CS70 – Spring 2026

Lecture 9 : Feb. 17

Last Lecture

- Computing inverses: extended Euclid
- Chinese Remainder Theorem: solution to $\begin{cases} x \equiv a_1 \pmod{n_1} \\ \vdots \\ x \equiv a_k \pmod{n_k} \end{cases}$
- Fermat's Little Theorem: $a^{p-1} \equiv 1 \pmod{p}$ $\forall a \not\equiv 0$
- Euler's Totient Theorem: $a^{\varphi(n)} \equiv 1 \pmod{n}$
  $\forall a$ s.t. $\gcd(a,n) = 1$

Today

- RSA Cryptosystem — an application of all the above

Cryptography

Alice wants to send a message to Bob securely (nobody else can read it)

Eve is a malicious eavesdropper who can monitor network traffic

$x \rightarrow E(x)$

msg.

encoding function

$D(E(x)) = x$

decoding function

Classical cryptography (pre-1970s)

Alice & Bob share in advance a secret (physical) "codebook" & use it for both encoding & decoding

Problems

- Eve might steal the codebook
- By observing a bunch of encrypted messages, Eve might break the code (cf Enigma in WW II)
- Alice & Bob need to meet/coordinate in advance

Public-key Cryptography (Rivest-Shamir-Adelman, 1978)

publishes a public key K
keeps a related private key P

uses K to encrypt msgs to Bob

decrypts messages using P

learns nothing from observing encrypted msgs

"Padlock" analogy: anyone can lock it (with K)
need special key (P) to unlock

Chooses 2 large primes p, q [1024 bits $\approx 10^{308}$ HUGE!]

Chooses integer e coprime to $(p-1)(q-1)$

Publishes public key $\boxed{(N, e) \quad \text{where } N = pq}$

Retains private key $\boxed{d = e^{-1} \pmod{(p-1)(q-1)}}$

To send a message x (an integer mod N) to Bob:

- computes $\boxed{E(x) = x^e \bmod N}$
- sends $y = E(x)$ to Bob

Decrypts incoming message y via

$$\boxed{D(y) = y^d \bmod N}$$

Example: $p = 5,\ q = 17 \Rightarrow N = pq = 85$

$(p-1)(q-1) = 64$

Choose $e = 5$ (coprime to 64) $\Rightarrow$ Public key $K = (85, 5)$

And $d = e^{-1} \pmod{64} = 13$ Private key

Message $x = 11$ (+ve integer < 85)

$E(x) = 11^5 \bmod 85 = \boxed{61} = y$

Send to Bob

$11^2 = 121 \equiv 36$

$11^4 \equiv 36^2 = 1296 \equiv 21$

$11^5 = 11^4 \times 11 \equiv 21 \times 11 = 231 \equiv 61$

$D(y) = 61^{13} \bmod 85 = \boxed{11}$

$61^2 = 3721 \equiv 66$

$61^4 = 66^2 = 4356 \equiv 21$

$61^8 = 21^2 = 441 \equiv 16$

$61^{13} = 61^8 \times 61^4 \times 61$

$\equiv 16 \times 21 \times 61$

$= 20,496 \equiv 11$

Important: Alice never knows p, q or d ! (or $(p-1)(q-1)$)

How do we know RSA works & is secure?

1. **Correctness**: Prove that $D(E(x)) = x \quad \forall x \in \{0, 1, \ldots N-1\}$

2. **Security**: Argue that Eve learns nothing by observing $E(x)$, N, e

3. **Efficiency**: Show that all computations by Alice & Bob can be done efficiently
   - choosing large primes
   - computing e, d, $E(x)$, $D(y)$

1. **Correctness**: Prove that $D(E(x)) = x \quad \forall x \in \{0,1,\ldots N-1\}$

Proof: Recall that $E(x) = x^e \bmod N$ and $D(x) = y^d \bmod N$ where $y = x^e$. Also $d = e^{-1} \pmod{(p-1)(q-1)}$.

So need to prove: $x^{ed} \equiv x \pmod N$

Equivalently: $x(x^{ed-1} - 1) \equiv 0 \pmod N$

So s.t.p that $p \mid x(x^{ed-1}-1)$ and $q \mid x(x^{ed-1}-1)$ ⊛

Note that $ed = k(p-1)(q-1)+1$ for some $k \in \mathbb{Z}$ [because $d = e^{-1} \bmod (p-1)(q-1)$]

So ⊛ becomes: $p \mid x(x^{k(p-1)(q-1)} - 1)$ and same for q

$\hookrightarrow (x^{p-1})^{k(q-1)}$

Case (i): $p \mid x$ ✓

Case (ii): $p \nmid x$ Then $\gcd(x,p) = 1$.

So by FLT: $x^{p-1} \equiv 1 \pmod p$

$\Rightarrow x^{k(p-1)(q-1)} \equiv 1 \pmod p \Rightarrow x^{k(p-1)(q-1)} - 1 \equiv 0 \pmod p$

$\Rightarrow p \mid (x^{k(p-1)(q-1)} - 1)$ ✓ Same argument for q □

Alternative Proof using Euler's Theorem

Recall: $x^{\varphi(m)} \equiv 1 \pmod{m}$ $\forall x$ s.t. $\gcd(m,x)=1$

where $\varphi(m)$ is the number of such x

When $m = N = pq$, we have $\varphi(N) = pq - p - q + 1$

$= (p-1)(q-1)$ WHY?

$N = pq$ numbers total: $\{0, 1, \ldots, N-1\}$

The numbers that are not coprime with N are: $\{0, p, 2p, \ldots (q-1)p\} \cup \{0, q, 2q, \ldots (p-1)q\}$

(q of them; p of them; $p+q-1$ of them)

So $\varphi(N) = pq - (p+q-1)$ ✓

So by Euler's Thm:

$$x^{ed} = x^{k(p-1)(q-1)+1} \equiv x \pmod{N} \quad \forall x \text{ s.t. } \gcd(x,N) = 1$$

[And for the few special x with $\gcd(x,N) \neq 1$, we have $p \mid x$ or $q \mid x$ and can argue as before]

2. Security: Argue that Eve learns nothing by observing $E(x), N, e$

How could Eve try to learn x from $E(x) = x^e \bmod N$?

(i) Try guessing x

Note: $\exists$ unique x s.t. $E(x) = y$ so Eve can check

Problem: $2^{1024} \approx 10^{308}$ possibilities for x 😢

(ii) Try to figure out the factors p, q – then she could compute $d = e^{-1} \bmod (p-1)(q-1)$ and decrypt $x = y^d \bmod N$

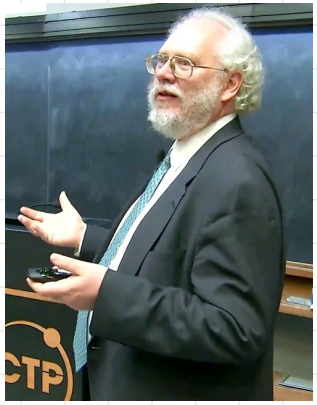
Problem: factoring large integers is very hard 😢

Note: computing $(p-1)(q-1)$ is equivalent to factoring N since $(p-1)(q-1) = N - p - q + 1$

(iii) Try to solve the equation $x^e \equiv y \pmod N$ for x

Problem: Equivalent to the discrete log problem, which in turn is very hard 😢

Caveat: A quantum computer can (in theory) efficiently factorize large numbers (Shor's Algorithm)



→ can break RSA (and other similar public key cryptosystems)

⟹ New field of "post-quantum cryptography"

3. Efficiency: Show that all computations by Alice & Bob can be done efficiently

(i) Computing e, d, $E(x)$, $D(y)$

e: anything coprime to $(p-1)(q-1)$ – often $2^{16}+1$ (prime)*

$d = e^{-1} \bmod (p-1)(q-1)$ – use extended Euclid

$E(x) = x^e \bmod N$

$D(y) = y^d \bmod N$

} use repeated squaring: if e, d have ≤ 1024 bits then only need $\sim 2k$ multiplications

Multiplications mod N take $\sim (1024)^2 \sim 1m$ bit ops.

*Still need to check that $\gcd((p-1)(q-1), e) = 1$.
This fails only if $e \mid (p-1)$ or $e \mid (q-1)$, which is very unlikely.
If it happens, just try a different e

3. Efficiency: Show that all computations by Alice & Bob can be done efficiently

(ii) Choosing large primes (say, 1024 bits)

- pick a random 1024-bit number n
- test if n is prime
- if not, try again

Prime Number Theorem: $\frac{\#\text{ of primes} < m}{m} \geq \frac{1}{\ln m}$

"on average, 1 in every $\ln m$ numbers is prime"

Since $\ln 2^{1024} \approx 710$, at least 1 in 710 1024-bit nos. is prime!

$\Rightarrow$ expect $\approx$ 710 attempts to find a prime

Primality Testing: How do we test if a large number n is prime?

Recall: Factoring n is very hard

But: Testing if n is prime is (fairly) easy!

Bad Primality Test:

for $2 \leq a \leq \sqrt{n}$

if $a \mid n$ then output "n is not prime"

output "n is prime"

Running time: $O(\sqrt{n}) \approx 10^{154}$ if $n \approx 10^{308}$ 😢

Note: Trying random a (as in Fermat Test on next slide) doesn't help because n may have only very few factors a!

Much Better: Fermat Test

pick $a \in \{2, 3, \ldots, n-1\}$ at random

if $\gcd(a,n) \neq 1$ output "n is not prime"

if $a^{n-1} \not\equiv 1 \pmod{n}$ output "n is not prime"

else output "n is prime"

If n is prime, output is always correct

If n is not prime, test may output wrong answer...
This happens if the random a we pick is not a witness: i.e., $\gcd(a,n) = 1$
and $a^{n-1} \equiv 1 \pmod{n}$
even though n is not prime

Good News: For almost all inputs n, at least half of the $a \in \{2, 3, \ldots, n-1\}$ are witnesses!

This means the Fermat Test is correct with probability at least $1/2$

Try 100 different a's $\Rightarrow$ Test is correct with prob. $\geq 1 - \frac{1}{2^{100}}$!

Carmichael Numbers: The inputs n that fail the Fermat Test are called Carmichael Numbers: they have no witnesses a.

First few CNs: 561, 1105, 1729, ...
Only ~ 1 in 2^{14} CNs up to 10^{21}

Miller-Rabin
Solovay-Strassen
⋮

Fact: There are efficient algorithms* (extensions of Fermat Test) that also work for CNs!