CS70 – SPRING 2026

LECTURE 6: FEB. 5

Last Lecture

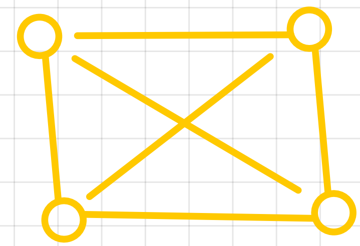
- Graphs: directed & undirected
- Paths, cycles, walks, tours
- Eulerian tours
- Trees, complete graphs

Today

- Planar graphs
- Hypercubes & connectivity

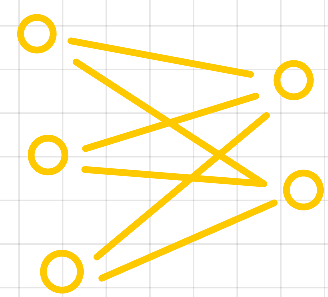
The Complete Graph

The <u>complete graph</u> on n vertices, K_n, is the graph that contains all possible edges (so # of edges is $\frac{n(n-1)}{2}$)



<u>Notes</u>:

1. K_n is unique but $\exists$ many (n^{n-2}) trees on n vertices
2. K_n is maximally connected (need to remove at least $n-1$ edges to disconnect); trees are minimally connected (removing any edge disconnects)
3. Complete <u>bipartite</u> graph $K_{n,m}$:



$$\#\text{ of edges} = nm$$

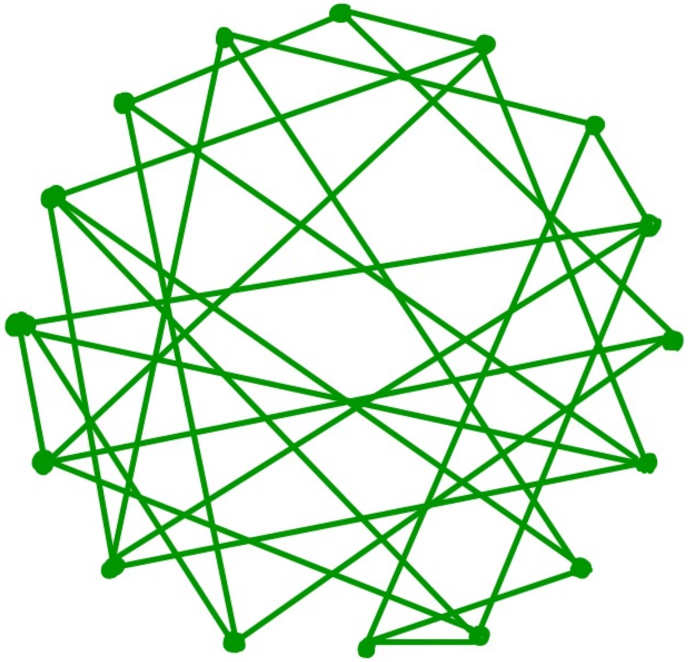
Defn: A graph is <u>planar</u> if it can be drawn on the plane so that none of its edges cross

Note: A planar graph may have many different planar embeddings/drawings

Q: Why planar graphs?

A:
- Easy to visualize
- Efficient algorithms
- Nice properties (e.g., colorable with 4 colors
  Appel/Haken 4 Color Theorem 1976)

⋮



Ex: Is this graph planar?

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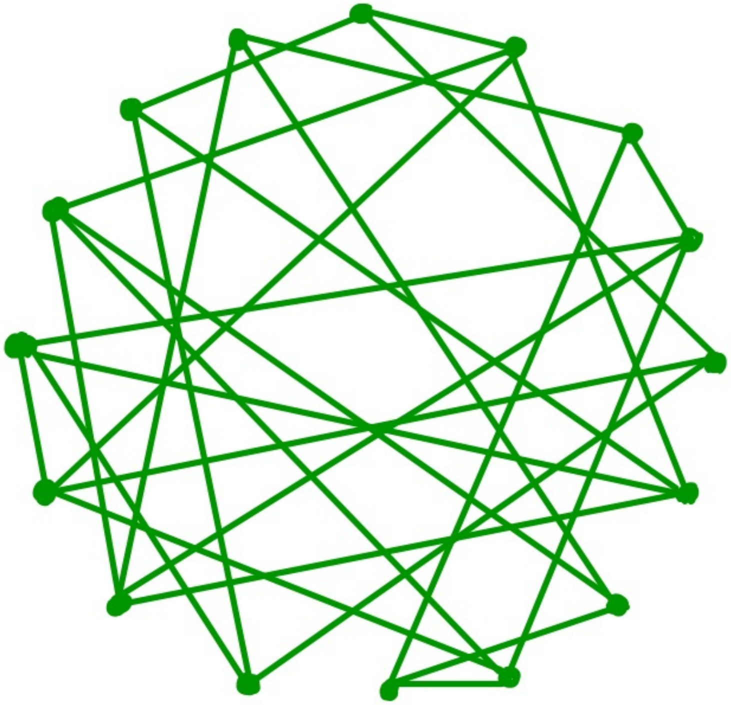
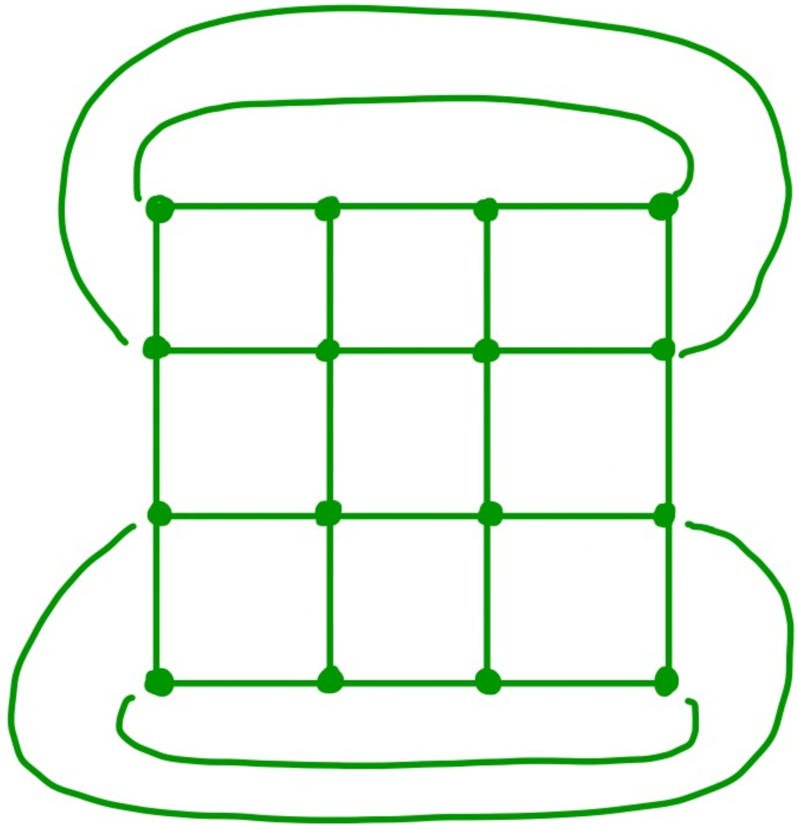
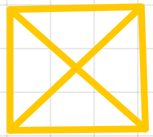


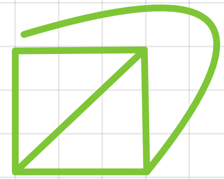


More Examples

K_4

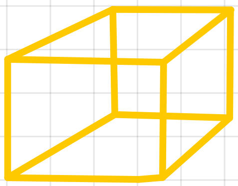


$\longrightarrow$

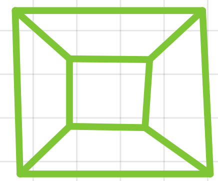


$f=4$, $e=6$, $v=4$ $\Big\}$ $v-e+f=2$

3-dim. cube

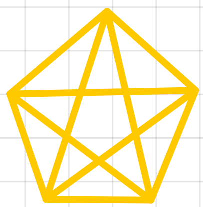


$\longrightarrow$



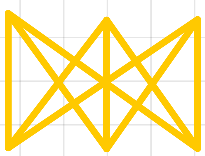
$f=6$, $e=12$, $v=8$ $\Big\}$ $v-e+f=2$

K_5



planar ?

$K_{3,3}$ ("utility graph")

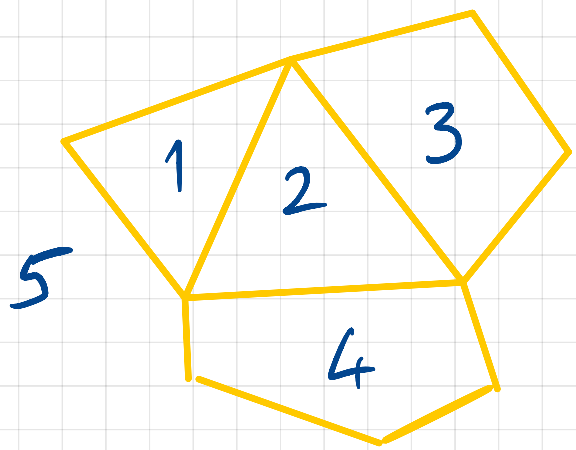


planar ?

Euler's Formula

Any planar drawing of a graph divides the plane into some number, f, of faces

E.g.



$f=5$

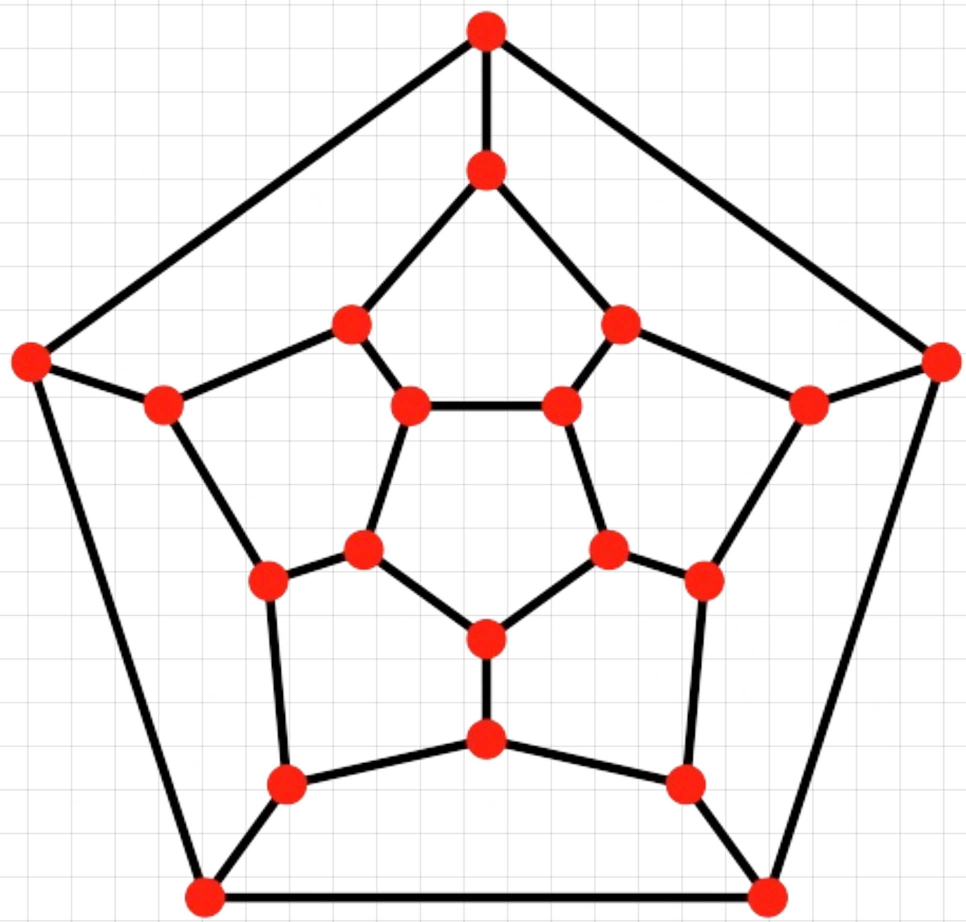
$$\left.\begin{array}{l} f=5 \\ v=9 \\ e=12 \end{array}\right\} \begin{array}{l} v-e+f \\ =9-12+5 \\ =2 \end{array}$$

Theorem [Euler's Formula]: Any planar drawing of a connected graph satisfies

$$\boxed{v-e+f=2}$$

where v, e are the numbers of vertices & edges, resp.

Note: The Greeks "knew" this for polyhedral graphs, but couldn't prove it!



Dodecahedron graph

$$\left.\begin{aligned} v &= 20 \\ e &= 30 \\ f &= 12 \end{aligned}\right\} \quad v - e + f = 2$$

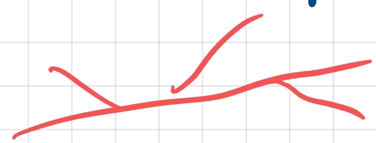
Theorem [Euler's Formula]: Any planar drawing of a connected graph satisfies

$$\boxed{v - e + f = 2}$$

where v, e are the numbers of vertices & edges, resp.

Proof: Induction on #faces, f

Base Case: $f = 1$

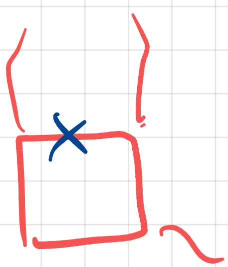


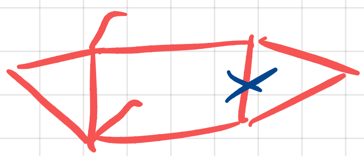
$\Rightarrow G$ is a tree ✓

$e = v - 1$ $\quad v - e + f = v - (v-1) + 1 = \boxed{2}$

Inductive Step: Assume formula holds for any planar drawing with $f-1$ faces $\quad (f \geq 2)$

Take any planar drawing of graph G with f faces, e edges, v vertices





$\rightarrow$ new drawing with $f-1$ faces, $e-1$ edges, v vertices

By induction hypothesis: $v - (e-1) + (f-1) = 2$

But this implies $v - e + f = 2$, as required ✓ $\square$

Applications of Euler's Formula

Formula says that # edges of planar graph is $e = v + f - 2$
How big can this be?

Corollary: Any connected planar graph with ≥ 2 edges satisfies $\boxed{e \leq 3v - 6}$

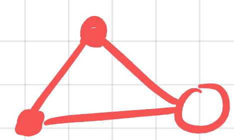
Proof: Draw G in the plane: satisfies $e = v + f - 2$

For each face F_i, let s_i = number of sides of F_i

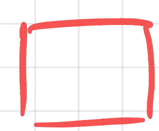
Then $\displaystyle\sum_{i=1}^{f} s_i = 2e$

Also, $s_i \geq \cancel{3}\,4$ $\forall i$ BIPARTITE $\Rightarrow 2e \geq \cancel{3}\,4f$

$e = v + f - 2 \leq v + \frac{2e}{\cancel{3}\,4} - 2$

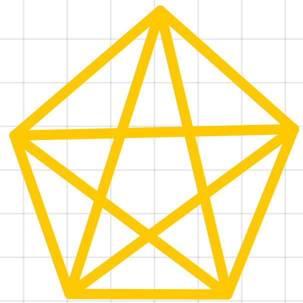


$\Rightarrow \boxed{e \leq 3v - 6}$ $\boxed{e \leq 2v - 4}$



Examples

K_5

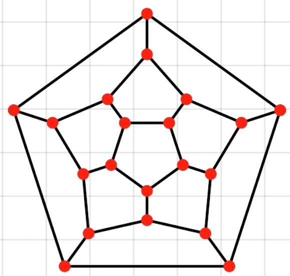


$$\left.\begin{array}{l} v = 5 \\ e = 10 \end{array}\right\} \quad e \overset{?}{\leq} 3v - 6$$

$$10 \leq 15 - 9 = 9 \quad \times$$

NOT planar

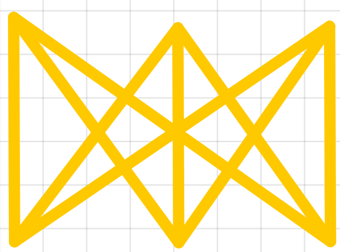
Dodecahedron



$$\left.\begin{array}{l} v = 20 \\ e = 30 \end{array}\right\} \quad 30 \overset{?}{\leq} 60 - 6 \quad \checkmark$$

MAY BE planar

$K_{3,3}$



$$\left.\begin{array}{l} v = 6 \\ e = 9 \end{array}\right\} \quad e \overset{?}{\leq} 3v - 6$$

$$9 \leq 18 - 6 = 12$$

$\checkmark$ MAY BE planar

$$e \overset{?}{\leq} 2v - 4$$

$$9 \leq 12 - 4 = 8 \quad \times$$

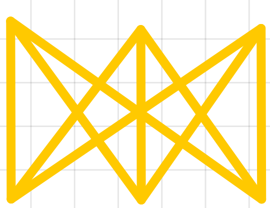
NOT planar

Strengthening Euler's Criterion

Euler's criterion says that if $e > 3v-6$ then graph cannot be planar — so planar graphs can't have too many edges

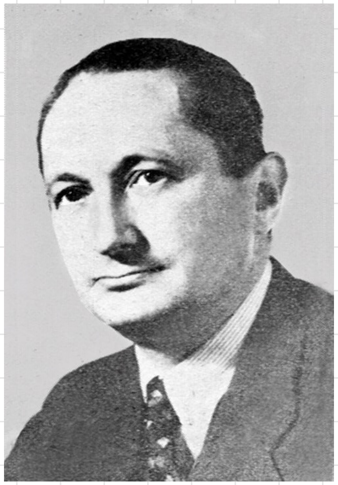
For special types of graph we can do better
E.g. sp. G is bipartite — then G has no triangles, so every face has $s_i \geq 4$ sides!
Replacing $3f$ by $4f$ in previous argument:

$$\boxed{e \leq 2v-4}$$

$K_{3,3}$ 

Kuratowski's Theorem

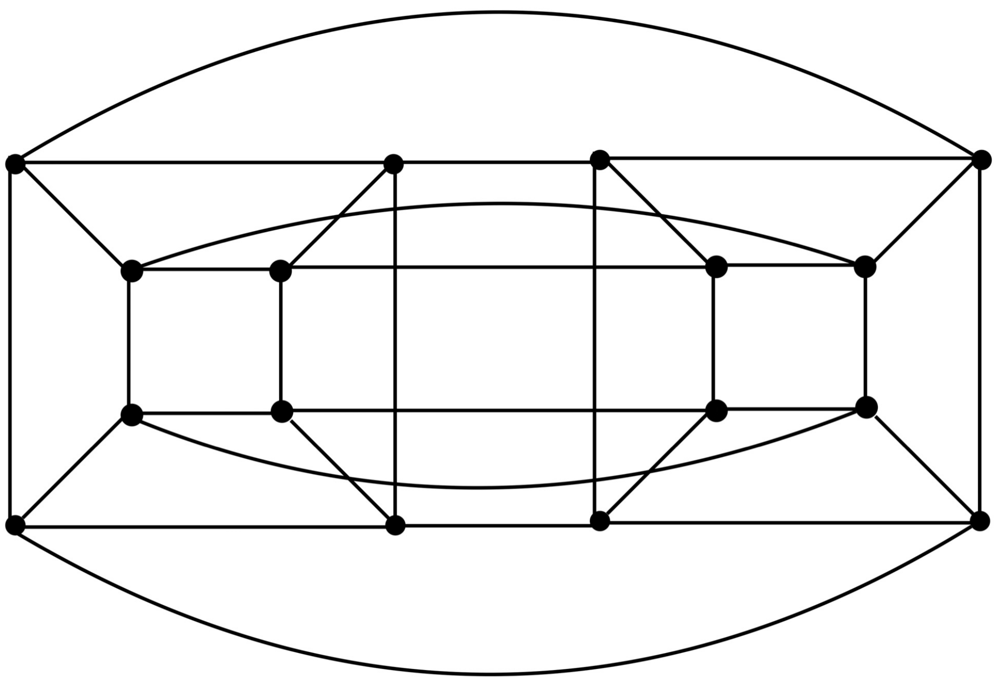
A graph G is planar $\iff$ G does not "contain" K_5 or $K_{3,3}$



"Proof": $\Rightarrow$ already seen!

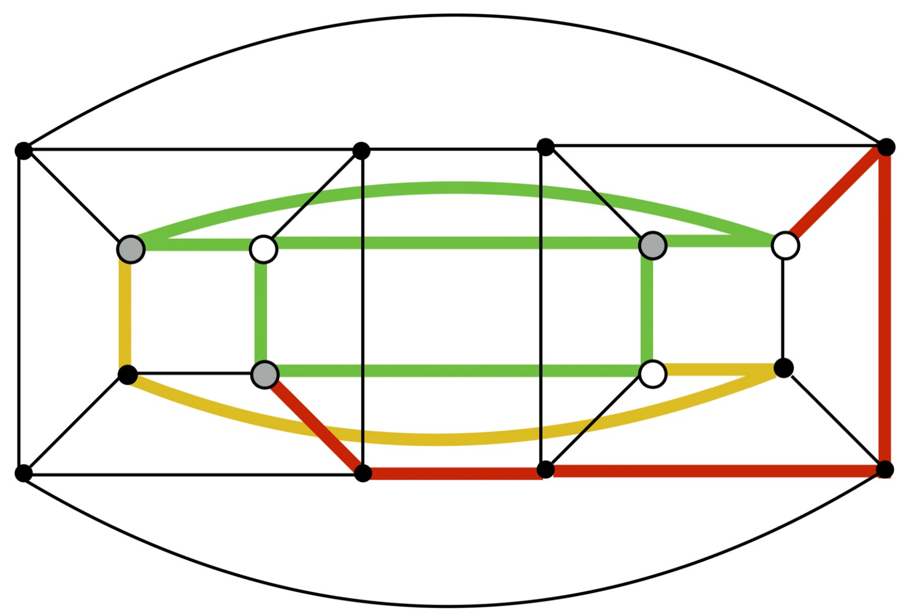
$\Leftarrow$ tricky!

Note: "G contains H" means we can find a copy of H inside G, where vertices of H are distinct vertices of G and edges of H are disjoint paths in G



4-dimensional hypercube

Copy of $K_{3,3}$ inside



Connectivity

Think of a graph G as a communication network

vertices $\longrightarrow$ nodes edges $\longrightarrow$ links

All nodes can communicate $\longrightarrow$ G must be connected

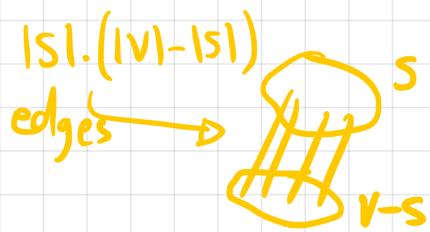
Links (edges) may fail !

1. G is a tree

diameter? (largest dist. between any pair of vertices)

#edges $= n-1$

very fragile : any edge failure disconnects !



2. G is a complete graph K_n

#edges $= \frac{n(n-1)}{2} \approx \frac{1}{2}n^2$!

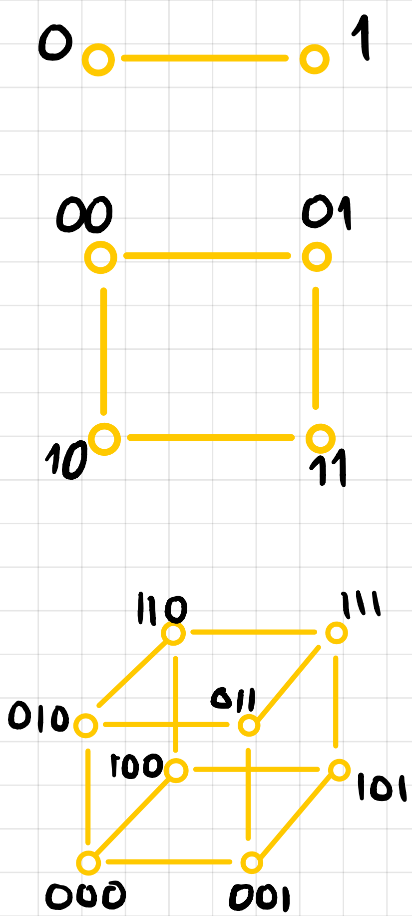
robust : need $\geq n-1$ edge failures to disconnect

Hypercubes

H_n : n-dimensional hypercube

vertices: $\{0,1\}^n$ (# vertices $= 2^n$)

edges: connect vertices that differ in 1 bit

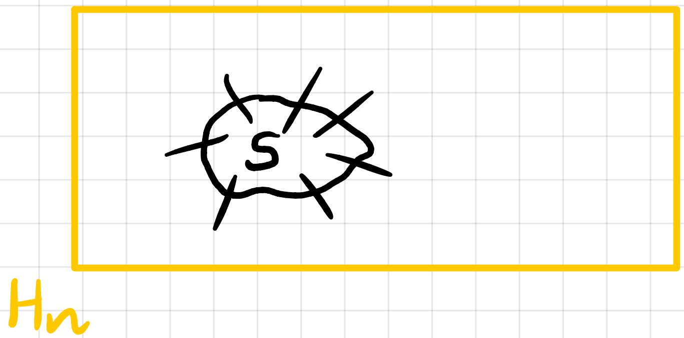


Note: $\boxed{H_n \text{ consists of 2 copies of } H_{n-1} \text{ with vertices matched up 1–1}}$

H_n has

2^n vertices	N
$\frac{n \cdot 2^n}{2} = n \cdot 2^{n-1}$ edges	$\frac{N}{2} \cdot \log_2 N$
all vertex degrees n	$\log_2 N$
diameter n	$\log_2 N$

Hypercubes are very well connected!



$S \subseteq V$: subset of vertices
$|S| \le \frac{|V|}{2}$

E_S: set of edges connecting S to $V-S$

<u>Theorem</u>: In H_n, for any set S as above, $|E_S| \geq |S|$

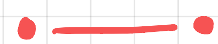
<u>Note</u>: Actually $|E_S| \geq \max\{n, |S|\}$ so v. small sets also OK.

(Ex.)

Theorem: In H_n, for any set S as above, $|E_S| \geq |S|$

Proof: Induction on n

Base case: $n = 1$. $|S| = 1$ $|E_S| = 1$ ✓



Inductive step: Assume true for H_k – prove for H_{k+1}

Let $S \subseteq V(H_{k+1})$ with $|S| \leq 2^k$

Write $S = S_0 \cup S_1$ where S_0, S_1 are in 0,1-subcubes

Assume w.l.o.g. $|S_0| \geq |S_1|$

Case (i): $|S_0| \leq 2^{k-1}$ & $|S_1| \leq 2^{k-1}$

Then can apply ind. hyp. within each subcube

$$\Rightarrow |E_S| \geq |S_0| + |S_1| = |S|$$ ✓

Theorem: In H_n, for any set S as above, $|E_S| \geq |S|$

Proof: Induction on n

Base case: $n = 1$. $|S| = 1$ $|E_S| = 1$ ✓

Inductive step: Assume true for H_k – prove for H_{k+1}

Let $S \subseteq V(H_{k+1})$ with $|S| \leq 2^k$

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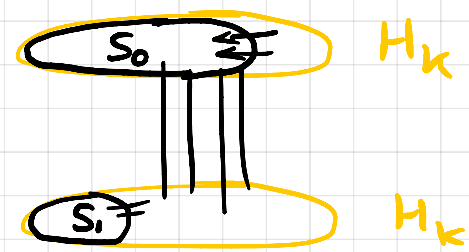

Case(ii): $|S_0| > 2^{k-1}$

Then $|S_1| = |S| - |S_0| < 2^{k-1}$

So ind. hyp. in 1-subcube gives $|S_1|$ edges

And ind. hyp. in 0-subcube applied to $V_0 - S_0$ gives $|V_0| - |S_0|$ edges

Finally, also get $|S_0| - |S_1|$ crossing edges (between subcubes)

So $|E_S| \geq |S_1| + |V_0| - |S_0| + |S_0| - |S_1| = |V_0| = 2^k \geq |S|$ ✓

Summary

- Planar graphs
- Euler's Formula : $v - e + f = 2$
- Corollary : $e \leq 3v - 6$ (or $e \leq 2v - 4$ for bipartite graphs)
- Two key non-planar graphs : K_5 & $K_{3,3}$ (Kuratowski's Theorem)
- Hypercubes H_n : well connected, good network model

Next Lecture

- Modular arithmetic