CS70 – SPRING 2026

LECTURE 6: FEB. 5

Last Lecture

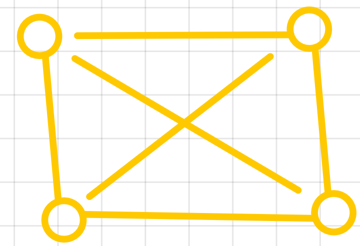
- Graphs: directed & undirected
- Paths, cycles, walks, tours
- Eulerian tours
- Trees, complete graphs

Today

- Planar graphs
- Hypercubes & connectivity

The Complete Graph

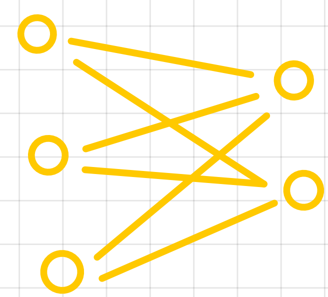
The <u>complete graph</u> on n vertices, K_n, is the graph that contains all possible edges
(so # of edges is)



K_4

<u>Notes</u>:

1. K_n is unique but $\exists$ many (n^{n-2}) trees on n vertices
2. K_n is maximally connected (need to remove at least $n-1$ edges to disconnect); trees are minimally connected (removing any edge disconnects)
3. Complete <u>bipartite</u> graph $K_{n,m}$:



$K_{3,2}$

# of edges =

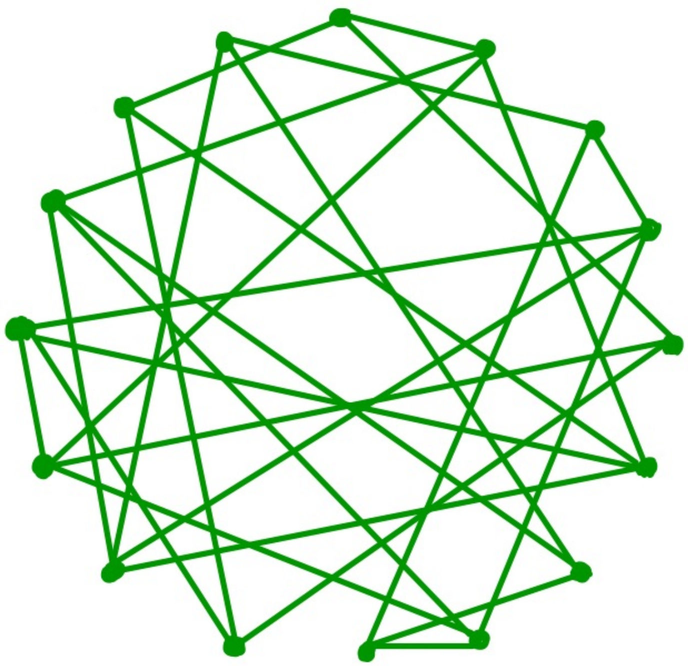
Defn: A graph is **planar** if it can be drawn on the plane so that none of its edges cross

Note: A planar graph may have many different planar embeddings/drawings

Q: Why planar graphs?

A:
- Easy to visualize
- Efficient algorithms
- Nice properties (e.g., colorable with 4 colors Appel/Haken 4 Color Theorem 1976)

⋮



Ex: Is this graph planar?

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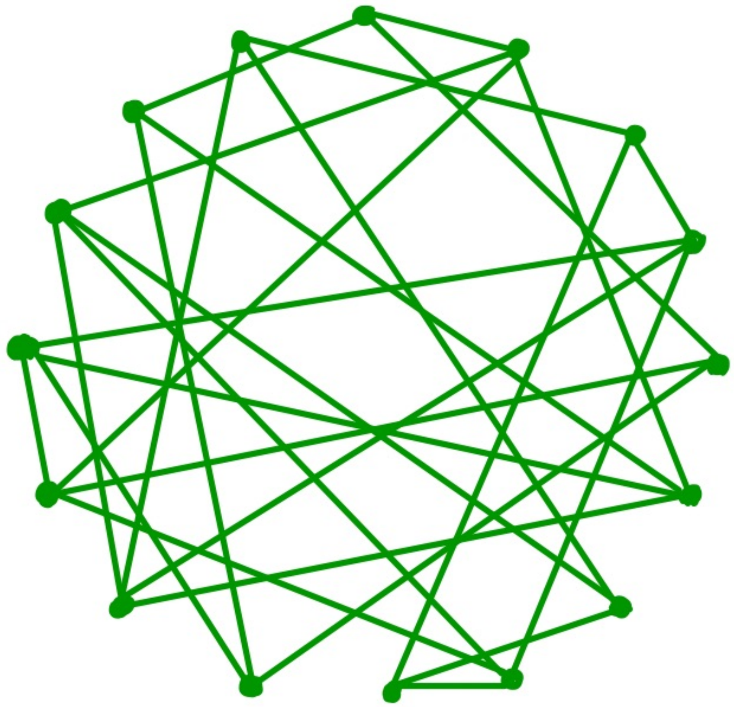
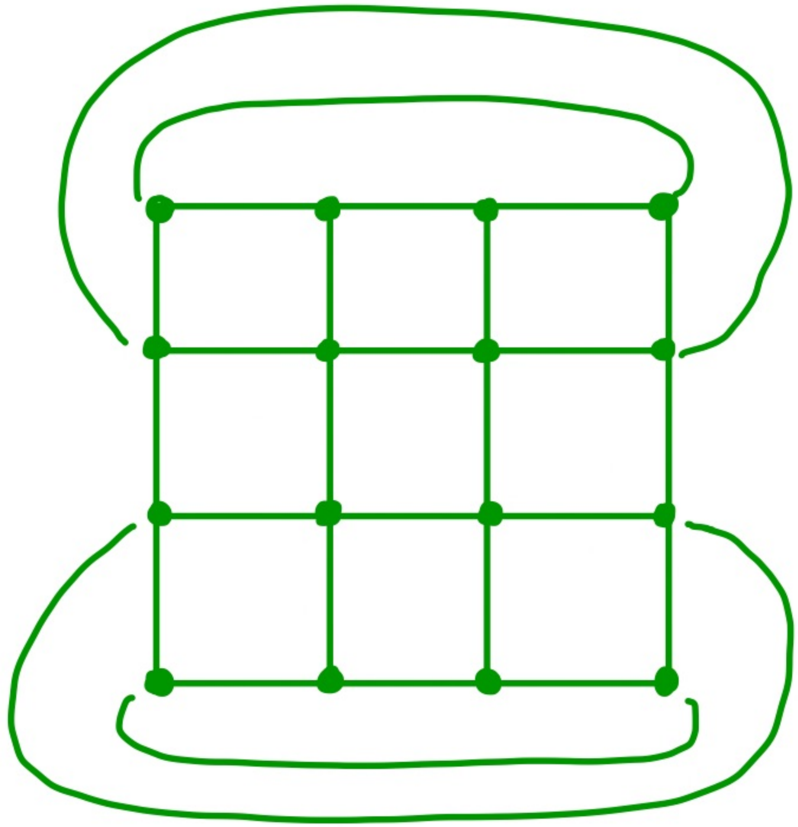
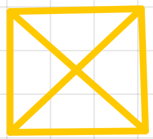


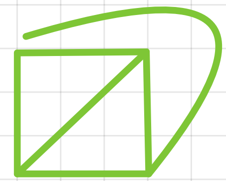


More Examples

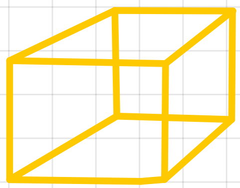
K_4



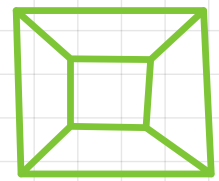
$\longrightarrow$



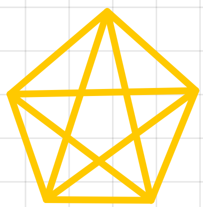
3-dim. cube



$\longrightarrow$

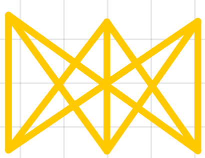


K_5



planar ?

$K_{3,3}$ ("utility graph")

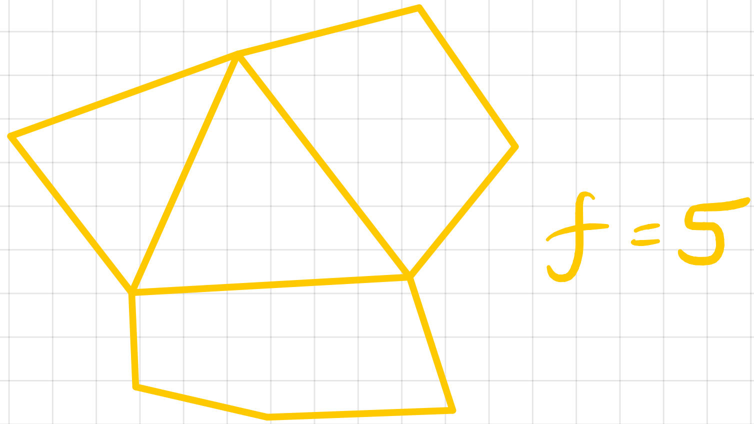


planar ?

Euler's Formula

Any planar drawing of a graph divides the plane into some number, f, of faces

E.g.

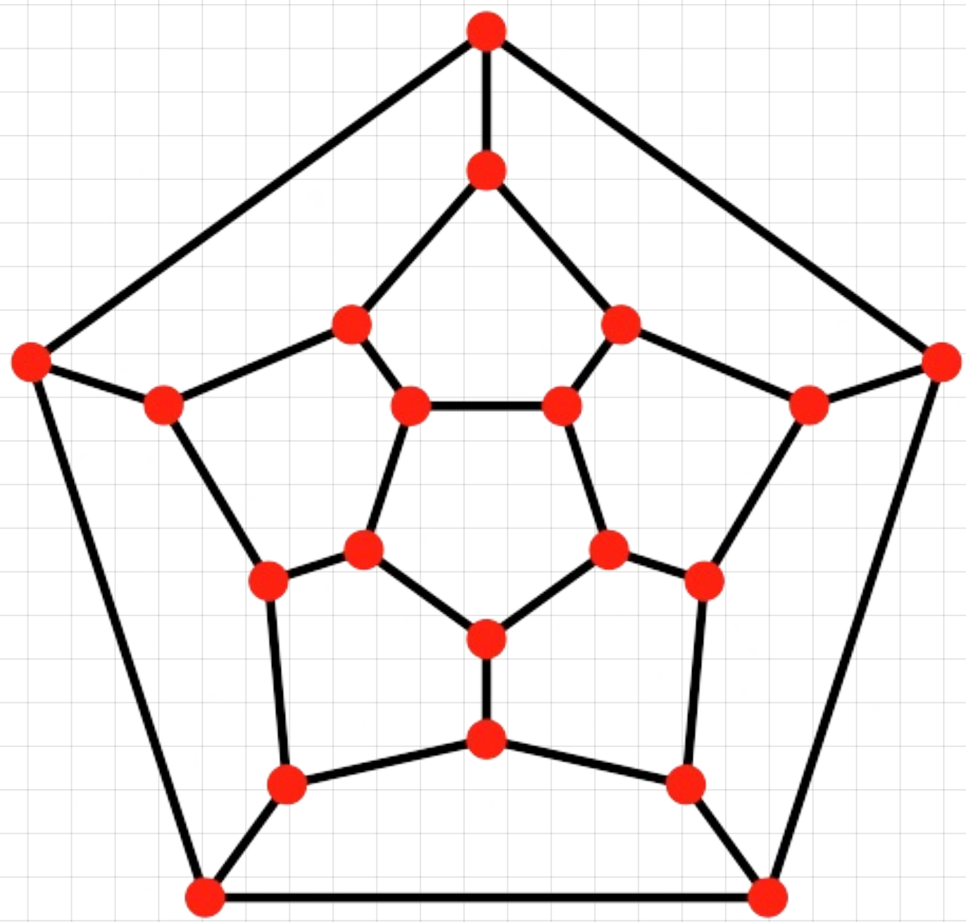


Theorem [Euler's Formula]: Any planar drawing of a connected graph satisfies

$$\boxed{v - e + f = 2}$$

where v, e are the numbers of vertices & edges, resp.

Note: The Greeks "knew" this for polyhedral graphs, but couldn't prove it!



Dodecahedron graph

$$\left.\begin{aligned} v &= 20 \\ e &= 30 \\ f &= 12 \end{aligned}\right\} \quad v - e + f = 2$$

Theorem [Euler's Formula]: Any planar drawing of a connected graph satisfies

$$\boxed{v - e + f = 2}$$

where v, e are the numbers of vertices & edges, resp.

Proof: Induction on #faces, f

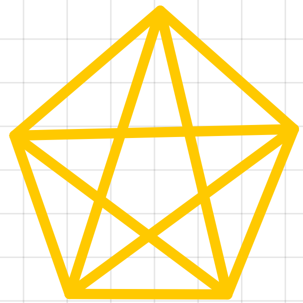
Applications of Euler's Formula

Formula says that # edges of planar graph is $e = v + f - 2$
How big can this be?

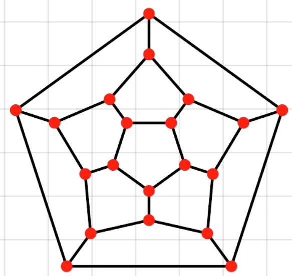
Corollary: Any connected planar graph with ≥ 2 edges satisfies $\boxed{e \leq 3v - 6}$

Examples

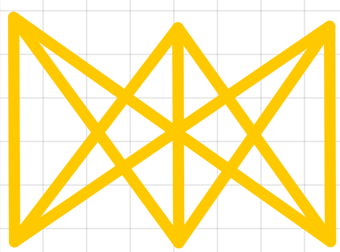
K_5



Dodecahedron



$K_{3,3}$

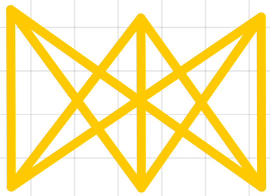


Strengthening Euler's Criterion

Euler's criterion says that if $e > 3v-6$ then graph cannot be planar — so planar graphs can't have too many edges

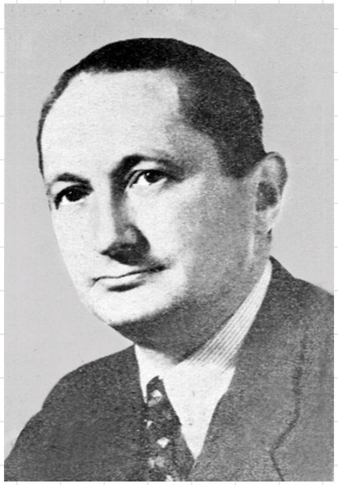
For special types of graph we can do better
E.g. sp. G is bipartite — then G has no triangles, so every face has $s_i \geq 4$ sides!
Replacing $3f$ by $4f$ in previous argument:

$$\boxed{e \leq 2v-4}$$

$K_{3,3}$ 

Kuratowski's Theorem

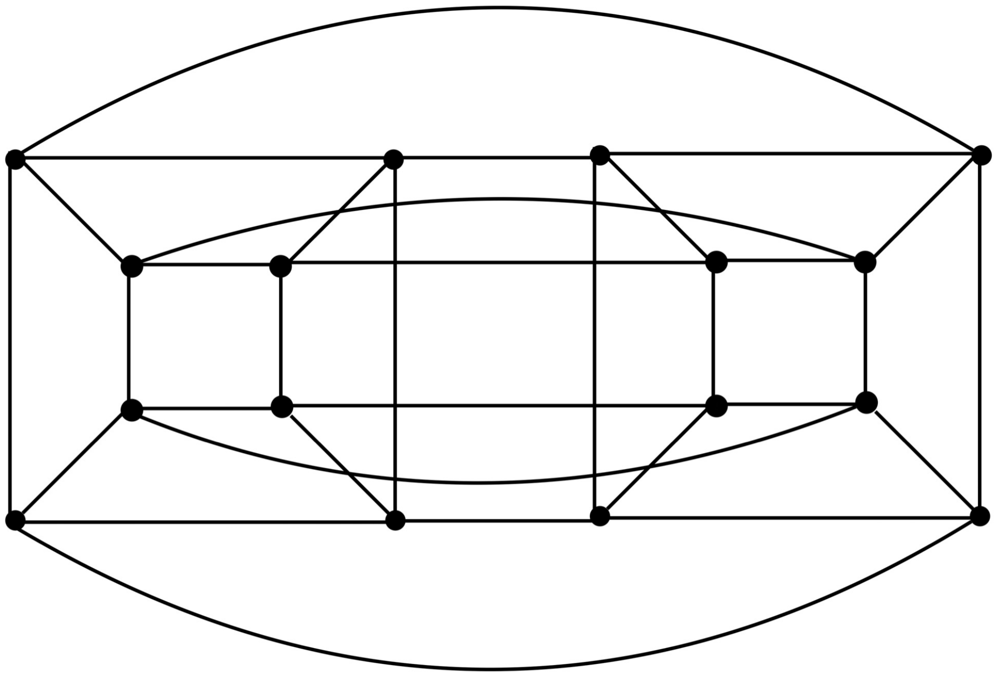
A graph G is planar $\iff$ G does not "contain" K_5 or $K_{3,3}$



"Proof": $\Rightarrow$ already seen!

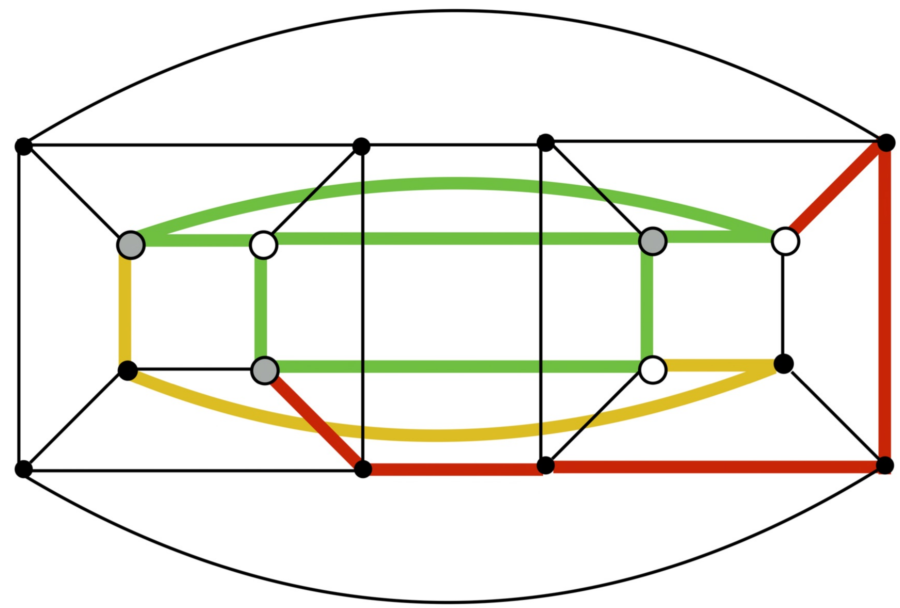
$\Leftarrow$ tricky!

Note: "G contains H" means we can find a copy of H inside G, where vertices of H are distinct vertices of G and edges of H are disjoint paths in G



4-dimensional hypercube

Copy of $K_{3,3}$ inside



Connectivity

Think of a graph G as a communication network

vertices $\longrightarrow$ nodes   edges $\longrightarrow$ links

All nodes can communicate $\longrightarrow$ G must be connected

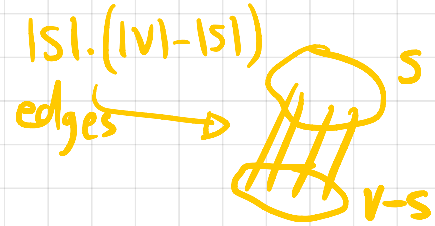
Links (edges) may fail !

1. G is a tree

   #edges $= n-1$

   very fragile : any edge failure disconnects !

2. G is a complete graph K_n



   #edges $= \frac{n(n-1)}{2} \approx \frac{1}{2}n^2$!

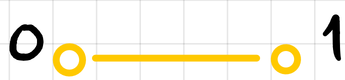
   robust : need $\geq n-1$ edge failures to disconnect

Hypercubes

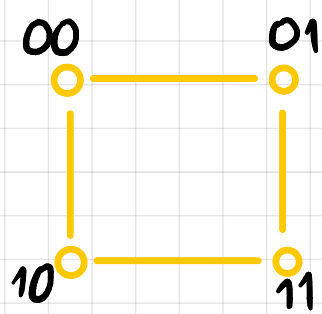
H_n : n-dimensional hypercube

vertices: $\{0,1\}^n$ $\quad$ (# vertices $= 2^n$)

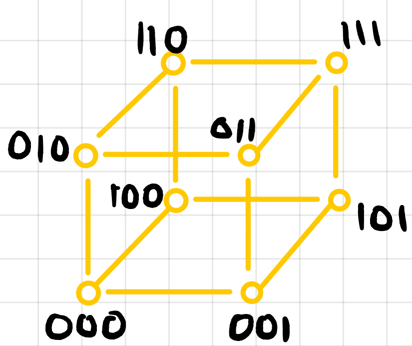
edges: connect vertices that differ in 1 bit



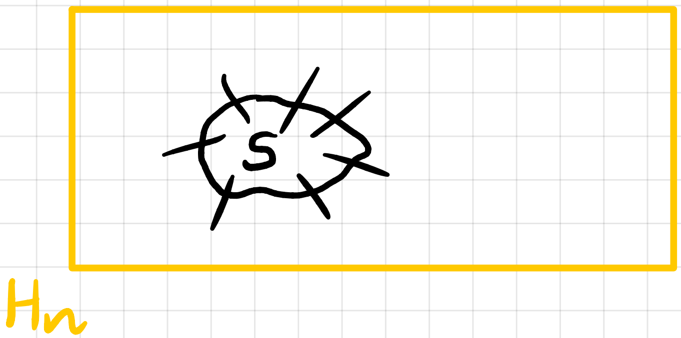
Note: H_n consists of 2 copies of H_{n-1} with vertices matched up 1-1



$$H_n \text{ has } \begin{cases} 2^n \text{ vertices} \\ \dfrac{n \cdot 2^n}{2} = n \cdot 2^{n-1} \text{ edges} \\ \text{all vertex degrees } n \\ \underline{\text{diameter}} \ n \end{cases}$$



Hyperconnected



$S \subseteq V$: subset of vertices
$|S| \le \frac{|V|}{2}$

E_S : set of edges connecting
S to $V-S$

<u>Theorem</u>: In H_n, for any set S as above, $|E_S| \geq |S|$

<u>Note</u>: Actually $|E_S| \geq \max\{n, |S|\}$ so v. small sets also OK.

Theorem: In H_n, for any set S as above, $|E_S| \geq |S|$

Proof: Induction on n

Base case: $n = 1$. $|S| = 1$ $|E_S| = 1$ ✓

Inductive step: Assume true for H_k – prove for H_{k+1}

Let $S \subseteq V(H_{k+1})$ with $|S| \leq 2^k$

Write $S = S_0 \cup S_1$ where S_0, S_1 are in 0,1–subcubes

Assume w.l.o.g. $|S_0| \geq |S_1|$

Case (i): $|S_0| \leq 2^{k-1}$ & $|S_1| \leq 2^{k-1}$

Then can apply ind. hyp. within each subcube

$$\Rightarrow |E_S| \geq |S_0| + |S_1| = |S|$$ ✓

Theorem: In H_n, for any set S as above, $|E_S| \geq |S|$

Proof: Induction on n

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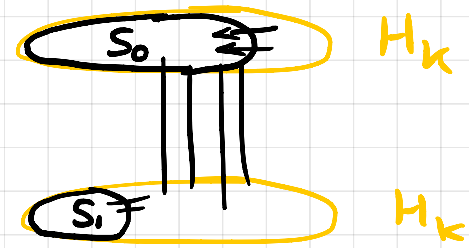

Case(ii): $|S_0| > 2^{k-1}$

Then $|S_1| = |S| - |S_0| < 2^{k-1}$

So ind. hyp. in 1-subcube gives $|S_1|$ edges

And ind. hyp. in 0-subcube applied to $V_0 - S_0$ gives $|V_0| - |S_0|$ edges

Finally, also get $|S_0| - |S_1|$ crossing edges (between subcubes)

So $|E_S| \geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0| = 2^k \geq |S|$ ✓

Summary

- Planar graphs
- Euler's Formula : $v - e + f = 2$
- Corollary : $e \leq 3v - 6$ (or $e \leq 2v - 4$ for bipartite graphs)
- Two key non-planar graphs : K_5 & $K_{3,3}$ (Kuratowski's Theorem)
- Hypercubes H_n : well connected, good network model

Next Lecture

- Modular arithmetic