchute
Island Gardens
Cutty Sark for Maritime Greenwich
Greenwich
Deptford Bridge
Elverson Road
Lewisham
North Greenwich
Gloucester Road
South Kensington
Sloane Square
Victoria
St. James's Park
Westminster
Charing Cross
Embankment
Waterloo
Southwark
London Bridge
Bermondsey
River Thames
Surrey Quays
West Kensington
Earl's Court
West Brompton
Fulham Broadway
Parsons Green
Putney Bridge
East Putney
Southfields
Wimbledon Park
Wimbledon
Imperial Wharf
Clapham Junction
Pimlico
Vauxhall
Lambeth North
Elephant & Castle
Borough
Oval
Kennington
Stockwell
Brixton
Clapham North
Clapham Common
Clapham South
Balham
Tooting Bec
Tooting Broadway
Colliers Wood
South Wimbledon
Morden
New Cross Gate
New Cross
Brockley
Honor Oak Park
Forest Hill
Sydenham
Crystal Palace
Penge West
Anerley
Norwood Junction
West Croydon
© Transport for London
This diagram is an evolution of the original design conceived in 1931 by Harry Beck

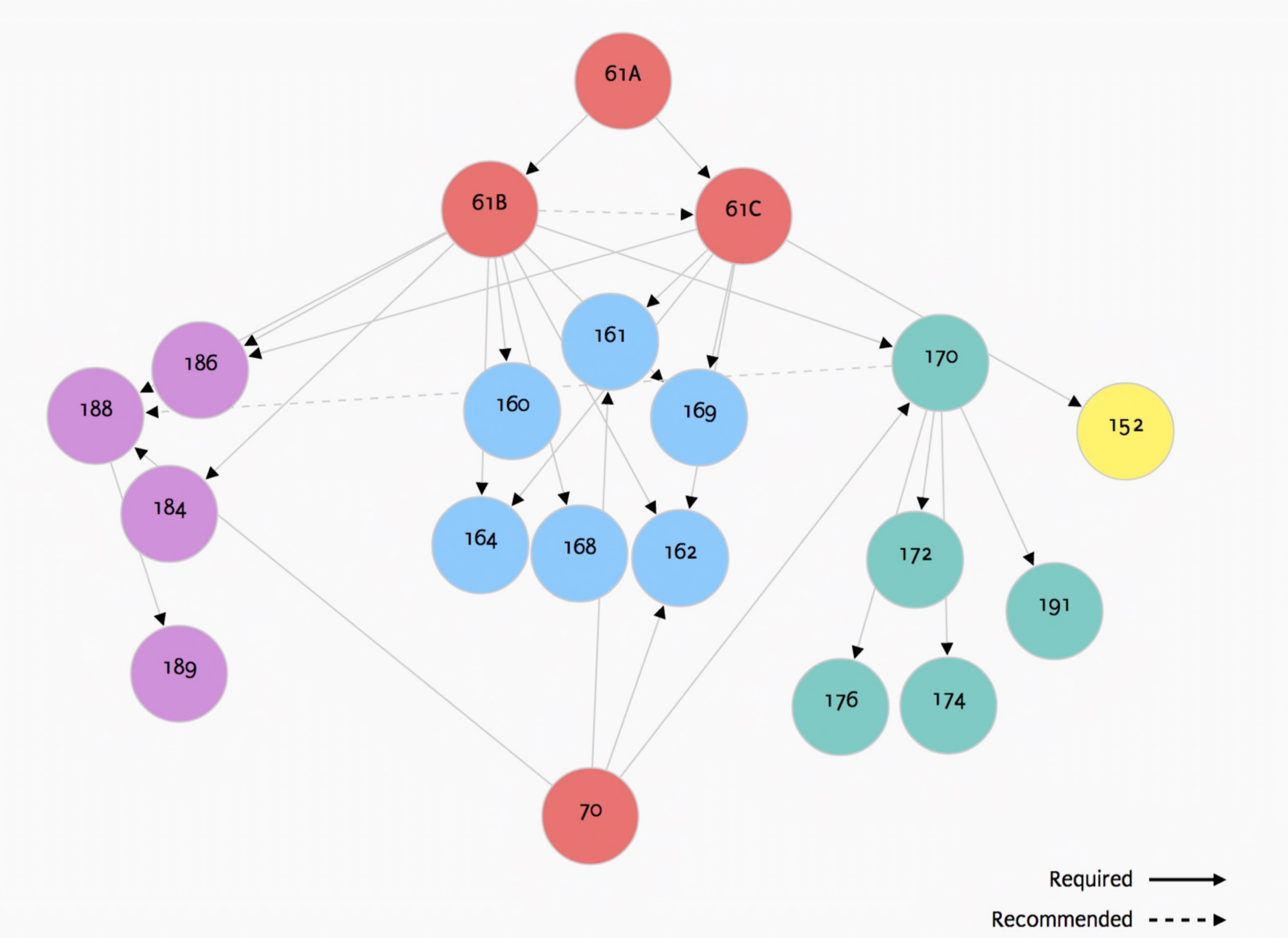


Renren Social Network: Chen Wen Xin

Examples of Directed Graphs

- internet (web links)
- course prerequisites
- biological networks
- social networks ("recognizes")

$\vdots$


61A
61B
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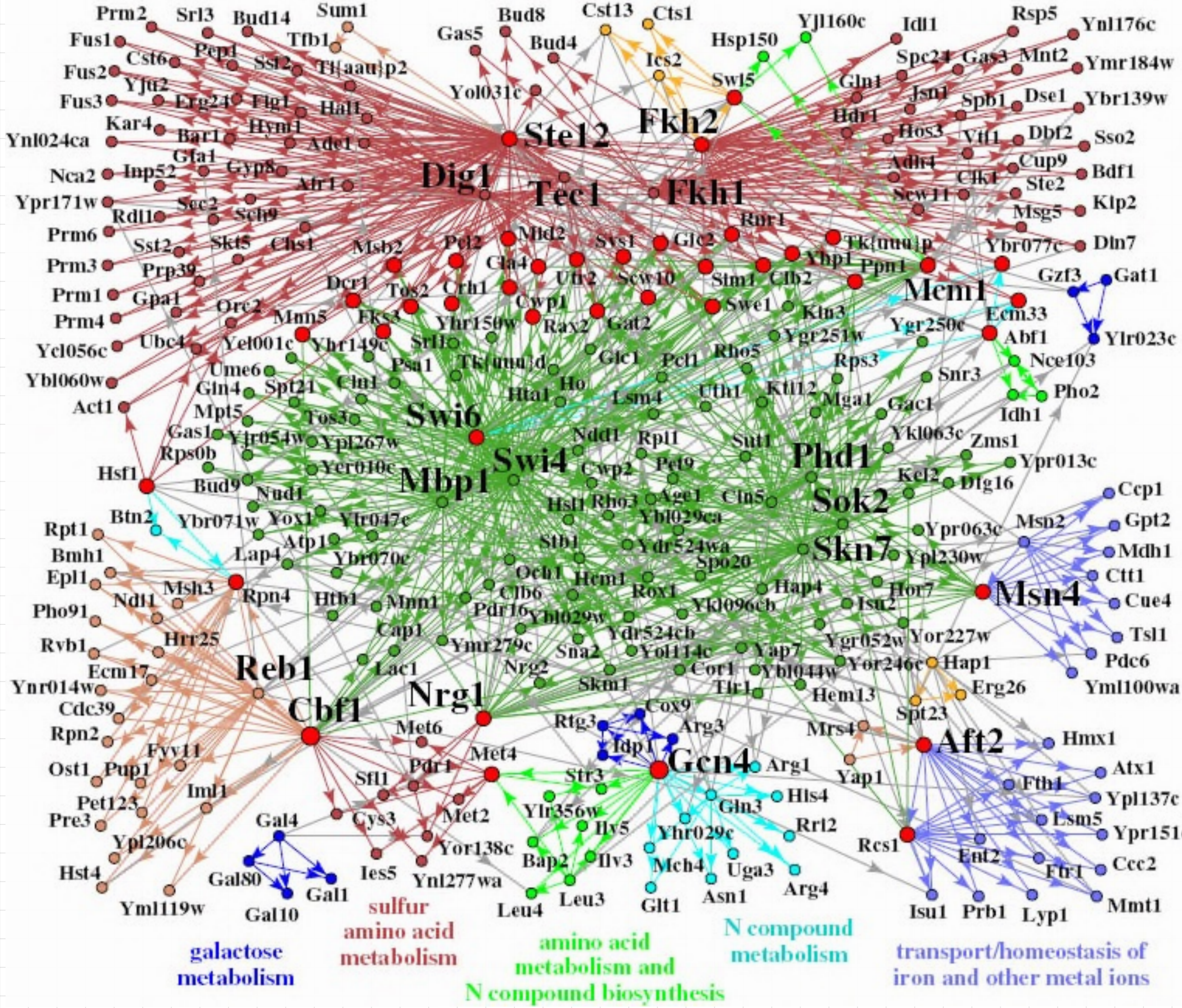
Required
Recommended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Reb1
Cbf1
Nrg1
Gcn4
Aft2
Rcs1
Met4
Prm2
Srl3
Bud14
Sum1
Bud8
Cst13
Cts1
Yjl160c
Idl1
Rsp5
Ynl176c
Fus1
Tfb1
Gas5
Bud4
Hsp150
Ics2
Swi5
Spc24
Gas3
Mnt2
Ymr184w
Cst6
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Inp52
Gyp8
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Clk1
Ste2
Ypr171w
Sec2
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Gpa1
Dcr1
Tos2
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Stm1
Clb2
Prm4
Orc2
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Yel001c
Yhr149c
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Gln4
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Utr1
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Act1
Mpt5
Tos3
Hta1
Lsm4
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Gac1
Pho2
Idh1
Gas1
Yjr054w
Ypl267w
Ndd1
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Ykl063c
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Cue4
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Ybl029w
Tsl1
Cap1
Ydr524cb
Rvb1
Ymr279c
Ygr052w
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Cox9
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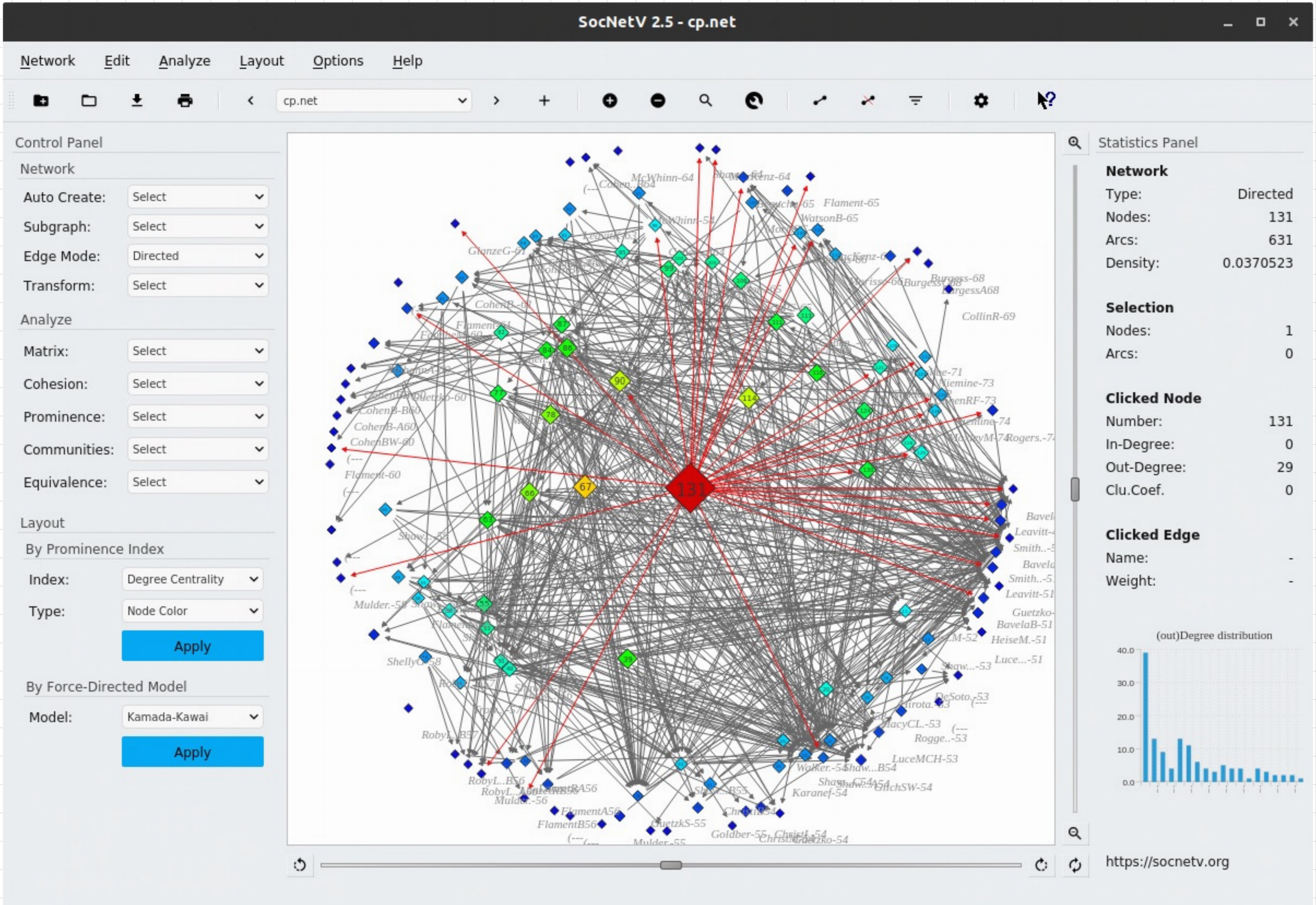
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Network
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Layout
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cp.net
Control Panel
Network
Auto Create: Select
Subgraph: Select
Edge Mode: Directed
Transform: Select
Analyze
Matrix: Select
Cohesion: Select
Prominence: Select
Communities: Select
Equivalence: Select
Layout
By Prominence Index
Index: Degree Centrality
Type: Node Color
Apply
By Force-Directed Model
Model: Kamada-Kawai
Apply
Statistics Panel
Network
Type: Directed
Nodes: 131
Arcs: 631
Density: 0.0370523
Selection
Nodes: 1
Arcs: 0
Clicked Node
Number: 131
In-Degree: 0
Out-Degree: 29
Clu.Coef. 0
Clicked Edge
Name: -
Weight: -
(out)Degree distribution
https://socnetv.org

Some Terminology

$e = \{u, v\}$ edge

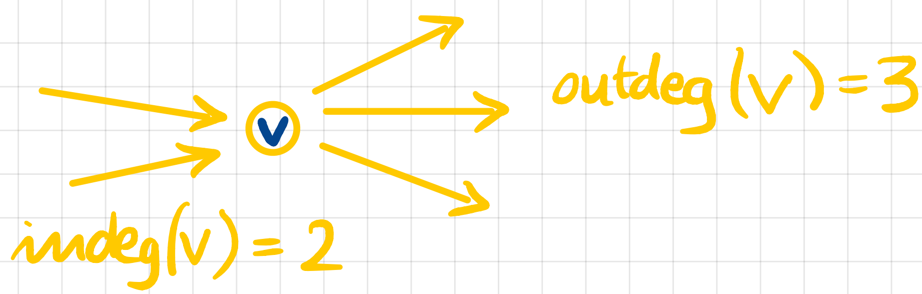
u, v are adjacent or neighbors

e is incident on u & v

degree of v = number of neighbors

If v has degree 0 it is isolated

For directed graphs:



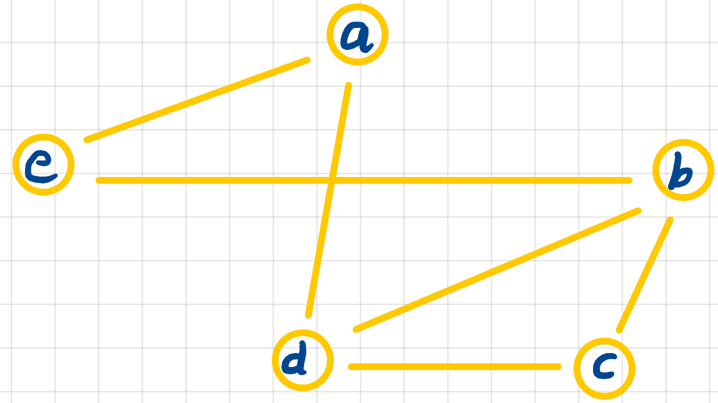
Q: What is the sum of all vertex degrees in G ?

A: $\sum_{v \in V} \deg(v) =$

More Terminology

A (simple) <u>path</u> is a sequence of edges
$\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \ldots, \{v_{k-1}, v_k\}$
s.t. all the v_i are distinct (except possibly $v_k = v_1$)

[Equivalently: $v_1 - v_2 - v_3 - \cdots - v_k$]



E.g.

path: $e - a - d - c$

cycle: $d - c - b - d$

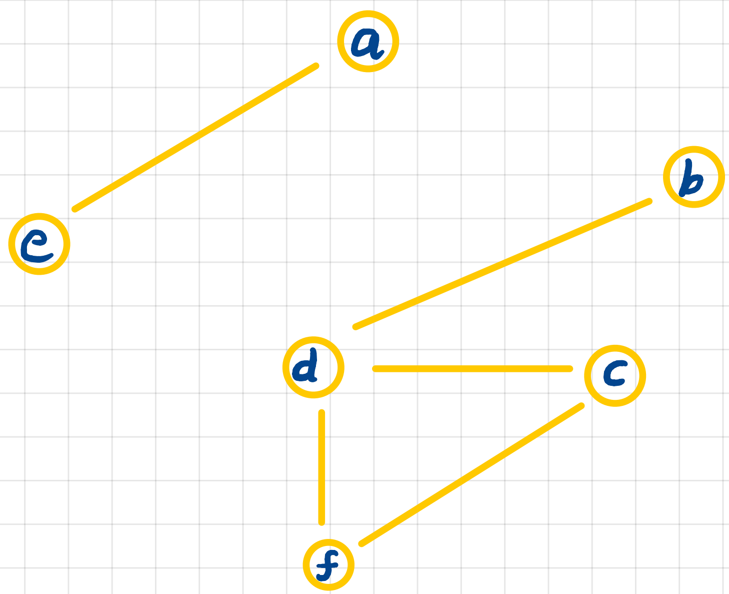
A <u>cycle</u> is a path where $v_1 = v_k$

A <u>walk</u> is any sequence of edges as above (repeated vertices/edges allowed)

A <u>tour</u> is a walk where $v_1 = v_k$

A graph is connected if there is a path from any vertex to any other vertex

Any graph can be decomposed into connected components

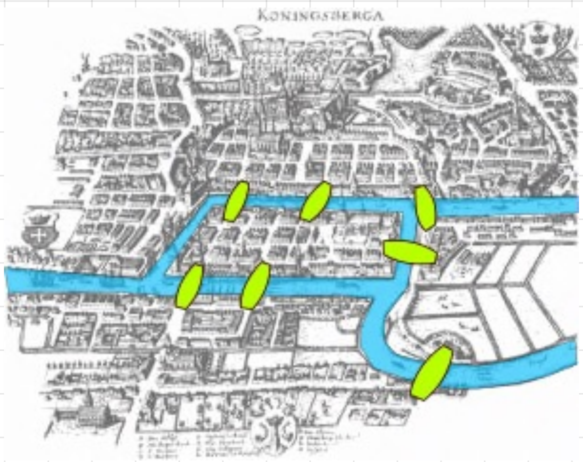


Graph Theory: Topics

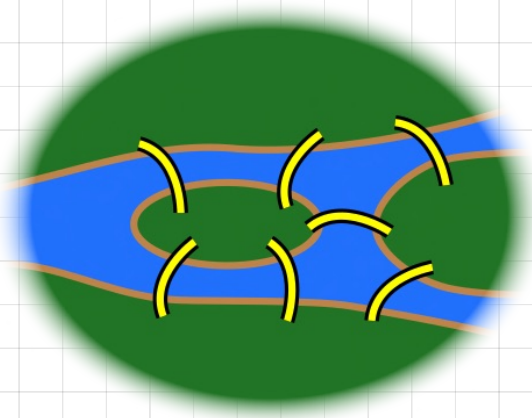
- Eulerian tours
- Trees & complete graphs
- Planar graphs
- Connectivity & hypercubes

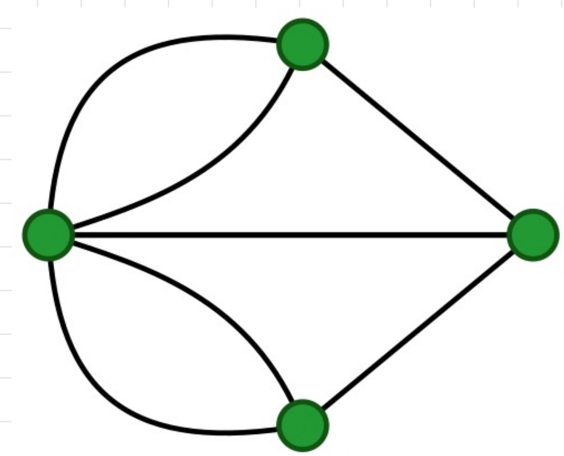
next lecture

Euler: Bridges of Königsberg (1736)



Can we cross all 7 bridges once & end up where we started?



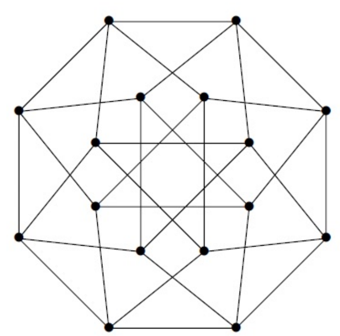


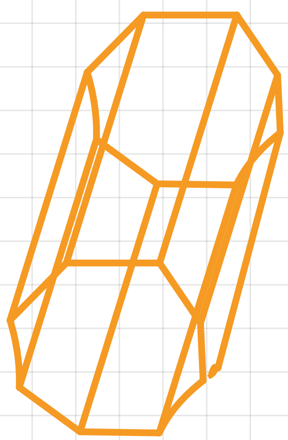
(multi)graph

Defn: An **Eulerian tour** of a graph is a walk that traverses each edge exactly once and ends up at the starting vertex

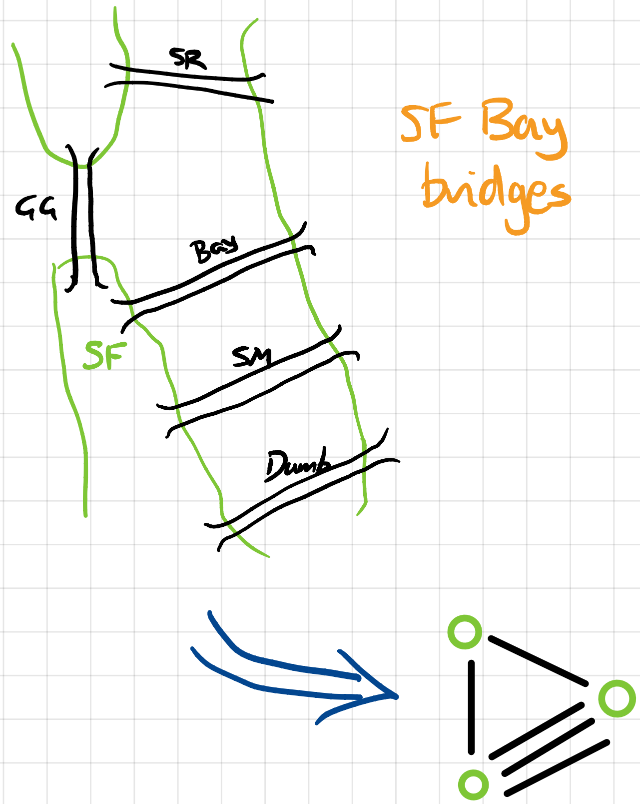
Theorem [Euler]: A connected graph has an Eulerian tour $\Longleftrightarrow$ the degree of every vertex is even

Examples





octagonal "lampshade"



Theorem [Euler]: A connected graph has an Eulerian tour $\Longleftrightarrow$ the degree of every vertex is even

Proof:

subroutine FindTour(G, s)

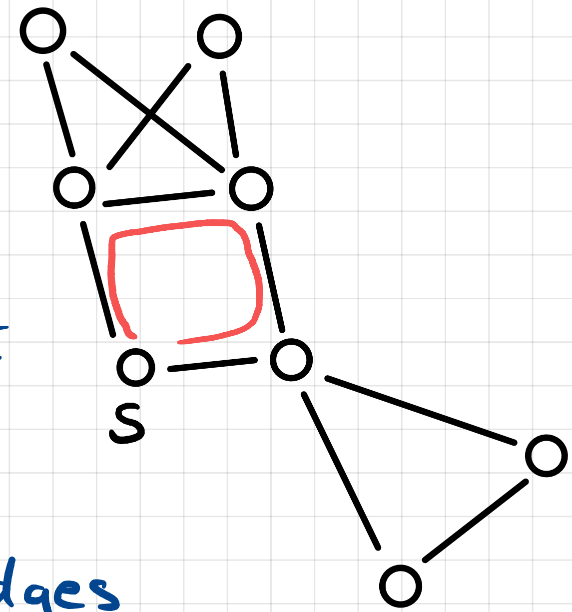
start at s

repeat

choose any untraversed edge incident on current vertex & traverse it

until get stuck

return the tour formed by traversed edges



Claim: FindTour(G, s) always gets stuck at s

Proof:

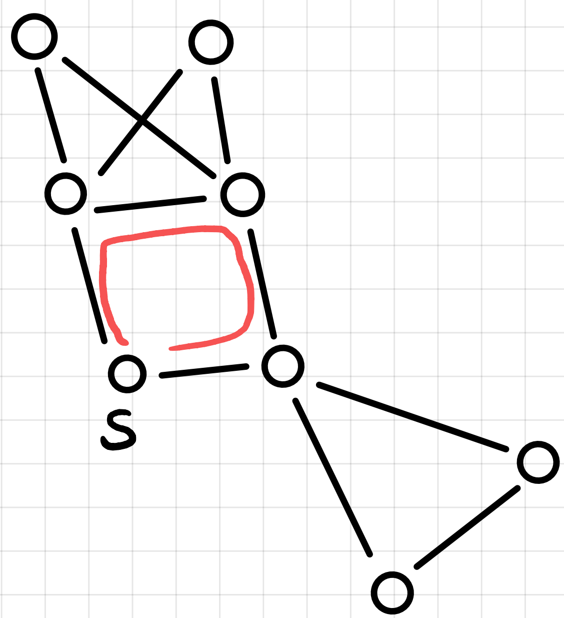
Euler(G, s)

$T =$ FindTour(G, s)

remove edges of T from G

let $\{G_1, \ldots, G_k\}$ be the connected components of the remaining graph, and s_i the first vertex of G_i visited by T

output (Splice(T, Euler(G_1, s_1), $\ldots$, Euler(G_k, s_k)))



Theorem: For any connected G with even degrees, Euler(G,s) outputs an Euler tour starting & ending at s

Proof: (Strong) induction on # edges m in G

Base: $m = 0$ ✓ (nothing to prove)

Ind. step:

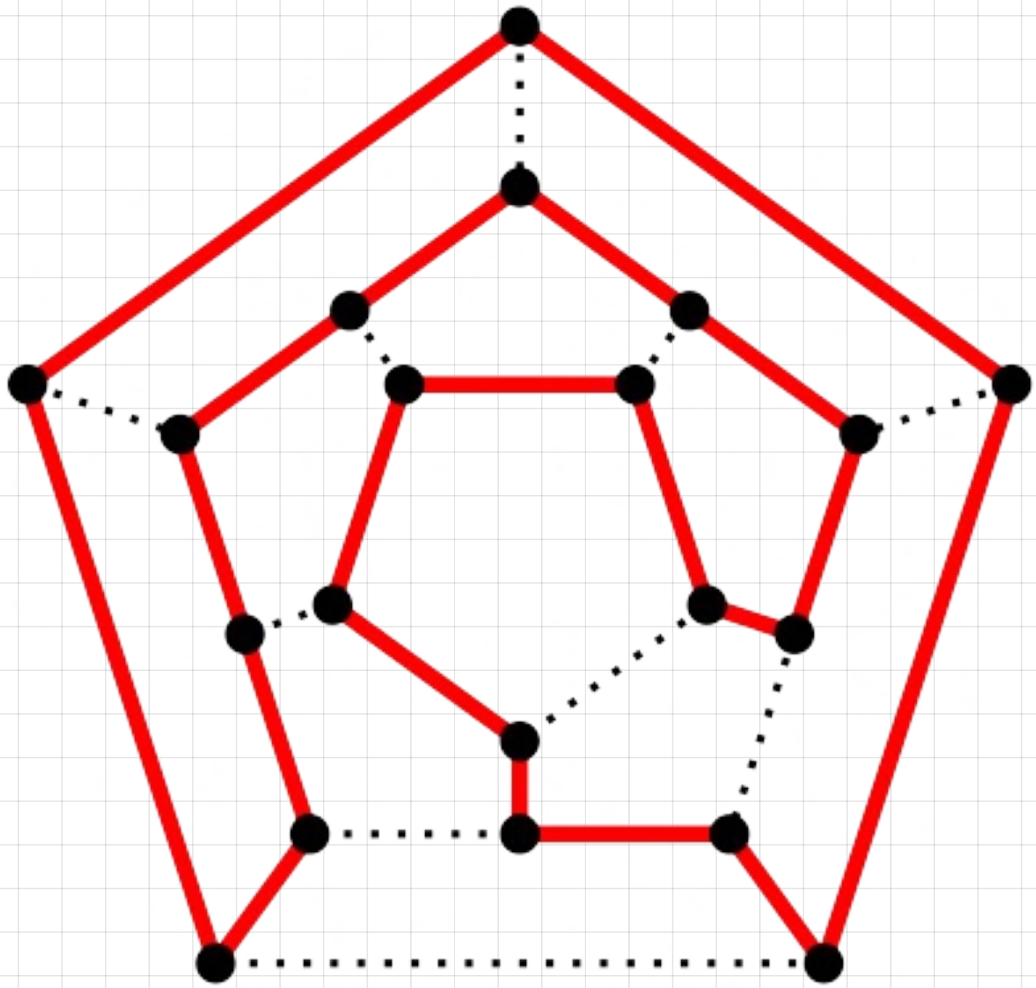
Eulerian Tours — Wrap-Up

1. **Exercise**: modify proof of Euler's Thm. to prove:

   Connected G has an Eulerian walk starting at s & ending at t $\iff$ all vertices of G except s & t have even degree

2. A Hamilton cycle in G is a cycle in G that visits every vertex exactly once

   - no simple characterization
   - NP-complete (related to Traveling Salesman)



Example of a graph with a Hamilton cycle
("dodecahedron" graph)

Graph Theory: Topics

- Eulerian tours
- Trees & complete graphs
- Planar graphs
- Connectivity & hypercubes

next lecture

Trees

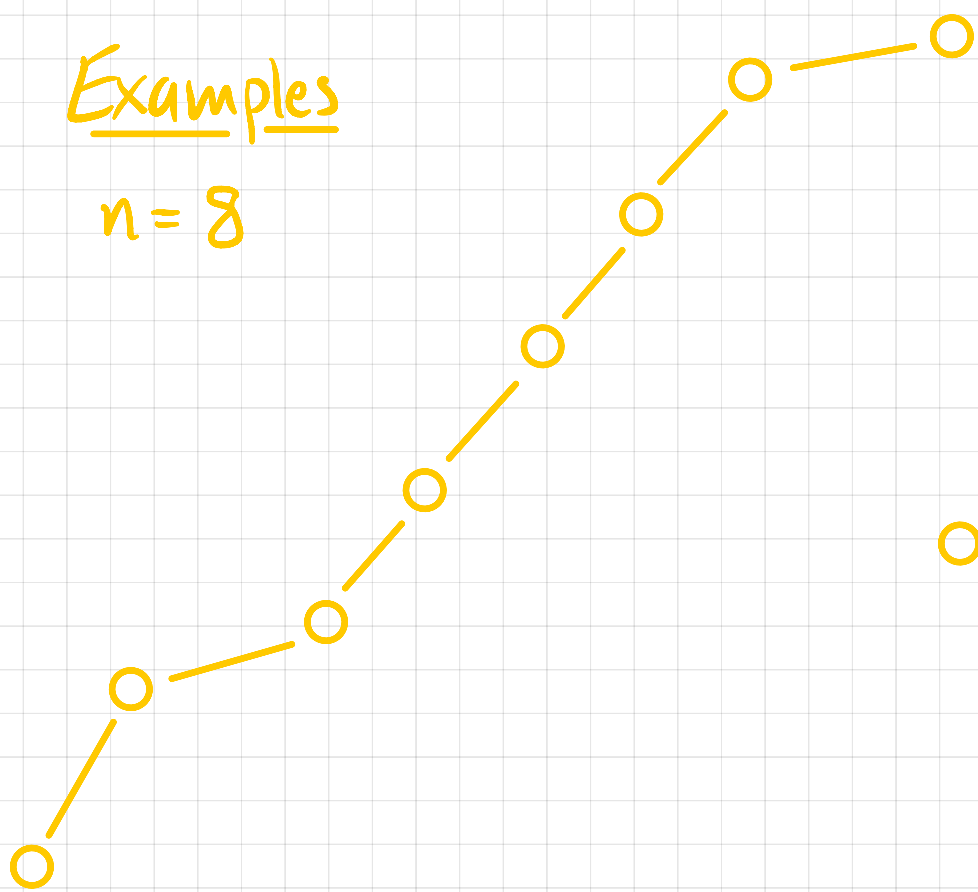
An undirected graph G with n vertices is a <u>tree</u> if either of the following equiv. conditions holds:

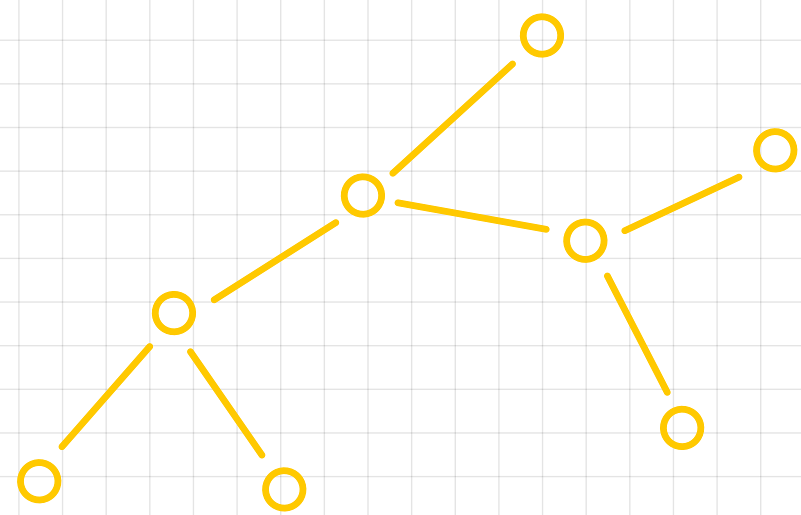
(i) G is connected & has $n-1$ edges

(ii) G is connected & has no cycles

Examples

$n=8$

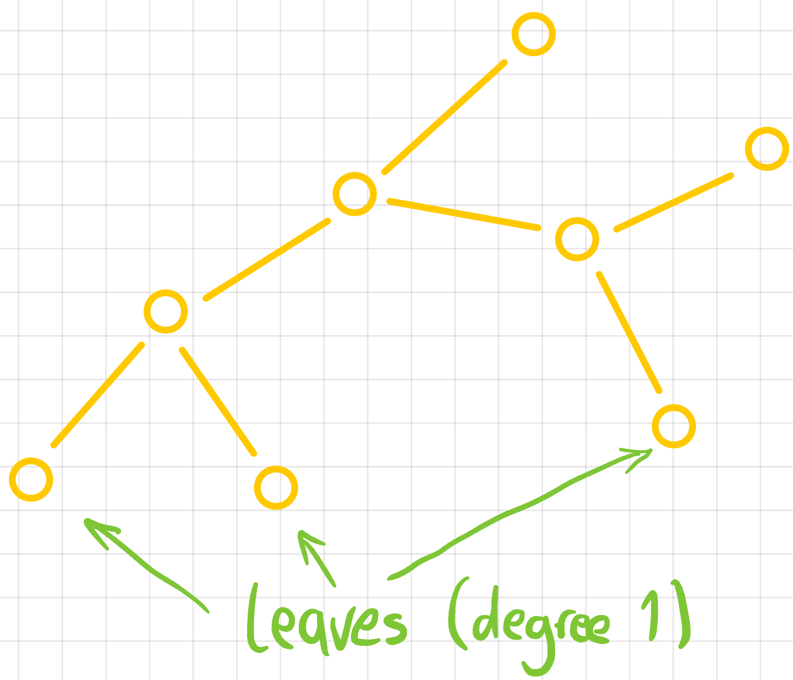


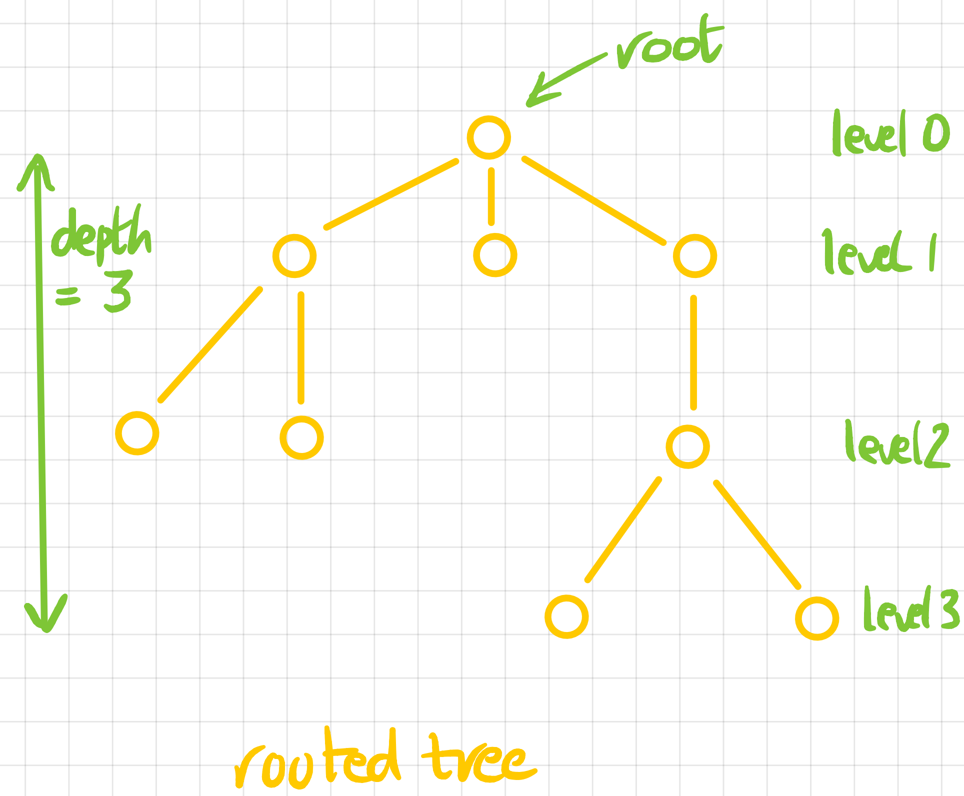


Trees show up in:

- data structures (BST, Red-Black trees, ...)
- phylogenetic trees (evolutionary biology, genealogy)
- epidemic models
- decision trees

⋮





Theorem : G is connected & has $n-1$ edges $\iff$ G is connected & has no cycles

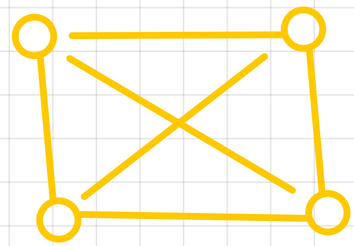
Proof :

Theorem: G is connected & has $n-1$ edges $\iff$ G is connected & has no cycles

Proof:

The Complete Graph

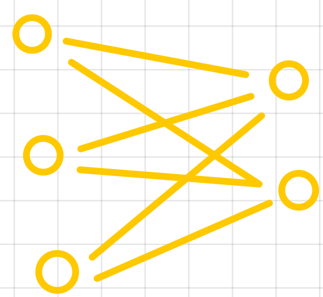
The <u>complete graph</u> on n vertices, K_n, is the graph that contains all possible edges (so # of edges is)



K_4

<u>Notes</u>:

1. K_n is unique but $\exists$ many (n^{n-2}) trees on n vertices
2. K_n is maximally connected (need to remove at least $n-1$ edges to disconnect); trees are minimally connected (removing any edge disconnects)
3. Complete <u>bipartite</u> graph $K_{n,m}$:



$K_{3,2}$

# of edges =

Summary

- Graphs: directed & undirected
- Paths, cycles, walks, tours
- Eulerian tours
- Trees, complete graphs

Next lecture

- Planar graphs
- Hypercubes & connectivity