CS 70 Final
Fall 2025 December 19th, 2025

Print Your Name: _____

Print Your Student ID: _____

Print Student name to your left: _____

Print Student name to your right: _____

You have three hours. There are 16 questions of varying credit. (154 points total)

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
Points:	4	19	9	10	7	4	18	7	13	3	7	12	17	5	13	6	154

For questions with **circular bubbles**, select only one choice (there is only one correct answer).

- ○ Unselected option (completely unfilled)
- ⊘ Don't do this (it will be graded as incorrect)
- ● Only one selected option (completely filled)

For questions with **square boxes**, you may select one or more choices (select all that apply).

- ■ You can select
- ■ multiple squares
- ☑ Don't do this (it will be graded as incorrect)

- There will be no clarifications. We will correct any mistakes post-exam in as fair a manner as possible. Please just answer the question as best you can and move on even if you feel it is a mistake.
- The questions vary in difficulty. In particular, the exam is not in the order of difficulty and quite accessible short answer and proof questions are late in the exam. No points will be given for a blank answer, and there will be no negative points on the exam. **So do really scan over the exam.**
- You may, without proof, use theorems and lemmas that were proven in the notes and/or in lecture, unless otherwise stated. That is, if we ask you to prove a statement, prove it from basic definitions, e.g., "$d \mid x$ means $x = kd$ for some integer k" is a definition.
- You may consult only two double sided sheets of notes on both sides. Apart from that, you may not look at books, notes, etc. Calculators, phones, computers, and other electronic devices are NOT permitted.
- Anything you write outside the answer boxes or you ~~cross out~~ will not be graded. If you write multiple answers or your answer is ambiguous, we will grade the **worst** interpretation.

Pledge

As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.

In particular, I acknowledge that:

- I alone am taking this exam. Other than with the course staff, I will not have any verbal, written, or electronic communication about the exam with anyone else while I am taking the exam or while others are taking the exam.
- I will not refer to any books, notes, large language models, or online sources of information while taking the exam, other than what the instructor has allowed.
- I will not take screenshots, photos, or otherwise make copies of exam questions to share with others.

Sign your name: _____

This page intentionally left (mostly) blank

The exam begins on the next page.

Q1 *Illogical Propositions* **(4 points)**

Q1.1 (2 points) $((P \wedge Q) \Rightarrow R) \equiv (P \Rightarrow (Q \Rightarrow R))$

◯ Always True ◯ Sometimes False

Q1.2 (2 points) $\forall x(P(x) \Rightarrow Q(x)) \equiv (\forall x \neg P(x)) \vee (\forall x Q(x))$

◯ Always True ◯ Sometimes False

Q2 *Divisors and Modular Arithmetic* **(19 points)**

Q2.1 (5 points) Suppose that we are given $a, b \in \mathbb{Z}$.

Prove that $\forall x, y \in \mathbb{Z}$, $ax + by$ is a multiple of $d = \gcd(a, b)$.

Q2.2 (2 points) 7 divides $4^{186} - 1$. [Hint: $186 = 31 * 6$]

◯ True ◯ False

Q2.3 (2 points) 5 divides $4^n - 1$ for all **even** $n \in \mathbb{N}$

◯ True ◯ False

Q2.4 (3 points) What is the last digit of the number 3^{47} (i.e. what number 0-9 is in the "one's place")?

Q2.5 (3 points) Alice implements RSA correctly with $N = 33$ and $e = 7$ and sends an RSA-encryption $y = 6$ to Bob. Unfortunately, Alice forgot that N should be big because you know that $33 = 3 * 11$. Knowing the encryption y, what was the original message x?

Q2.6 (4 points) Find $6^7 + 7^6 \pmod{91}$. **(Hint: use CRT, $91 = 7 * 13$.)**

Q3 *Polypourri* **(9 points)**

Q3.1 (3 points) Working in mod 6, how many distinct polynomials of degree *exactly* two are there?

Q3.2 (3 points) Over GF(7), find an equivalent polynomial $g(x)$ (i.e. outputs the same values when given the same inputs) to $f(x) = 9x^{66} + 7x^{58} + 5x^{55} + 5x^{24} + 10x^{19}$ such that g has degree strictly less than 7 with coefficients from GF(7). That is, it should be the case that $\forall x \in$ GF(7), $f(x) \equiv g(x) (\mathrm{mod}\, 7)$. [Hint: Your solution will be a simple polynomial.]

Q3.3 (3 points) You receive $n + 2k$ points that come from a degree n-1 polynomial. You know there can be up to k corruptions. How many messages can you recover using Berlekamp–Welch?

Q4 *Graphs Speedrun* **(10 points)**

Q4.1 (2 points) K_4 is planar.

◯ True ◯ False

Q4.2 (2 points) K_5 has an Eulerian tour.

◯ True ◯ False

Q4.3 (2 points) All dice can be viewed as planar graphs wrapped around a sphere. A 12-sided die thus has 12 *faces* to roll. If we know there are 20 pointy *vertices* to 12-sided dice, then how many edges are there on a 12-sided die?

Q4.4 (4 points) Prove that if a graph has a vertex that has odd degree, there must be another vertex that has odd degree.

Q5 *Uncountable!?* **(7 points)**

State whether each set is Finite, Countably Infinite, or Uncountably Infinite.

Q5.1 (1 point) The set of all complex numbers $\mathbb{C}$

◯ Finite ◯ Countably Infinite ◯ Uncountably Infinite

Q5.2 (1 point) The set of all prime numbers

◯ Finite ◯ Countably Infinite ◯ Uncountably Infinite

Q5.3 (1 point) The set of real numbers between and including 0.001 and 0.1 (i.e the interval $[0.001, 0.1]$)

◯ Finite ◯ Countably Infinite ◯ Uncountably Infinite

Q5.4 (1 point) The set of complex numbers $a + bi$ such that $a, b \in \mathbb{Z}$

◯ Finite ◯ Countably Infinite ◯ Uncountably Infinite

Q5.5 (1 point) The set of edges of the complete bipartite graph $K_{100,100}$

◯ Finite ◯ Countably Infinite ◯ Uncountably Infinite

Q5.6 (1 point) The set of all integer powers of 2

◯ Finite ◯ Countably Infinite ◯ Uncountably Infinite

Q5.7 (1 point) The set of all polynomials with degree at most two on $\mathbb{R}$ that pass through $(1, 1)$ and $(2, 2)$

◯ Finite ◯ Countably Infinite ◯ Uncountably Infinite

Q6 *All About Compute* **(4 points)**

Two programs F and G are considered *equivalent* if for every input, either both halt with the same output or both don't halt (i.e. $\forall x, F(x) = G(x)$). We will show that deciding whether two arbitrary programs are equivalent is uncomputable:

By contradiction, suppose there exists a procedure *EQUIVALENT(F,G)* that always halts and returns TRUE iff programs F and G are equivalent. We want to construct a program *HALT(P,x)* that decides whether program P halts on input x, thus solving the Halting Problem.

Fill in the blanks below to implement *HALT* using *EQUIVALENT*.

```
def HALT(P,x):

    def F(y):
        ----(1)----
        ----(2)----

    def G(y):
        return 0

    return EQUIVALENT(----(3)----, ----(4)----)
```

1:

2:

3:

4:

Q7 *Let's Count!* **(18 points)**

Throughout this question, you may leave your answers unsimplified (i.e. you can leave binomial coefficients, factorials, exponents, etc. as is), but you should not use any summation or product notation (i.e. you may not use Σ or Π)

Q7.1 (3 points) A card deck has 52 cards; each of four suits has thirteen ranks (A, 2–10, J, Q, K). A "poker hand" is an unordered set of five cards. How many poker hands contain exactly one 6 and one 7?

Q7.2 (3 points) Suppose we draw 3 cards without replacement. Let A be the event that we draw at least one ace. Let C_1, C_2, and C_3 be the event that we draw exactly one ace, exactly two aces, and exactly three aces, respectively. Let F, S, and T be the events that the first, second, and third card drawn was an ace. Is it easier to write $P(A)$ in terms of C_i or F, S, T? Explain why your choice is easier.

- ○ C_i
- ○ F, S, T

Agnes is distributing 40 indistinguishable girl scout cookies to her 3 minions, Bob, Kevin, and Stuart, to help her sell them! However, each minion can only carry at most 15 cookies.

Q7.3 (3 points) If there were no carrying limit, how many ways are there of distributing the 40 cookies?

Q7.4 (3 points) How many ways are there of distributing the 40 cookies such that Bob's limit is exceeded?

Q7.5 (2 points) How many ways are there such that both Bob's and Kevin's limits are exceeded?

Q7.6 (4 points) How many ways are there to distribute the 40 cookies while respecting the limits of the minions? Give your answer terms of a, b, and c, which are the correct answers to 7.3, 7.4, and 7.5.

Q8 *The Paradoxical Survey* **(7 points)**

In a given town, $\frac{3}{4}$ of the residents have black hair and $\frac{2}{3}$ are female. Having black hair and being female are independent. Suppose the mayor selects two people at random from the town with replacement.

The mayor tells Manuel that at least one of the two people is a black haired female.

Manuel is told to guess whether both people have black hair. Given the information from the mayor, what is the probability that both people have black hair? Make sure to show your work.

Hint: The answer is not $\frac{9}{16}$.

Answer:

Q9 *Flipping Random Variables* **(13 points)**

Flip a biased coin infinitely many times. Each flip is heads with probability p, tails otherwise, independently of others. For each of the following situations, give the name of the distribution and its parameters. The first is provided for you as an example. Recall the common discrete distributions in this class are Bernoulli, Binomial, Geometric, Uniform, and Poisson.

Q9.1 (0 points) First three flips are heads.

$X \sim \text{Bernoulli}(p^3)$

Q9.2 (2 points) Number of heads in the first 70 flips.

$X \sim$

Q9.3 (2 points) If there's exactly one head in the first 20 flips, the flip number where the first head occurs.

$X \sim$

Q9.4 (3 points) Gap between first two tails. For example, the sequence THHHT... would have a gap of length 3. This distribution is some standard random variable minus 1.

$X \sim \qquad -1$

Q9.5 (3 points) Number of flips until the same flip occurs twice in a row. Assume for this problem that $p = \frac{1}{2}$. This distribution is 1 plus some standard random variable.

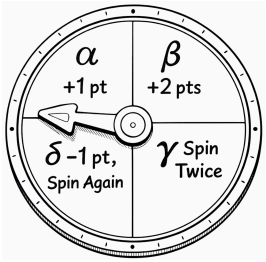
$X \sim 1 +$

Q9.6 (3 points) Number of runs in the first 50 flips. Define a "run" as an uninterrupted sequence of the same flip. For example if the flips are `HHHHHTTTHTTH...`, we have the run `HHHHH`, followed by the run `TTT`, followed by the run `H`, followed by the run `TT`, ... Assume for this problem that $p = \frac{1}{2}$. This distribution is 1 plus some standard random variable.

$X \sim 1 +$

Q10 *The Wheel* **(3 points)**

JQ loves the kids menu at the CS70 Restaurant, which has a game on the back. Each player gets exactly one spin on a wheel, though some of the options on the wheel allow additional spins. The player's score is the sum of the results of their spin. The possible outcomes for each spin are equally likely and are listed below:



α: Earn 1 point.
β: Earn 2 points.
γ: Spin two additional times.
δ: Lose 1 point and spin again.

Examples:

1. The player gets α on their first spin, so they earn 1 point and the game is over.
2. The player gets δ on their first spin, so their score becomes −1. The player gets γ, so now they get to spin twice. On the first of these spins, they get γ again, so increase their remaining spins to three. The player gets α, then β, then α, for a total score of −1 + 1 + 2 + 1 = 3. Overall spin sequence: $\delta\gamma\gamma\alpha\beta\alpha$.

Let X be the score of a player. What is $E[X]$?

○ 0 ○ 1 ○ 2 ○ 3 ○ 4 ○ ∞

Q11 *Summing* **(7 points)**

Q11.1 (3 points) Suppose $X \sim \text{Normal}(1, 2)$ and $Y \sim \text{Normal}(2, 1)$ are independent random variables. What is the distribution of $Z = 2X - Y + 1$? State its name and specify its parameter(s).

$Z \sim$

Q11.2 (2 points) Suppose we have two independent random variables $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\lambda)$. Let $Z = X + Y$. Which of the following are true?

☐ $Z \sim \text{Poisson}(2\lambda)$ ☐ $Z = 2P$, where $P \sim \text{Poisson}(\lambda)$ ○ None of the above

Q11.3 (2 points) Suppose we have n independent random variables $X_i \sim \text{Poisson}(\lambda)$. Let $S = \sum_{i=1}^{n} X_i$. Which of the following are true? Check all that apply.

☐ For all n, $S \sim \text{Poisson}(\lambda n)$ ○ None of the above

☐ In the large n limit, $\frac{S-\lambda n}{\sqrt{\lambda n}}$ converges to $\text{Normal}(0, 1)$

Q12 *The Waiting Game* **(12 points)**

Sam and Andy are waiting for students in their office hours. They've observed that students arrive at a rate of one student every 30 minutes, and decide to model students arrivals as a Poisson random variable.

Q12.1 One day, nobody has shown up after 50 minutes. They're debating whether they should leave early to eat pineapple pizza. What is the probability that at least one student arrives in the last ten minutes?

Q12.2 Suppose Jay observes the same rate of student arrivals. However, due to his excellent teaching skills, once a student arrives at his OH, they do not leave.

If nobody is in the room at the start of the hour, what is the expected number of students in the room at the end of the hour?

Suppose we model the departure of BART trains as an exponential random variable. On average, BART trains leave Richmond every 15 minutes.

Q12.3 If Ishan shows up at a random time, what is his expected wait for a BART train to leave?

Q12.4 A station agent informs Ishan that the most recent train left 5 minutes ago; what is his expected wait for a BART train?

Q12.5 For this part, trains arrive at some rate λ_t trains per hour. Passengers arrive at the station at some rate λ_p people per hour. Once passengers arrive, they take the first train that leaves.

Let A_p represent the first arrival time for a passenger and A_t represent the first arrival time for a train. The joint PDF of these exponential random variables is:

$$f_{A_p,A_t}(a_p, a_t) = \lambda_p \lambda_t e^{-\lambda_p a_p - \lambda_t a_t}.$$

for $a_p, a_t \geq 0$ and 0 elsewise.

Write an expression for $P(A_t < A_p)$ but do not solve; this is the probability that a train arrives before any passengers.

Chill out zone

Q13 *Bingo* **(17 points)**

At the MLK Student Union (bound) Open House, every booth in the building is represented by exactly one square on a 5×5 bingo card (25 squares total). For each booth, Melody decides **independently** whether to visit it. She visits any given booth with probability p and skips it with probability $1 - p$. A square is **marked** if Melody visits its booth.

There are 12 possible **bingo lines**: the 5 rows, the 5 columns, and the 2 diagonals. Let X denote the total number of bingo lines that are completely marked. Note that there is no "free square".

Q13.1 (3 points) Compute the expectation $E[X]$.

Q13.2 (2 points) Consider two different bingo lines that share exactly one square. What is the probability that both get completely marked?

Q13.3 (2 points) Consider two different bingo lines that do not share any squares. What is the probability that both get completely marked?

Q13.4 (5 points) Let a be the answer to Q13.1, let b be the answer to Q13.2, and let c be the answer to Q13.3.

Compute the variance $\mathrm{Var}(X)$. Feel free to leave your answer in terms of a,b, and/or c.

Q13.5 (2 points) Students win a sticker if they have at least 1 bingo line. Give an upper bound for the probability that Melody achieves at least one bingo line, i.e. $P[X \geq 1]$, using the union bound.

Q13.6 (3 points) Students win a pair of socks if they have at least 2 bingo lines. Give an upper bound for the probability that Melody achieves at least two bingo lines, i.e. $P[X \geq 2]$, using Chebyshev's inequality. Give your answer in terms of $E[X]$ and $\mathrm{Var}(X)$.

Q14 *Covariance Proof* **(5 points)**

Consider two events A, B with nonzero probability and indicators $\boldsymbol{I}_A, \boldsymbol{I}_B$ for them.

Show that, if $\mathrm{Cov}(\boldsymbol{I}_A, \boldsymbol{I}_B) > 0$, then $P(A \mid B) > P(A)$.

Q15 *Continuous Random Variable Potpourri* **(13 points)**

Q15.1 (2 points) Suppose we have continuous random variables $X \sim \text{Gaussian}(3, 6)$, $Y \sim \text{Exponential}(3)$. What is $P(X = Y)$?

$$P(X = Y) =$$

Q15.2 (2 points) For $X \sim \text{Gaussian}(3, 6)$, which of the following is true about $P(X < 4)$?

◯ < 0.5 ◯ $= 0.5$ ◯ > 0.5 ◯ Not enough information

Suppose we have three independent continuous random variables over the interval between 0 and 12, i.e. $A, B, C \sim \text{Unif}(0, 12)$. Let X be an indicator that C is between A and B.

Q15.3 (2 points) What is $P(X = 1 |\ A, B)$? In other words, if we know A and B, what is the probability that C is between them? Give your answer in terms of A and B and potentially the absolute value function.

$$P(X = 1|A, B) =$$

Q15.4 (2 points) Suppose we don't know A and B. What is $P(X = 1)$? In other words, if we pick three random numbers in the range $[0, 12]$, what is the chance that the third one we pick is between the other two?

Hint: There's a way to do this without any integrals.

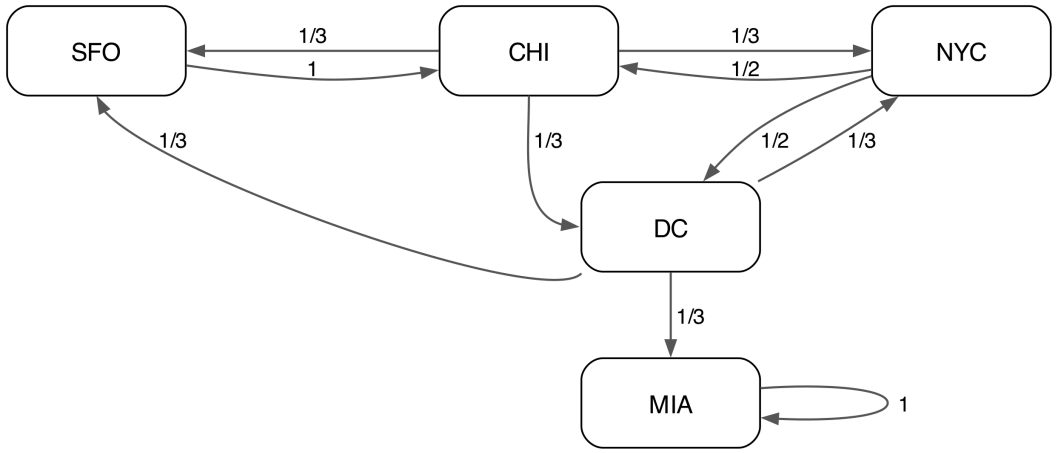
$$P(X = 1) =$$

Q15.5 (5 points) What is $E[|A - B|]$? Hint: There's a way to solve this problem using the previous two parts that avoid the need for any integrals. Warning: This problem is particularly challenging.

$$E[|A - B|] =$$

Q16 *Markov Trains* **(6 points)**

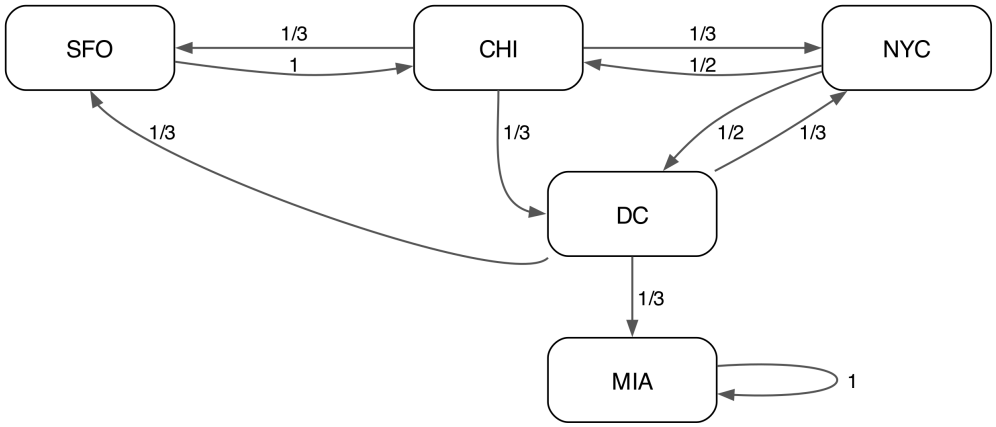
Tom is exploring America using trains. He's attempting to make it from Chicago (CHI) to Miami (MIA). However, America's trains are incredibly unreliable. Whenever Tom boards a train, he does not know its destination. The probability of a train being bound for a station is uniformly distributed among the outgoing routes.



Q16.1 (3 points) Tom is trying to avoid running into his uncle in New York City (NYC). What is the probability that Tom will make it to Miami from Chicago without visiting NYC?

Write out a system of equations that you could use to find the answer, but do not solve the equations.

Q16.2 (3 points)



Tom needs this trip to be over before his finals week; however his uncle doesn't intend to make that easy. Every time Tom ends up in NYC, his uncle will force him to stay there for three days. In all other cities, Tom will spend one day there before moving on to his next destination.

Assume train travel is instant and takes zero days.

What is the expected number of days Tom takes to arrive in Miami (starting from CHI and including the initial day spent there)? Write out a system of equations that you could use to find the answer, but do not solve the equations.

And with that, we're done! Have a great winter break, and please enjoy this wolf shirt.